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RAPID CONVERGENCE TO EQUILIBRIUM IN ONE DIMENSIONAL STOCHASTIC ISING MODELS

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We consider one dimensional stochastic Ising models with finite range interactions. For such processes we first prove that the semi-group of the process converges exponentially fast on the L^2 space of the Gibbs states. Under the additional hypothesis that the flip rates are attractive, we prove that the semigroup acting on the cylinder functions converges to equilibrium exponentially fast in the uniform norm.

0. Introduction. The stochastic Ising model is a Markov process which has been used to model the usual Ising model in nonequilibrium situations and in particular to study the convergence to equilibrium. The details of the mechanism of the stochastic Ising model are chosen so that the model satisfies a detailed balance equation, which means that when started in equilibrium the process is time reversible. Unfortunately, this requirement is not enough to determine the mechanism uniquely even when the spins are required to flip one at a time, and the author knows of no physical argument which can be used to pick one of the infinitely many possible mechanisms as the "correct" one. For this reason it would be nice to be able to draw conclusions about the stochastic Ising model merely from information about the Gibbs states (or more specifically about the potential which determines the Gibbs states) instead of having to use information about a particular choice of the stochastic Ising model. In this paper we attempt to do this in the case of one dimensional stochastic Ising models with finite range interactions. Admittedly this is the simplest situation; nevertheless, our techniques yield new results in this case and they show what type of information about the Gibbs states in two or more dimensions is needed to carry out this program in higher dimensions. We will point out the places where the one dimensionality is critical for our argument when we come to them. Presumedly our arguments which rely on one dimensionality can be replaced by other arguments which work at all temperatures strictly above the critical temperature; however, we have only been able to find such replacements at temperatures which are bounded away from the critical temperature from below. Since we do not have complete results in two or more dimensions and since the arguments are simpler in one dimension, we restrict ourselves to one dimension in this paper.

In order to state precisely what we prove we need some notation. Let Z be the integers and let $\{J_R: R \subset Z\}$ be a finite range translation invariant potential (i.e.

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for each finite subset $R \subset Z$, J_R is a real number with $J_R = 0$ if the diameter of R is larger than L (the range) and $J_{R+k} = J_R$ for all $R \subset Z$ and all $k \in Z$.)

If $R \subset Z$ is a finite subset we define $\chi_R(\eta)$ for $\eta \in E \equiv \{-1, 1\}^Z$ by

$$\chi_R(\eta) = \textstyle\prod_{j \in R} \eta(j).$$

If $\Delta \subset Z$ let $E(\Delta) = \{-1, 1\}^\Delta$. For finite Δ and $\sigma \in E(\Delta)$ and $\eta \in E$ we define

$$\rho_\Delta(\sigma \mid \eta) = \frac{\exp[-\sum_R J_R \chi_R(\eta\sigma)]}{\sum_\xi \exp[-\sum_R J_R \chi_R(\eta\xi)]}, \tag{0.1}$$

where the summation over R is over those finite subsets $R \subset Z$ for which $R \cap \Delta \neq \phi$, the summation over ξ is over $\xi \in E(\Delta)$, and the configuration $\eta\sigma$ is given by

$$\eta\sigma(j) = \begin{cases} \eta(j) & \text{if } j \notin \Delta. \\ \sigma(j) & \text{if } j \in \Delta. \end{cases}$$

The unique (since we are in one dimension and the range is finite) Gibbs state for the potential $\{J_R\}$ is the measure μ on E with the property that for all finite $\Delta \subset Z$, and all $\eta \in E(\Delta)$,

$$\rho_\Delta(\eta \mid \cdot) \text{ is a version of } P_\mu(\eta \mid M_\Delta)(\cdot),$$

where for arbitrary $\Delta \subset Z$, M_Δ is the σ-algebra generated by $\{\eta(k): k \notin \Delta\}$, and $P_\mu(\eta \mid M_\Delta)(\cdot)$ is the conditional probability under μ that the configuration inside Δ is η given the configuration outside of Δ.

Give E the product topology and let $C(E)$ be the space of continuous real valued functions on E. Let D be the set of local observables or cylinder functions on E. That is

$$D = \{f \in C(E): \text{there is a finite } \Delta \subset Z \text{ such that if } \eta = \eta' \text{ on } \Delta \text{ then } f(\eta) = f(\eta')\}.$$

We define the generator, Ω, of the stochastic Ising model as follows. First let $\{d_k: k \in Z\}$ be a collection of functions in D with the property that $d_k(\eta) = d_0(\tau^k\eta)$, where $\tau^k\eta \in E$ is given by $(\tau^k\eta)(j) = \eta(j+k)$. Assume in addition that there is an $\alpha > 0$ such that $d_k(\eta) \geq \alpha$ for all $k \in Z$ and all $\eta \in E$ and that d_k does not depend on the kth coordinate of η. This last assumption means that if ${}^k\eta$ is the configuration obtained by flipping the spin at k then $d_k(\eta) = d_k({}^k\eta)$. Otherwise the functions d_k are arbitrary. This is where the nonuniqueness enters in the stochastic Ising model. Now define

$$c_k(\eta) = d_k(\eta)\rho_{\{k\}}(-\eta(k) \mid \eta) \tag{0.2}$$

and define Ω on D by the formula

$$\Omega f(\eta) = \textstyle\sum_{k \in Z} c_k(\eta)[f({}^k\eta) - f(\eta)]. \tag{0.3}$$

The closure of Ω generates a positive contraction semigroup, $T_t: C(E) \to C(E)$, which has μ as its only stationary measure (see [8] and [9]). Since μ is stationary for $\{T_t: t \geq 0\}$ it follows that $\{T_t: t \geq 0\}$ can be extended to a semigroup of positive contractions on $L^2(\mu)$ (see [6]). We denote the extension by $\{T_t\ t \geq 0\}$ also. It will be clear from the context whether we are thinking of it as a semigroup on $C(E)$ or on $L^2(\mu)$. The flip rates (c_k's in (0.3)) were chosen to make this semigroup self-adjoint on $L^2(\mu)$. This is just another way of expressing the fact that the stochastic Ising model is time reversible when it is in equilibrium. See [6] for proofs of these statements and [1], [2], [5], or [7] for further discussion of the stochastic Ising model.

We use the following notation:

$\| \cdot \|_2$ is the usual L^2 norm on the space $L^2(\mu) \equiv L^2$.

$\| \cdot \|_u$ is the uniform norm on $C(E)$.

If $f \in L^2(\mu)$ then $\langle f \rangle = \int_E f(\eta)\mu(d\eta)$.

The main result of Section one is the following theorem.

THEOREM 0.4. *For any one dimensional stochastic Ising model as described above there is a constant $\gamma > 0$ such that for all $f \in L^2$*

$$\| T_t f - \langle f \rangle \|_2 \le e^{-\gamma t} \| f - \langle f \rangle \|_2. \tag{0.5}$$

If the flip rates, $\{c_k: k \in Z\}$, are attractive then we can strengthen the conclusion of Theorem (0.4) to a pointwise statement. Before giving that result we define what it means for the flip rates to be attractive. Notice that E is a lattice under the ordering $\eta \le \eta'$ if $\eta(k) \le \eta'(k)$ for all $k \in Z$. We say that $f \in C(E)$ is increasing if $\eta \le \eta'$ implies that $f(\eta) \le f(\eta')$ and similarly for decreasing. The flip rates are attractive if c_k is an increasing function on $\{\eta: \eta(k) = -1\}$ and a decreasing function on $\{\eta: \eta(k) = 1\}$. Note that if $d_k(\eta) = 1$ and $J_R \le 0$ if $|R| = 2$ and $J_R = 0$ if $|R| > 2$ (i.e. except for the self interaction ($|R| = 1$) there are only ferromagnetic pair potentials), then the flip rates $\{c_k: k \in Z\}$ are attractive.

Section two is devoted to the proof of the following theorem.

THEOREM 0.6. *For any one dimensional stochastic Ising model with finite range potential and attractive flip rates there is a constant $\delta > 0$ such that for all $f \in D$ there is a constant $A_f < \infty$ and*

$$\| T_t f - \langle f \rangle \|_u < A_f e^{-\delta t}. \tag{0.7}$$

1. L^2 convergence. D is a dense set in L^2, but for $f \in D$ and $t \ge 0$, $T_t f$ will not be in D. However there is a larger set

$$D_1 = \{f \in C(E): \textstyle\sum_{k \in Z} \sup_\eta |f({}^k\eta) - f(\eta)| < \infty\}$$

on which Ω is still defined by (0.3) and for which $T_t: D_1 \to D_1$ for all $t \ge 0$ (see [9]). Our first goal is to find an expression for $\int f\Omega f \, d\mu$, $f \in D_1$, which allows us to get some information about the spectrum of Ω.

Recall the notation used in (0.1). For singleton sets we write k instead of $\{k\}$.

LEMMA 1.1. *For $f \in D_1$,*

$$\Omega f(\eta) = \textstyle\sum_{k \in Z} \sum_{\xi \in E(k)} (f(\eta\xi) - f(\eta))\, d_k(\eta)\rho_k(\xi \mid \eta).$$

PROOF. If $f \in D$ this follows immediately from the definitions. For $f \in D_1$ it follows since Ω is a closed operator. □

LEMMA 1.2. *For* $f, g \in D_1$,

$$\int g\Omega f\, d\mu = -\sum_{k\in Z} \int \left(\sum_{\xi\in E(k)} (g(\eta\xi) - g(\eta))\rho_k(\xi \mid \eta)\right) \left(\sum_{\xi\in E(k)} (f(\eta\xi) - f(\eta))\rho_k(\xi \mid \eta)\right) d_k(\eta)\mu(d\eta). \tag{1.3}$$

PROOF.

$$\int g\Omega f\, d\mu = \sum_k \int g(\eta) \sum_\xi (f(\eta\xi) - f(\eta))\rho_k(\xi \mid \eta)\, d_k(\eta)\mu(d\eta). \tag{1.4}$$

Now conditioning the integral in the kth term on $M_{\{k\}}$ we get

$$\sum_K \int \sum_{\sigma\in E(k)} g(\eta\sigma) \sum_\xi (f(\eta\xi) - f(\eta\sigma))\rho_k(\xi \mid \eta\sigma)\, d_k(\eta\sigma)\rho_k(\sigma \mid \eta)\mu(d\eta). \tag{1.5}$$

Next note that $\rho_k(\xi \mid \eta) = \rho_k(\xi \mid \eta\sigma)$, $d_k(\eta) = d_k(\eta\sigma)$ and that for all η

$$\sum_\sigma \sum_\xi (f(\eta\xi) - f(\eta\sigma))\rho_k(\xi \mid \eta)\rho_k(\sigma \mid \eta) = 0.$$

Thus the expression in (1.5) is equal to

$$\begin{aligned}
&\sum_k \int \sum_\sigma \left(g(\eta\sigma) - \sum_\xi g(\eta\xi)\rho_k(\xi \mid \eta)\right) \\
&\qquad\cdot \left(\sum_\xi (f(\eta\xi) - f(\eta\sigma))\rho_k(\xi \mid \eta)\right) d_k(\eta)\rho_k(\sigma \mid \eta)\mu(d\eta) \\
&= -\sum_k \int \left(\sum_\xi (g(\eta\xi) - g(\eta))\rho_k(\xi \mid \eta)\right) \\
&\qquad\cdot \left(\sum_\xi (f(\eta\xi) - f(\eta))\rho_k(\xi \mid \eta)\right) d_k(\eta)\mu(d\eta). \quad \square
\end{aligned}$$

From Lemma (1.2) we obtain, for $f \in D_1$

$$\int f\Omega f\, d\mu = -\sum_k \int \left(\sum_\xi (f(\eta\xi) - f(\eta))\rho_k(\xi \mid \eta)\right)^2 d_k(\eta)\mu(d\eta). \tag{1.6}$$

It is possible to simplify this somewhat (see [6]), however, it will be most useful to us left in this form.

Zero is an eigenvalue for Ω and the corresponding eigenvectors are the constants. The fact that 0 is a simple eigenvalue is related to the uniqueness of the Gibbs state. (see Theorem 1.12 of [6]).

By the spectral theorem we may write the L^2 semigroup T_t as

$$T_t = \int_0^\infty e^{-\lambda t}\, dE(\lambda),$$

where $E(\lambda)$ is the resolution of the identity corresponding to $-\Omega$. (i.e. $-\Omega = \int_0^\infty \lambda\, dE(\lambda)$). We will prove Theorem 0.4 by showing that there is a gap of length $\gamma > 0$ between 0 and the rest of the spectrum of $-\Omega$.

LEMMA 1.7. *If there is a* $\gamma_0 > 0$ *such that for all* $f \in D_1$

$$\sum_k \int (\sum_\xi (f(\eta\xi) - f(\eta))\rho_k(\xi \mid \eta))^2 \mu(d\eta) \geq \gamma_0 \| f - \langle f \rangle \|_2^2 \tag{1.8}$$

then for all $f \in L^2$,

$$\| T_t f - \langle f \rangle \|_2 \leq e^{-\gamma t} \| f - \langle f \rangle \|_2, \tag{1.9}$$

where $\gamma = \gamma_0 \alpha > 0$. (See the assumptions about d_k in the introduction.)

PROOF. Since D_1 is dense in L^2 and T_t is continuous on L^2, (1.9) will follow for all $f \in L^2$ if we can prove it for all $f \in D_1$. By (1.6) and (1.8) for all $f \in D_1$ we have

$$-\int f \Omega f \, d\mu \geq \gamma \| f - \langle f \rangle \|_2^2. \tag{1.10}$$

Since $T_t 1 = 1$ it suffices to prove (1.9) with $\langle f \rangle = 0$.

Now D_1 is contained in the domain of Ω and $T_t : D_1 \to D_1$, hence

$$\frac{d}{dt} \| T_t f \|_2^2 = \frac{d}{dt} (T_t f, T_t f) = 2(T_t f, \Omega T_t f) < -2\gamma \| T_t f \|_2^2.$$

The conclusion follows immediately from this. □

The results up to this point were already clear in [6]. The inequality in (1.8) is some sort of mixing condition on the Gibbs state μ. All previous successful attempts to check (1.8) have proceeded by checking the conclusion of Theorem 0.6 for some particular choice of flip rates and then using the spectral theorem together with (1.6). Even in one dimension this can be done only in special cases. The goal of the rest of this section is to show that (1.8) holds for all finite range one dimensional systems. To do this we first define auxiliary operators Ω_N on D_1 as follows. Let $\Delta(k, N) = \{j \in Z : | k - j | < N\}$. Then for $f \in D_1$ define

$$\Omega_N f(\eta) = (2N + 1)^{-1} \sum_{k \in Z} \sum_{\xi \in E(\Delta(k,N))} (f(\eta\xi) - f(\eta))\rho_{\Delta(k,N)}(\xi \mid \eta).$$

For each N, the closure of Ω_N generates a positive contraction semigroup, $T_t^{(N)}$, on $C(E)$. The Markov process whose semigroup is $T_t^{(N)}$ has the following intuitive description. At each site $k \in Z$ there is an independent Poisson process with rate $1/(2N + 1)$. At the times of the jumps in this process the configuration in the interval $\Delta(k, N)$ is changed to ξ with probability $\rho_{\Delta(k,N)}(\xi \mid \eta)$, where η was the configuration of the system immediately before the jump occurred.

The proof of the next lemma is exactly the same as the proof of Lemma (1.2) except that we condition on $M_{\Delta(k,N)}$ instead of $M_{\{k\}}$.

LEMMA 1.11. *For* $f, g \in D_1$

$$\int g \Omega_N f \, d\mu = -\sum_k (2N + 1)^{-1} \int (\sum_{\xi \in E(\Delta(k,N))} (g(\eta\xi) - g(\eta))\rho_{\Delta(k,N)}(\xi \mid \eta)) \cdot (\sum_{\xi \in E(\Delta(k,N))} (f(\eta\xi) - f(\eta))\rho_{\Delta(k,N)}(\xi \mid \eta))\mu(d\eta).$$

Just as for Ω, Lemma 1.11 implies that the semigroups $\{T_t^{(N)}: t \geq 0\}$ are self-adjoint on L^2 and that zero is a simple eigenvalue of Ω_N.

Suppose that we can show that for some N there is a $\gamma_1 > 0$ such that

$$\text{(1.12)} \qquad \begin{gathered} \text{for all } f \in D \text{ there is a constant } A_f < \infty \text{ such that} \\ \| T_t^{(N)} f - \langle f \rangle \|_u \leq A_f e^{-\gamma_1 t}. \end{gathered}$$

In Lemma 1.7 we reduced the proof of Theorem 0.4 to checking the inequality (1.8). We will now check (1.8) assuming that (1.12) holds.

LEMMA 1.13. *If* (1.12) *holds then for all* $f \in D_1$,

$$-\int f \Omega_N f \, d\mu \geq \gamma_1 \| f - \langle f \rangle \|_2^2.$$

PROOF. Let $\{E_N(\lambda): \lambda > 0\}$ be the resolution of the identity corresponding to $-\Omega_N$. The lemma will follow if we can show the $E_N(\lambda) - E_N(0) = 0$ for all $0 < \lambda < \gamma_1$. We proceed by contradiction. Suppose $0 < \lambda' < \gamma_1$ and $h \in L^2$ are such that $(E_N(\lambda') - E_N(0))h = h$, and that $\| h \|_2 = 1$. Since D is dense in L^2 there is an $f \in D$ such that $\| f - h \|_2 < 1/2$ and since $(h, 1) = ((E_N(\lambda') - E_N(0))h, E_N(0)1) = ((E_N(0) - E_N(0))h, 1) = 0$ we may assume that $(f, 1) = 0$. Therefore

$$\begin{aligned} \| (E_N(\lambda') - E_N(0)) f \|_2 &= \| (E_N(\lambda') - E_N(0))(f - h) + (E_N(\lambda') - E_N(0)) h \|_2 \\ &> 1 - \| (E_N(\lambda') - E_N(0))(f - h) \|_2 > 1/2. \end{aligned}$$

Hence for $t > 0$

$$\begin{aligned} A_f^2 e^{-2\gamma_1 t} > \| T_t^{(N)} f \|_u^2 > \| T_t^{(N)} f \|_2^2 &= \int_0^\infty e^{-2\lambda t} \, d(E_N(\lambda) f, f) \\ &\geq \int_0^{\lambda'} e^{-2\lambda t} \, d(E_N(\lambda) f, f) \geq e^{-2\lambda' t} \| E_N(\lambda') f \|_2^2 \geq e^{-2\lambda' t}/4. \end{aligned}$$

That is $e^{-2(\gamma_1 - \lambda)t} \geq 1/4 \, A_f^2$ for all $t \geq 0$. Since $\gamma_1 > \lambda'$ this is a contradiction. □

LEMMA 1.15. *For all* $N = 2, 3, \cdots$ *there is a constant* $\gamma_N > 0$ *such that for all* $f \in C(E)$, $k \in Z$, *and*, $\eta \in E$

$$\text{(1.16)} \qquad \begin{aligned} &\textstyle\sum_{j \in \Delta(k,N)} \sum_{\sigma \in E(\Delta(k,N))} \left(\sum_{\xi \in E(j)} (f((\eta\sigma)\xi) - f(\eta\sigma)) \rho_j(\xi \mid \eta\sigma) \right)^2 \rho_{\Delta(k,N)}(\sigma \mid \eta) \\ &\quad > \gamma_N \textstyle\sum_{\sigma \in E(\Delta(k,N))} \left(f(\eta\sigma) - \sum_{\xi \in E(\Delta(k,N))} f(\eta\xi) \rho_{\Delta(k,N)}(\xi \mid \eta) \right)^2 \rho_{\Delta(k,N)}(\sigma \mid \eta). \end{aligned}$$

PROOF. Since η is held fixed and is the same on both sides of (1.16) it suffices to prove (1.16) for functions f which only depend on $\sigma \in E(\Delta(k, N))$. Note that $\{\chi_F: F \subseteq \Delta(k, N)\}$ is a basis for the functions on $E(\Delta(k, N))$ and moreover it is an orthonormal basis with respect to the uniform probability measure on $E(\Delta(k, N))$.

Set

$$\Gamma_N = \inf_{\xi,\eta} 2^{(2N+1)} \rho_{\Delta(0,N)}(\xi \mid \eta)$$

and

$$\Gamma^N = \sup_{\xi,\eta} 2^{(2N+1)} \rho_{\Delta(0,N)}(\xi \mid \eta)$$

and note that if $f = \sum_{F\subset\Delta(k,N)} a_F \chi_F$, then for $\xi \in E(j)$ and $\sigma \in E(\Delta(k, N))$

$$f(\sigma\xi) - f(\sigma) = \begin{cases} -2 \sum_{F\subset\Delta(k,N),F\ni j} a_F\chi_F(\sigma) & \text{if } \xi = -\sigma(j) \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\begin{aligned}
&\textstyle\sum_{j\in\Delta(k,N)} \sum_{\sigma\in E(\Delta(k,N))} (\sum_{\xi\in E(j)} (f(\sigma\xi) - f(\sigma))\rho_j(\xi \mid \eta\sigma))^2 \rho_{\Delta(k,N)}(\sigma \mid \eta) \\
&\quad\geq (\Gamma_0)^2\Gamma_N \textstyle\sum_{j\in\Delta(k,N)} \sum_\sigma (\sum_\xi (f(\sigma\xi) - f(\sigma))/2)^2 2^{-(2N+1)} \\
&\quad= (\Gamma_0)^2\Gamma_N \textstyle\sum_j \sum_\sigma (-\sum_{F\ni j} a_F\chi_F)^2 2^{-(2N+1)} \\
&\quad= (\Gamma_0)^2\Gamma_N \textstyle\sum_j \sum_{F\ni j} a_F^2 \geq (\Gamma_0)^2\Gamma_N \sum_{F\subset\Delta(k,N),F\neq\varphi} a_F^2 \\
&\quad= (\Gamma_0)^2\Gamma_N \textstyle\sum_\sigma (f(\sigma) - a_\varphi)^2 2^{-(2N+1)} \\
&\quad\geq \frac{(\Gamma_0)^2\Gamma_N}{\Gamma^N} \textstyle\sum_\sigma (f(\sigma) - a_\varphi)^2 \rho_{\Delta(k,N)}(\sigma \mid \eta) \\
&\quad\geq \frac{(\Gamma_0)^2\Gamma_N}{\Gamma^N} \textstyle\sum_\sigma (\sum_{\xi\in E(\Delta(k,N))} (f(\xi) - f(\sigma))\rho_{\Delta(k,N)}(\xi \mid \eta))^2 \rho_{\Delta(k,N)}(\sigma \mid \eta),
\end{aligned}$$

and the lemma is proved with $\gamma_N = (\Gamma_0)^2\Gamma_N/\Gamma^N$. □

LEMMA 1.17. *If* (1.12) *holds for some N then there is a $\gamma_0 > 0$ such that* (1.8), *and hence* (1.9) *holds.*

PROOF. By Lemmas (1.11) and (1.13) there is a $\gamma > 0$ such that for all $f \in D_1$

$$\textstyle\sum_k (2N + 1)^{-1} \displaystyle\int (\textstyle\sum_{\xi\in E(\Delta(k,N))} (f(\eta\xi) - f(\eta))\rho_{\Delta(k,N)}(\xi \mid \eta))^2 \mu(d\eta) \geq \gamma_1 \| f - \langle f \rangle \|_2^2.$$

By conditioning on $M_{\Delta(k,N)}$ and using Lemma (1.15) we see that there is a $\gamma_N > 0$ such that

$$\begin{aligned}
&(2N + 1)^{-1} \textstyle\sum_k \displaystyle\int (\textstyle\sum_\xi (f(\eta\xi) - f(\eta))\rho_{\Delta(k,N)}(\xi \mid \eta))^2 \mu(d\eta) \\
&\quad= (2N + 1)^{-1} \textstyle\sum_k \displaystyle\int \textstyle\sum_\sigma (\sum_\xi (f((\eta\sigma)\xi) - f(\eta\sigma))\rho_{\Delta(k,N)}(\xi \mid \eta\sigma))^2 \\
&\qquad\qquad \cdot \rho_{\Delta(k,N)}(\sigma \mid \eta)\mu(d\eta) \\
&\quad< (2N + 1)^{-1}(\gamma_N)^{-1} \textstyle\sum_k \displaystyle\int \textstyle\sum_{j\in\Delta(k,N)} \sum_\sigma (\sum_{\xi\in F(j)} (f((\eta\sigma)\xi) - f(\eta\sigma))\mu_j(\xi \mid \eta\sigma))^2 \\
&\qquad\qquad \cdot \rho_{\Delta(k,N)}(\sigma \mid \eta)\mu(d\eta)
\end{aligned}$$

$$= (\gamma_N)^{-1} \textstyle\sum_j \sum_{k \in \Delta(k,N)} (2N+1)^{-1} \displaystyle\int \textstyle\sum_\sigma \left(\sum_\xi \left(f((\eta\sigma)\xi) - f(\eta\sigma)\right)\rho_j(\xi \mid \eta\sigma)\right)^2 \cdot \rho_{\Delta(k,N)}(\sigma \mid \eta)\mu(d\eta)$$

$$= (\gamma_N)^{-1} \textstyle\sum_j \displaystyle\int \left(\textstyle\sum_\xi \left(f(\eta\xi) - f(\eta)\right)\rho_j(\xi \mid \eta)\right)^2 \mu(d\eta).$$

Thus we may take $\gamma_0 = \gamma_1\gamma_N > 0$. □

To finish the proof of Theorem (0.4) we need to show that (1.12) holds for some N. For our proof of (1.12) we first make a further assumption which we will verify at the end of this section.

ASSUMPTION 1.18. For some N and every $k \in Z$ and $(\eta^1, \eta^2) \in E^2$ there is a measure $R_{k,N}(\ \mid (\eta^1, \eta^2))$ on $E(\Delta(k, N))^2$ such that; for each $A \subset E(\Delta(k, N))^2$, $R_{k,N}(A \mid (\ ,\ ))$ is

$$\sigma((\eta^1(j), \eta^2(j)): j \in \Delta(k, N+L)\backslash\Delta(k, N)) \text{ measurable},$$

$$R_{k,N}(A \mid (\eta^1, \eta^2)) = R_{0,N}(\tau^k A \mid \tau^k(\eta^1, \eta^2)),$$

where τ^k shifts all coordinates by k,

$$R_{k,N}(A \times E(\Delta(k, N)) \mid (\eta^1, \eta^2)) = \rho_{\Delta(k,N)}(A \mid \eta^1),$$

$$R_{k,N}(E(\Delta(k, N)) \times A \mid (\eta^1, \eta^2)) = \rho_{\Delta(k,N)}(A \mid \eta^2),$$

$$\textstyle\sum_{j \in \Delta(k,N)} \sup_{(\eta^1,\eta^2)} R_{k,N}(\sigma^1(j) \neq \sigma^2(j) \mid (\eta^1, \eta^2)) < N/L, \tag{1.19}$$

$$R_{k,N}(\sigma^1(j) \neq \sigma^2(j) \mid (\eta^1, \eta^2) = 0 \quad \text{if } j \in \Delta(k, N) \text{ and } \eta^1 \text{ and } \eta^2 \text{ on } \Delta(k, N+L)\backslash\Delta(k, N). \tag{1.20}$$

Assuming that there is an $R_{k,N}$ as in (1.18), (1.19), and (1.20) we couple together two copies of the process generated by Ω_N, one starting from η^1 and the other from η^2 by means of $\{R_{k,N}: k \in Z\}$. Specifically let $\mathscr{U}$ be the operator defined on $D(E \times E)$ = cylinder functions in $C(E \times E)$ by the formula

$$\mathscr{U}f(\eta^1, \eta^2) = \frac{1}{2N+1} \textstyle\sum_k \sum_{(\sigma^1,\sigma^2)} \left(f(\eta^1\sigma^1, \eta^2\sigma^2) - f(\eta^1, \eta^2)\right)R_{k,N}((\sigma^1, \sigma^2) \mid (\eta^1, \eta^2)). \tag{1.21}$$

By standard results on coupling (see [10]), the closure of $\mathscr{U}$ generates a semigroup S_t on $C(E \times E)$ such that if $f_i(\eta^1, \eta^2) = h(\eta^i)$ for some $h \in C(E)$, $i = 1, 2$, then $S_t f_i(\eta^1, \eta^2) = T_t^{(N)} h(\eta^i)$. We denote the coupled process by (η_t^1, η_t^2).

LEMMA 1.22. *For all* $(\eta^1, \eta^2) \in E^2$

$$\sup_k P_{(\eta^1,\eta^2)}(\eta_t^1(k) \neq \eta_t^2(k)) \leq e^{-t/2}. \tag{1.23}$$

PROOF. Let $A(k, j) = \Delta(k, N) \cap \{i \in Z: j \in \Delta(i, N+L)\backslash\Delta(i, N)\}$. Note that

$A(k, j) = \varphi$ if $|k - j| > 2N + L$. Now define

$$a_{k,j} = \begin{cases} \sum_{i \in A(k,j)} (2N+1)^{-1} \sup_{(\eta^1, \eta^2)} & \\ R_{i,N}(\sigma^1(k) \neq \sigma^2(k) \mid (\eta^1, \eta^2)) & \text{if } \ |k-j| < 2N + L \\ 0 & \text{otherwise.} \end{cases}$$

By assumption (1.18), $a_{k,j} = a_{0,j-k}$, and since $|\{j : i \in A(0, j)\}| \leq L$ we have

$$\begin{aligned} \sum_j a_{0,j} &= \sum_{j=-2N-L}^{2N+L} \sum_{i \in A(0,j)} (2N+1)^{-1} \sup_{(\eta^1,\eta^2)} \\ &\quad \cdot R_{i,N}(\sigma^1(0) \neq \sigma^2(0) \mid (\eta^1, \eta^2)) \\ &= \sum_{j=-2N-L}^{2N+L} \sum_{i \in A(0,j)} (2N+1)^{-1} \sup_{(\eta^1,\eta^2)} \\ &\quad \cdot R_{0,N}(\sigma^1(-i) \neq \sigma^2(-i) \mid (\eta^1, \eta^2)) \\ &\leq L \sum_{i=-N}^{N} (2N+1)^{-1} \sup_{(\eta^1,\eta^2)} \\ &\quad \cdot R_{0,N}(\sigma^1(-i) \neq \sigma^2(-i) \mid (\eta^1, \eta^2)) < 1/2. \end{aligned} \tag{1.24}$$

The last inequality in (1.24) follows from (1.19).

Let $\tilde{a} = \sum_j a_{0,j}$ and set $P_{k,j} = a_{k,j}/\tilde{a}$. Then by (1.20)

$$\begin{aligned} &\frac{d}{dt} P_{(\eta^1,\eta^2)}(\eta_t^1(k) \neq \eta_t^2(k)) \\ &= (2N+1)^{-1} \sum_{i=k-N}^{k+N} E[R_{i,N}(\sigma^1(k) \neq \sigma^2(k) \mid (\eta_t^1, \eta_t^2))] \\ &\qquad - P_{(\eta^1,\eta^2)}(\eta_t^1(k) \neq \eta_t^2(k)) \\ &\leq (2N+1)^{-1} \sum_{i=k-N}^{k+N} \sum_{\{j : i \in A(k,j)\}} [\sup_{(\xi^1,\xi^2)} R_{i,N}(\sigma^1(k) \neq \sigma^2(k) \mid (\xi^1, \xi^2)) \\ &\qquad \times P_{(\eta^1,\eta^2)}(\eta_t^1(j) \neq \eta_t^2(j))] \\ &\qquad - P_{(\eta^1,\eta^2)}(\eta_t^1(k) \neq \eta_t^2(k)) \\ &= \sum_j a_{k,j} P_{(\eta^1,\eta^2)}(\eta_t^1(j) \neq \eta_t^2(j)) - P_{(\eta^1,\eta^2)}(\eta_t^1(k) \neq \eta_t^2(k)) \\ &= \tilde{a} \sum_j P_{k,j}(P_{(\eta^1,\eta^2)}(\eta_t^1(j) \neq \eta_t^2(j)) - P_{(\eta^1,\eta^2)}(\eta_t^1(k) \neq \eta_t^1(k))) \\ &\quad - (1 - \tilde{a}) P_{(\eta^1,\eta^2)}(\eta_t^1(k) \neq \eta_t^2(k)). \end{aligned} \tag{1.25}$$

Note that $P_{k,j}$ is the transition matrix of a random walk on Z. Thus by standard arguments one easily concludes that

$$\begin{aligned} &P_{(\eta^1,\eta^2)}(\eta_t^1(k) \neq \eta_t^2(k)) \\ &\qquad \leq e^{-(1-\tilde{a})t} e^{-\tilde{a}t} \sum_{n=0}^{\infty} \frac{(\tilde{a}t)^n}{n!} \sum_j P_{k,j}^n P_{(\eta^1,\eta^2)}(\eta_t^1(j) \neq \eta_t^2(j)) \leq e^{-t/2}. \quad \square \end{aligned}$$

COROLLARY (1.26) *For all $f \in D$ there is a constant A_f such that*

$$\| T_t^{(N)} f - \langle f \rangle \|_u < A_f e^{-t/2}. \tag{1.27}$$

PROOF. Fix $f \in D$. Let $\Delta \subset Z$ be the smallest subset of Z such that $\eta = \eta'$ on Δ implies that $f(\eta) = f(\eta')$. We show that (1.27) holds with $A_f = 2 \| f \|_u |\Delta|$.

Since for all $t \geq 0$ and $\eta^1, \eta^2, \sigma \in E$, $T_t^{(N)}f(\sigma) = E_{(\sigma,\eta^2)}[f(\eta_t^1)] = E_{(\eta^1,\sigma)}[f(\eta_t^2)]$ and μ is stationary for $T_t^{(N)}$, we have

$$\langle f \rangle = \int E_{(\eta^1,\eta^2)}[f(\eta_t^2)]\mu(d\eta^2)$$

and

$$T_t^{(N)}f(\eta) = \int E_{(\eta,\eta^2)}[f(\eta_t^1)]\mu(d\eta^2).$$

Thus

$$\begin{aligned} | T_t^{(N)}f(\eta) - \langle f \rangle | &= \left| \int E_{(\eta,\eta^2)}[f(\eta_t^1) - f(\eta_t^2)]\mu(d\eta^2) \right| \\ &< 2 \| f \|_u \textstyle\sum_{k\in\Delta} \displaystyle\int P_{(\eta,\eta^2)}(\eta_t^1(k) \neq \eta_t^2(k))\mu(d\eta^2) \\ &< 2 \| f \|_u | \Delta | e^{-t/2}. \quad \square \end{aligned}$$

REMARK (1.28) The argument up to this point has not depended on dimension in any way. Dimension only comes in during the verification of Assumption (1.18)–(1.20). The method we use below yields the desired result in one dimension for all finite range translation invariant potentials. If one were to restrict himself to translation invariant finite range ferromagnetic pair potentials then a sufficient condition to imply (1.18)—(1.20) would be the following:

(1.29) There exists a constant $A < \infty$ and $\delta > 0$ such that for all cubes $\Delta \subset Z^d$, all $j, k \in \Delta$, and all $\eta \in E$

$$| \textstyle\sum_{\sigma\in E(\Delta)} \sigma(j)\sigma(k)\rho_\Delta(\lambda \mid \eta) - \sum_{\sigma\in E(\Delta)} \sigma(j)\rho_\Delta(\sigma \mid \eta) \sum_{\xi\in E(\Delta)} \xi(k)\rho_\Delta(\xi \mid \eta) | < Ae^{-\delta|k-j|}.$$

While the author believes (1.29) to be true at all temperatures strictly above the critical temperature, he has not been able to prove it. Note however that (1.29) is a statement purely about the Gibbs state and has nothing to do with the stochastic Ising model.

We prove that Assumption (1.18)—(1.20) holds by first reducing the question to a similar one concerning finite state space Markov chains. To do this we let $S = \{-1, 1\}^{\{1,2,\cdots,L\}}$ and define Q on $S \times S$ by

$$Q(\eta, \sigma) = \exp[-\textstyle\sum' J_R\chi_R(\eta\sigma)],$$

where the summation $\sum'$ is over $R \subset [1, 2, \cdots 2L\}$ such that $R \cap \{1, \cdots, L\} \neq \varphi$ and $\eta\sigma$ is the configuration which is given by

$$\eta\sigma(k) = \begin{cases} \eta(k) & \text{if } 1 \leq k \leq L \\ \sigma(k - L) & \text{if } L + 1 \leq k \leq 2L. \end{cases}$$

Let λ be the largest eigenvalue of Q and $r(\sigma)$ be the corresponding right eigenvector. Since the entries of Q are strictly positive λ is a simple eigenvalue which is strictly larger than the absolute value of any other eigenvalue of Q, and we may assume that $r(\sigma) > 0$ for all σ. Now define

$$P(\eta, \sigma) = Q(\eta, \sigma)r(\sigma)/(\lambda r(\eta)). \tag{1.30}$$

$P(\eta, \sigma)$ is the transition matrix of a Markov chain on S. Also if $\eta_0, \eta_1, \cdots, \eta_n \in S$ then it is easily checked that

$$P(\eta_0, \eta_1)P(\eta_1, \eta_2) \cdots P(\eta_{n-1}, \eta_n)/P^n(\eta_0, \eta_n) = \rho_\Delta(\sigma \mid \eta), \tag{1.31}$$

where $\Delta = \{L + 1, \cdots, (n-1)L\}$, and

$$\eta(k) = \begin{cases} \eta_0(k) & \text{if } 1 \le k \le L \\ \eta_n(k - (n-1)L) & \text{if } (n-1)L < k \le nL \\ \text{arbitrary} & \text{otherwise,} \end{cases}$$

and σ is the configuration in $E(\{L + 1, \cdots, (n-1)L\})$ obtained by stringing together $\eta_1, \eta_2, \cdots, \eta_{n-1}$. Thus Assumption (1.18)–(1.20) follows from the following lemma.

LEMMA 1.32. *Let S be a finite set and $P(x, y)$ be a strictly positive transition matrix on S. Then for all $\delta > 0$ there is an N and for every pair $(a, a'), (b, b') \in S^2$ there is a measure $R(\ \mid (a, a'), (b, b'))$ on $S^N \times S^N$ such that*

$$\begin{aligned} &R((a_1, \cdots, a_N) \times S^N \mid (a, a'), (b, b')) \\ &\qquad = P(a, a_1)P(a_1, a_2) \cdots P(a_N, b)/P^{N+1}(a, b) \end{aligned} \tag{1.33}$$

$$\begin{aligned} &R(S^N \times (a_1', \cdots, a_N') \mid (a, a'), (b, b')) \\ &\qquad = P(a', a_1')P(a_1', a_2') \cdots P(a_N', b')/P^{N+1}(a', b') \end{aligned} \tag{1.34}$$

and

$$\textstyle\sum_{k=1}^N \sup_{(a,a'),(b,b')} R(a_k \ne a_k' \mid (a, a'), (b, b')) < \delta N. \tag{1.35}$$

PROOF. If we were conditioning only on the left end of the interval this would be an immediate consequence of any of several standard couplings (see [3]). Similarly if we were conditioning only on the right end we could use the reversed chain and any standard coupling. Since we are conditioning on both ends we will do the coupling in three pieces. A standard one for the section of the interval on the left end, a similar one for a section on the right end, and a coupling in the center which is concentrated on the diagonal if both of the end couplings succeed and is just product measure if either end coupling fails. The reader who is familiar with coupling arguments could certainly supply the details of the proof and may want to skip them here since they involve lots of notation which will not be used again in this paper.

Let

$$P_L(m, n, a, a_m, b) = P^m(a, a_m)P^{m+n}(a_m, b)/P^{2m+n}(a, b)$$

and

$$P_R(m, n, a_m, a_{m+n}, b) = P^n(a_m, a_{m+n})P^m(a_{m+n}, b)/P^{n+m}(a_m, b).$$

If π is the stationary measure for P then since $P^n(x, y) \to \pi(y)$ as $n \to \infty$ we may take m_0 and n_0 so large that

$$\sup_{a,b} \textstyle\sum_{a_{m_0}} |P_L(m_0, n_0, a, a_{m_0}, b) - \pi(a_{m_0})| < \delta/3, \tag{1.36}$$

$$\sup_{a_{m_0},b} \textstyle\sum_{a_{m_0+n_0}} |P_R(m_0, n_0, a_{m_0}, a_{m_0+n_0}, b) - \pi(a_{m_0+n_0})| < \delta/3, \tag{1.37}$$

and

$$2m_0/(2m_0 + n_0) < \delta/3. \tag{1.38}$$

To simplify the notation fix $m = m_0$, $n = n_0$, and set $P_L(a, a_m, b) = P_L(m, n, a, a_m, b)$ and $P_R(a_m, a_{m+n}, b) = P_R(m, n, a_m, a_{m+n}, b)$.

Now set

$$Q_L((a, a'), (a_m, a'_m), (b, b'))$$
$$= \begin{cases} P_L(a, a_m, b) \wedge P_L(a', a'_m, b') & \text{if } \ a_m = a'_m \\ [(P_L(a, a_m, b) - P_L(a', a_m, b'))]^+ & \\ \times [(P_L(a', a'_m, b') - P_L(a, a'_m, b))]^+/Z((a, a'), (b, b')) & \text{if } \ a_m \neq a'_m, \end{cases}$$

where $Z((a, a'), (b, b')) = 1 - \sum_c P_L(a, c, b) \wedge P_L(a', c, b')$.

Define $Q_R((a_m, a'_m), (a_{m+n}, a'_{m+n}), (b, b'))$ similarly except replace (a, a') by (a_m, a'_m); (a_m, a'_m) by (a_{m+n}, a'_{m+n}); and P_L by P_R throughout.

Note that

$$\textstyle\sum_{a'_m} Q_L((a, a'), (a_m, a'_m), (b, b')) = P_L(a, a_m, b) \tag{1.39}$$

$$\textstyle\sum_{a_m} Q_L((a, a'), (a_m, a'_m), (b, b')) = P_L(a', a'_m, b') \tag{1.40}$$

$$\textstyle\sum_{a'_{m+n}} Q_R((a_m, a'_m), (a_{m+n}, a'_{m+n}), (b, b')) = P_R(a_m, a_{m+n}, b) \tag{1.41}$$

and

$$\textstyle\sum_{a_{m+n}} Q_R((a_m, a'_m), (a_{m+n}, a'_{m+n}), (b, b')) = P_R(a'_m, a'_{m+n}, b'). \tag{1.42}$$

Now set

$$U((a_m, a'_m), (a_{m+1}, a'_{m+1}), \cdots, (a_{m+n}, a'_{m+n}))$$
$$= \begin{cases} \left(\prod_{k=m}^{m+n-1} P(a_k, a_{k+1})\right)/P^n(a_m, a_{m+n}) & \text{if } \ a_k = a'_k \quad k = m, \cdots, m+n \\ \dfrac{\prod_{k=m}^{m+n-1} P(a_k, a_{k+1})}{P^n(a_m, a_{m+n})} \, \dfrac{\prod_{j=m}^{m+n-1} P(a'_j a'_{j+1})}{P^n(a'_m, a'_{m+n})} & \text{if } \ a_m \neq a'_m \quad \text{or} \quad a_{m+n} \neq a'_{m+n} \\ 0 & \text{otherwise}, \end{cases}$$

and note that

$$\textstyle\sum_{a'_{m+1}, \cdots, a'_{m+n-1}} U((a_m, a'_m), \cdots, (a_{m+n}, a'_{m+n})) = \left(\prod_{k=m}^{m+n-1} P(a_k, a_{k+1})\right)/P^n(a_m, a_{m+n}), \tag{1.43}$$

and similarly for the summation over $a_{m+1}, \cdots, a_{m+n-1}$.

Next set $a_0 = a$, $a_0' = a'$, $a_{n+2m+1} = b$, $a'_{n+2m+1} = b'$, and

$$V_L((a_0, a_0'), (a_1, a_1'), \cdots, (a_m, a_m'), (a_{n+2m+1}, a'_{n+2m+1})) = \frac{\prod_{k=0}^{m-1} P(a_k, a_{k+1})}{P^m(a_0, a_m)} \frac{\prod_{j=0}^{m-1} P(a_j', a_{j+1}')}{P^m(a_0', a_m')} \cdot Q_L((a_0, a_0'), (a_m, a_m'), (a_{n+2m+1}, a'_{n+2m+1}))$$

and

$$V_R((a_m, a_m'), (a_{m+n}, a'_{m+n}), (a_{m+n+1}, a'_{m+n+1}), \cdots, (a_{n+2m+1}, a'_{m+2n+1})) = Q_R((a_m, a_m'), (a_{m+n}, a'_{m+n}), (a_{n+2m+1}, a'_{n+2m+1})) \times \frac{\prod_{k=1}^{m+1} P(a_{m+n+k-1}, a_{m+n+k})}{P^{m+1}(a_{n+m}, a_{n+2m+1})} \frac{\prod_{j=1}^{m+1} P(a'_{m+n+k-1}, a'_{m+n+k})}{P^{m+1}(a'_{n+m}, a'_{n+2m+1})}.$$

Finally take $N = n_0 + 2m_0$ and set

$$R((a_1, a_2, \cdots, a_N), (a_1', a_2', \cdots, a_N') \mid (a, a'), (b, b')) = V_L((a, a'), (a_1, a_1'), \cdots, (a_m, a_m'), (b, b'))U((a_m, a_m'), \cdots, (a_{m+n}, a'_{m+n})) \times V_R((a_m, a_m'), (a_{m+n}, a'_{m+n}), \cdots, (a_N, a_N'), (b, b')).$$

It now follows immediately from the definitions of V_L, V_R and U together with (1.39)—(1.41) that (1.33) and (1.34) hold. To see that (1.35) holds note that for $m < k < m + n$.

$$\begin{aligned} R(a_k \neq a_k' \mid (a, a'), (b, b')) &\leq R(a_m \neq a_m' \mid (a, a'), (b, b')) \\ &\quad + R(a_{m+n} \neq a'_{m+n} \mid (a, a'), (b, b')) \\ &= \textstyle\sum_{a_m \neq a_m'} Q_L((a, a'), (a_m, a_m'), (b, b')) \\ &\quad + \textstyle\sum_{a_m, a_m'} \sum_{a_{m+n} \neq a'_{m+n}} Q_R((a_m, a_m'), (a_{m+n}, a'_{m+n}), (b, b')) \\ &\quad \cdot Q_L((a, a'), (a_m, a_m'), (b, b')) \\ &= 1 - \textstyle\sum_c P_L(a, c, b) \wedge P_L(a', c, b') \\ &\quad + \textstyle\sum_{a_m, a_m'} (1 - \sum_c P_R(a_m, c, b) \wedge P_R(a', c, b)) \\ &\quad \cdot Q_L((a, a'), (a_m, a_m'), (b, b')). \end{aligned}$$

But

$$\begin{aligned} \textstyle\sum_c P_L(a, c, b) \wedge P_L(a', c, b') &= \tfrac{1}{2}[\textstyle\sum_c (P_L(a, c, b) + P_L(a', c, b')) - \sum_c \lvert P_L(a, c, b) - P_L(a', c, b') \rvert] \\ &= 1 - \tfrac{1}{2} \textstyle\sum_c \lvert P_L(a, c, b) - P_L(a', c, b') \rvert > 1 - (\delta/3). \end{aligned}$$

The last inequality follows from (1.36) and is uniform in (a, a'), and (b, b').

Similarly from (1.37)

$$\textstyle\sum_c P_R(a_m, c, b) \wedge P_R(a'_m, c, b') > 1 - (\delta/3)$$

uniformly in (a_m, a'_m), and (b, b').

Thus for $m \leq k \leq m + n$, $R(a_k \neq a'_k\ (a, a'), (b, b')) < 2\delta/3$.
For $1 \leq k < m$ or $m + n < k \leq N$, $R(a_k \neq a'_k \mid (a, a'), (b, b')) \leq 1$, and therefore

$$\textstyle\sum_{k=1}^{N} R(a_k \neq a'_k \mid (a, a'), (b, b')) < 2m_0 + (2\delta/3)\, n_0 < \delta N,$$

where the last inequality follows from (1.38). □

2. The attractive case. Recall that we defined what we mean by attractive flip rates just before the statement of Theorem (0.6). The property of attractive systems which we need is preservation of monotone functions by the semigroup, T_t. That is if $f \in C(E)$ satisfies $f(\eta) \leq f(\eta')$ whenever $\eta \leq \eta'$ then for all $t \geq 0$, $T_t f(\eta) \leq T_t f(\eta')$ whenever $\eta \leq \eta'$ (see [4]).

Suppose that the semigroup T_t preserves monotone functions on E. Since

$$\chi_\Delta(\eta) = \textstyle\prod_{k\in\Delta}\eta(k) = \prod_{k\in\Delta}(-1 + (\eta(k) + 1)) = \sum_{B\subset\Delta} (-1)^{|B|} \prod_{k\in B}(1 + \eta(k))$$

is a linear combination of monotone functions of the type $\omega_B(\eta) = \pi_{k\in B}(1 + \eta(k))$, and every function $f \in D$ is a finite linear combination of χ_Δ's, it follows that every $f \in D$ is a finite linear combination of ω_B's. Also for all finite B, $2^{|B|} \sum_{k\in B} \omega_k(\eta) - \omega_B(\eta)$ is an increasing function. Thus letting $\underline{+1}$ be the maximal element of E and $\underline{-1}$ be the minimal element of E we have

$$0 \leq T_t(2^{|B|} \textstyle\sum_{k\in B} \omega_k(\cdot) - \omega_B(\cdot))(\underline{+1}) - T_t(2^{|B|} \sum_{k\in B} \omega_k(\cdot) - \omega_B(\cdot))(\underline{-1}). \tag{2.1}$$

Hence because of translation invariance

$$\begin{aligned} 0 &\leq T_t\omega_B(\underline{+1}) - T_t\omega_B(\underline{-1}) \leq 2^{|B|} \textstyle\sum_{k\in B} (T_t\omega_k(\underline{+1}) - T_t\omega_k(\underline{-1})) \\ &= |B|\, 2^{|B|}(T_t\omega_0(\underline{+1}) - T_t\omega_0(\underline{-1})). \end{aligned} \tag{2.2}$$

Also since for all $\eta \in E$, $T_t\omega_B(\underline{-1}) \leq T_t\omega_B(\eta) \leq T_t\omega_B(\underline{+1})$, and

$$\langle \omega_B \rangle = \int \omega_B \, d\mu = \int T_t\omega_B \, d\mu,$$

we will have proved Theorem (0.6) if we show that there is an $A < \infty$ and a $\delta > 0$ such that

$$T_t\omega_0(\underline{+1}) - T_t\omega_0(\underline{-1}) \leq Ae^{-\delta t}. \tag{2.3}$$

In order to prove (2.3) we compare the stochastic Ising model on Z with finite ones on $\Delta(0, n)$, $n = 1, 2, \cdots$. We define the finite stochastic Ising models by first defining $\pi_n^+: E \to E$ by

$$\pi_n^+(\eta)(k) = \begin{cases} \eta(k) & \text{if } k \in \Delta(0, n) \\ +1 & \text{if } k \notin \Delta(0, n). \end{cases} \tag{2.4}$$

Then define

$$c_k^{n+}(\eta) = \begin{cases} c_k(\tau_n^+(\eta)) & \text{if } k \in \Delta(0, n) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\Omega_n^+ f(\eta) = \sum_k c_k^{n+}(\eta)(f({}^k\eta) - f(\eta)).$$

For each n, Ω_n^+ generates a semigroup T_t^{n+} and it is easily checked just as in Lemma (1.3) that

$$\begin{aligned}(2.5)\qquad &\int f(\eta)\Omega_n^+ g(\eta)\rho_{\Delta(0,n)}(d\eta \mid \underline{+1}) \\ &= -\sum_{k\in\Delta(0,n)} \int (\sum_{\xi\in E(k)} (g(\eta\xi) - g(\eta))\rho_k(\xi \mid \pi_n^+\eta)) \\ &\qquad (\sum_{j\in E(k)} (f(\eta\xi) - f(\eta))\rho_k(\xi \mid \pi_n^+\eta))d_k^{n+}(\eta)\rho_{\Delta(0,n)}(d\eta \mid \underline{+1}),\end{aligned}$$

where $d_k^{n+}(\eta) = d_k(\pi_n^+(\eta))$.

If we replace + by − everywhere in the above paragraph then we get operators T_t^{n-} and Ω_n^- such that (2.5) with + replaced by − still holds.

We need the following fact, which can be found proved in [4].

If f is increasing on E then for all n and all $\eta \in E$

$$(2.6)\qquad T_t^{n-}f(\eta) \le T_t f(\eta) \le T_t^{n+} f(\eta).$$

Our goal is to prove (2.3) and the method will be to prove that the semigroups $T_t^{n\pm}$ converge uniformly exponentially fast in the L^2 space of their stationary measures. Then we use this to obtain a statement about pointwise rates of convergence and use (2.6) to bound the expression on the left side of (2.3) by a term going to zero exponentially fast and a term involving integrals with respect to $\rho_{\Delta(0,n)}(\cdot \mid \underline{\pm 1})$. Finally we note that the dependence on $\underline{\pm 1}$ in the integral of ω_0 with respect to $\rho_{\Delta(0,n)}(\cdot \mid \underline{\pm 1})$ goes to zero exponentially fast in n and then we let t and n go to infinity simultaneously.

We denote the L^2 space of $\rho_{\Delta(0,n)}(\cdot \mid \underline{\pm 1})$ by $L^2(n, \pm)$. If $f \in L^2(n, \pm)$ then set $\langle f\rangle_n^\pm = \int f(\eta)\rho_{\Delta(0,n)}(d\eta \mid \pm 1)$ and denote the norm of $L^2(n, \pm)$ by $\|\cdot\|_n^\pm$.

LEMMA (2.7). *There is a* $\delta_1 > 0$ *such that for all* $n = 1, 2, \cdots$ *and all* $f \in L^2(n, \pm)$,

$$(2.8)\qquad -\int f(\eta)\Omega_n^\pm f(\eta)\rho_{\Delta(0,n)}(d\eta \mid \underline{\pm 1}) > \delta_1(\|f - \langle f\rangle_n^\pm\|_n^\pm)^2.$$

PROOF.

$$\begin{aligned}&-\int f(\eta)\Omega_n^\pm f(\eta)\rho_{\Delta(0,n)}(d\eta \mid \underline{\pm 1}) \\ &\quad= \sum_{k\in\Delta(0,n)} \int (\sum_{\xi\in E(k)} (f(\eta\xi) - f(\eta))\rho_k(\xi \mid \eta))^2 d_k^{n\pm}(\eta)\rho_{\Delta(0,n)}(d\eta \mid \pm 1) \\ &\quad> \inf d_0^{n\pm}(\eta) \sum_{k\in\Delta(0,n)} \int (\sum_\xi (f(\eta\xi) - f(\eta))\rho_k(\xi \mid \eta))^2\rho_{\Delta(0,n)}(d\eta \mid \underline{\pm 1}).\end{aligned}$$

Now from (0.1) it is easily seen that there is an $\alpha_1 > 0$ such that for all σ, $\sigma' \in E$, all n, and all $\eta \in E(\Delta(0, n))$

$$\rho_{\Delta(0,n)}(\eta \mid \sigma) > \alpha_1 \rho_{\Delta(0,n)}(\eta \mid \sigma'). \tag{2.9}$$

Letting $\alpha_2 = \inf_\eta d_0^{n\pm}(\eta)$, we have

$$\begin{aligned} -\int & f(\eta)\Omega_n^{\pm} f(\eta)\rho_{\Delta(0,n)}(d\eta \mid \pm 1) \\ &\geq \alpha_1\alpha_2 \textstyle\sum_{k\in\Delta(0,n)} \displaystyle\int \left(\textstyle\sum_{\xi\in E(k)} (f(\eta\xi) - f(\eta))\rho_k(\xi \mid \eta\sigma)\right)^2 \rho_{\Delta(0,n)}(d\eta \mid \sigma) \end{aligned} \tag{2.10}$$

for all $\sigma \in E$. Integrating the right side of (2.10) with respect to $\mu(d\sigma)$ and applying the results of section one we obtain

$$-\int f(\eta)\Omega_n^{\pm} f(\eta)\rho_{\Delta(0,n)}(d\eta \mid \underline{\pm 1}) > \alpha_1\alpha_2\gamma \,\| f - \langle f\rangle \|_2^2$$

for some $\gamma > 0$. Finally by (2.9)

$$\begin{aligned} \| f - \langle f\rangle \|_2^2 &= \inf_c \int (f(\eta) - c)^2 \mu(d\nu) \\ &= \inf_c \int\int (f(\eta) - c)^2 \rho_{\Delta(0,n)}(d\eta \mid \sigma)\mu(d\sigma) \\ &\geq \inf_c \int (f(\eta) - c)^2 \alpha_1 \rho_{\Delta(0,n)}(d\eta \mid \underline{\pm 1})\mu(d\sigma) \\ &= \alpha_1(\| f - \langle f\rangle_n^{\pm} \|_n^{\pm})^2. \end{aligned}$$

Thus the lemma is proved with $\delta_1 = \alpha_1\alpha_2\gamma > 0$. □

LEMMA 2.11. *There is a constant $\lambda < \infty$ such that for all $n = 1, 2, \cdots$ and all $t \geq 0$*

$$| T_t^{n\pm}\omega_0(\underline{\pm 1}) - \langle \omega_0\rangle_n^{\pm} | \leq 2e^{\lambda n - \delta_1 t}.$$

PROOF. Since $\| \omega_0 \|_u = 2$, Lemma (2.7) implies that

$$\| T_t^{n\pm}\omega_0 - \langle \omega_0\rangle_n^{\pm} \|_n^{\pm} < 2e^{-\delta_1 t}.$$

From (0.1) one sees that there is a $\lambda < \infty$ such that for all n, $\rho_{\Delta(0,n)}(\underline{\pm 1} \mid \underline{\pm 1}) > e^{-2\lambda n}$. Thus

$$\begin{aligned} | T_t^{n\pm}\omega_0(\underline{\pm 1}) - \langle \omega_0\rangle_n^{\pm} | &= \left[(T_t^{n\pm}\omega_0(\underline{\pm 1}) - \langle \omega_0\rangle_n^{\pm})^2 \left(\frac{\rho_{\Delta(0,n)}(\underline{\pm 1} \mid \underline{\pm 1})}{\rho_{\Delta(0,n)}(\underline{\pm 1} \mid \underline{\pm 1})}\right)\right]^{1/2} \\ &< e^{\lambda n}\left(\int (T_t^{n\pm}\omega_0(\eta) - \langle \omega_0\rangle_n^{\pm})^2 \rho_{\Delta(0,n)}(d\eta \mid \underline{\pm 1})\right)^{1/2} \\ &< e^{\lambda n} 2e^{-\delta_1 t}. \quad \square \end{aligned}$$

LEMMA (2.12). *There is a $\delta_2 > 0$ such that*

$$0 \leq \int \omega_0(\eta)\rho_{\Delta(0,n)}(d\eta \mid \pm 1) - \int \omega_0(\eta)\rho_{\Delta(0,n)}(d\eta \mid \underline{-1}) < 2e^{-\delta_2 n}.$$

PROOF. This is well known for one-dimensional Gibbs states with finite range potentials. A proof can be given by considering consecutive intervals of length L and reducing the problem to one involving finite state space Markov chains with strictly positive transition matrices as in (1.30) and (1.31). Once this reduction is made the proof is an elementary computation based on the Frobenius Theorem. We leave the details to the reader.

PROOF OF (2.3). By the monotonicity of $T_t\omega_0(\cdot)$, (2.6), and Lemmas (2.11) and (2.12), for all $n \geq 1$ and all $t \geq 0$ we have

$$
\begin{aligned}
0 &\leq T_t\omega_0(\underline{+1}) - T_t\omega_0(\underline{-1}) \leq T_t^{n+}\omega_0(\underline{+1}) - T_t^{n-}\omega_0(\underline{-1}) \\
&\leq \left| T_t^{n+}\omega_0(\underline{+1}) - \int \omega_0(\eta)\rho_{\Delta(0,n)}(d\eta \mid \underline{+1}) \right| \\
&\quad + \left| T_t^{n-}\omega_0(\underline{-1}) - \int \omega_0(\eta)\rho_{\Delta(0,n)}(d\eta \mid \underline{-1}) \right| \qquad (2.13)\\
&\quad + \left| \int \omega_0(\eta)\rho_{\Delta(0,n)}(d\eta \mid \underline{+1}) - \int \omega_0(\eta)\rho_{\Delta(0,n)}(d\eta \mid \underline{-1}) \right| \\
&< 4e^{\lambda n - \delta_1 t} + 2e^{-\delta_2 n}.
\end{aligned}
$$

Letting $n = (\delta_1 t)/(\lambda + \delta_2)$ we bound the right side of (2.13) by $Ae^{-\delta t}$ where $\Delta = 4 + 2e^{\delta_2}$, and $\delta = \delta_1\delta_2/(\lambda + \delta_2)$. □

REMARK (2.14). While we have used the assumption that the dimension is one several places in this section, only in Lemma (2.12) is this assumption crucial. In that lemma one would have to replace $e^{-\delta_2 n}$ with $e^{-\delta_2 n^d}$ in d dimensions. In other words if one could check Assumption (1.18)—(1.20) one could give different proof of everything in this section up to Lemma (2.12). With the new version of Lemma (2.12) the conclusion of Theorem (0.6) would be

$$\| T_t f - \langle f \rangle \|_u < A_f e^{-\delta t^{1/d}}$$

(again assuming (1.18)—(1.20)).

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