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# LARGE DEVIATION LOCAL LIMIT THEOREMS FOR ARBITRARY SEQUENCES OF RANDOM VARIABLES[1]

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The results of W. Richter (*Theory Probab. Appl.* (1957) **2** 206–219) on sums of independent, identically distributed random variables are generalized to arbitrary sequences of random variables $T_n$. Under simple conditions on the moment generating function of $T_n$, which imply that $T_n/n$ converges to zero, it is shown, for arbitrary sequences $\{m_n\}$, that $k_n(m_n)$, the probability density function of $T_n/n$ at $m_n$, is asymptotic to an expression involving the large deviation rate of $T_n/n$. Analogous results for lattice valued random variables are also given. Applications of these results to statistics appearing in nonparametric inference are presented. Other applications to asymptotic distributions in statistical mechanics are pursued in another paper.

**1. Introduction.** Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $E(X_1) = 0$, $\text{Var}(X_1) = 1$. Let $\psi(s)$ be the cumulant generating function (c.g.f.) and $\gamma(u) = \sup_{s \geq 0}[us - \psi(s)]$ be the large deviation rate of $X_1$. Let $S_n = X_1 + \cdots + X_n$. Under some mild conditions on $\psi$, Richter (1957) obtained an asymptotic expression for the probability density function, $f_n$, of $S_n/n$ involving the Cramer series. A close examination of the asymptotic expression for $f_n$, for the case of a nonlattice valued random variable $X$ in his paper, reveals that it can be rewritten as

$$f_n(x_n) = [n/2\pi]^{1/2} e^{-n\gamma(x_n)}[1 + O(|x_n|)], \tag{1.1}$$

whenever $x_n = o(1)$ and $\sqrt{n}x_n > 1$. The purpose of this paper is to obtain similar large deviation local limit theorems for arbitrary sequences of random variables which are not necessarily sums of i.i.d. random variables, thereby increasing the applicability of Richter's theorem.

Let $\{T_n, n \geq I\}$ be an arbitrary sequence of nonlattice random variables with an analytic moment generating function (m.g.f.) $\phi_n(z)$, nonvanishing in the region $\Omega = \{z \in \mathbb{C}: |\text{Real}(z)| < a\}$, where $\mathbb{C}$ is the set of complex numbers and $a > 0$. Let

$$\psi_n(z) = (1/n)\log \phi_n(z) \quad \text{for} \quad z \in \Omega \quad \text{and} \tag{1.2}$$

$$\gamma_n(u) = \sup_{s \in (-a,a)}[us - \psi_n(s)] \quad \text{for real } u. \tag{1.3}$$

The main Theorem 2.1 in Section 2 states that under some standard conditions

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on $\psi_n$, which guarantee the existence of the density function, $k_n$, of $T_n/n$ and further imply that $T_n/n$ converges to 0 in probability, we can write

$$k_n(m_n) = \left[\frac{n}{2\pi\psi_n''(\tau_n)}\right]^{1/2} e^{-n\gamma_n(m_n)}\left[1 + O\left(\frac{1}{n}\right)\right], \tag{1.4}$$

where $\{m_n\}$ is any sequence of real numbers and $\tau_n$ is defined by $\psi_n'(\tau_n) = m_n$. When $T_n$ is taken to be the sum of $n$ i.i.d. random variables the above theorem reduces to Richter's result.

Our extension of Richter's theorem to arbitrary sequences of random variables $T_n$ may be compared to Steinebach's (1978) extension of the earlier work on large deviations of Chernoff (1952) and Bahadur and Ranga Rao (1960) for sums of i.i.d. random variables. Steinebach's (1978) results on large deviations of an arbitrary sequence of random variables $T_n$, which extended earlier work of Sievers (1969), Plachky and Steinebach (1975) used conditions based solely on the moment generating function of $T_n$. In this sense Theorem 2.1 is comparable with Steinebach's (1978) results. We also present analogous results for lattice valued random variables.

Some other results on large deviations related to the main results of this paper are Sievers (1975), Vandermaele and Veraverbeke (1982) and Ellis (1984). This last reference contains a generalization of Chernoff's Theorem for $R^d$-valued random variables by applying the techniques of Gärtner.

The proofs of our theorems follow the classical pattern of proofs in this area, including that of Richter's theorem. The p.d.f. of $T_n/n$ is first expressed in terms of its Laplace transform. Next, the claimed asymptotic expression for this p.d.f. is extracted leaving a remainder term. The inverse transformation for Laplace transforms still allows one to pick the value of the real argument in that transform. We pick the value of the real argument in the appropriate way to use a saddle point approximation. Our proof differs from Richter's proof at this stage in two respects. We split the integral in the remainder term into two parts which depend on $n$ (see (2.13)), and we use the Cauchy formula for derivatives of an analytic function (see (2.5)) to obtain sharper estimates of all the quantities involved. This allows us to generalize Richter's theorem.

In Section 3 we present various applications of the theorems of Section 2, for statistics occurring in nonparametric inference. Prominent examples are the Wilcoxon signed rank statistic and Kendall's tau statistic for testing independence in a bivariate population. For example consider Kendall's tau $W_n$. We show that

$$P(W_n = 0) = \frac{6}{\sqrt{\pi n(n-1)(2n+5)}}\cdot\left[1 + O\left(\frac{1}{n}\right)\right] \tag{1.5}$$

as $n$ tends to $\infty$ through the appropriate sequence for which 0 is in the range of $W_n$.

**2. Local limit theorems for arbitrary sequence of random variables.** This section contains the main theorems of this paper, namely Theorems 2.1 and 2.2. We develop some notations before stating these theorems. Let

$\{T_n, n \geq 1\}$ be a sequence of nonlattice valued random variables with m.g.f. $\phi_n(z)$ which is analytic and nonvanishing for $z$ in $\Omega = \{z: |\text{Real}(z)| < a\}$ with $a > 0$. Let $\psi_n$ and $\gamma_n$ be as defined in (1.2) and (1.3). Let $I = (-a, a)$ and $I_1 = (-a_1, a_1)$ for some $a_1$ with $0 < a_1 < a$. Let $\{m_n\}$ be a sequence of real numbers and let $G_{n,\tau}(t) = \psi_n(\tau) + itm_n - \psi_n(\tau + it)$, for $\tau \in I_1$. The following theorems provide an asymptotic expansion for the density function of $T_n/n$ in terms of the large deviation rate $\gamma_n$.

THEOREM 2.1. *Assume the following conditions for* $T_n$:

(A) *There exists* $\beta < \infty$ *such that* $|\psi_n(z)| < \beta$, *for* $z \in \Omega$ *and* $n \geq 1$.

(B) *There exists* $\alpha > 0$ *and* $\tau_n \in I_1$ *such that* $\psi_n'(\tau_n) = m_n$ *and* $\psi_n''(\tau) \geq \alpha$ *for* $\tau \in I_1$ *and* $n \geq 1$.

(C) *There exists* $\eta > 0$ *such that for any* $0 < \delta < \eta$,

$$\inf_{|t| \geq \delta} \text{Real}(G_n(t)) = \min[\text{Real}(G_n(\delta)), \text{Real}(G_n(-\delta))], \quad \textit{for} \quad n \geq 1, \quad \textit{where}$$

$$G_n(t) = G_{n,\tau_n}(t).$$

(D) *There exists* $p, \ell > 0$ *such that*

$$\sup_{\tau \in I} \int_{-\infty}^{\infty} |\phi_n(\tau + it)/\phi_n(\tau)|^{\ell/n}\, dt = O(n^p).$$

*Then*

$$k_n(m_n) = \left[\frac{n}{2\pi\psi_n''(\tau_n)}\right]^{1/2} e^{-n\gamma_n(m_n)}\left[1 + O\left(\frac{1}{n}\right)\right]. \tag{2.1}$$

For lattice valued random variables $T_n$ we have the following analogous theorem.

THEOREM 2.2. *Let* $T_n$ *take values in the set* $\{a_n + kh_n: k = 0, \pm 1, \pm 2, \cdots\}$. *Let* $\{m_n = (a_n + k_n h_n)/n\}$ *be a sequence of real numbers, where* $\{k_n\}$ *is a sequence of integers. Assume that conditions* (A), (B) *of Theorem* 2.1 *hold and replace conditions* (C), (D) *by the following*:

(C′) *There exists* $\eta > 0$ *such that for any* $0 < \delta < \eta$,

$$\inf_{\delta \leq |t| \leq \pi/|h_n|} \text{Real}(G_n(t)) = \min[\text{Real}(G_n(\delta)), \text{Real}(G_n(-\delta))] \quad \textit{for} \quad n \geq 1.$$

(D′) *There exists* $p, \ell > 0$ *such that*

$$\sup_{\tau \in I} \int_{-\pi/h_n}^{\pi/h_n} |\phi_n(\tau + it)/\phi(\tau)|^{\ell/n}\, dt = O(n^p).$$

*Then*

$$\frac{\sqrt{n}}{|h_n|} \Pr(T_n = nm_n) = \left[\frac{1}{2\pi\psi_n''(\tau_n)}\right]^{1/2} e^{-n\gamma_n(m_n)}\left[1 + O\left(\frac{1}{n}\right)\right]. \tag{2.2}$$

We now make a few observations which will explain the implications of the Conditions (A)–(D). The proofs of the above two theorems will be given after Lemma 2.10.

REMARK 2.3. Condition (A) of Theorem 2.1 implies, as is shown later, that the first and the second derivatives of $\psi_n$ at 0 are bounded in $n$, which in turn implies that $(T_n - E(T_n))/n$ is converging in probability to 0.

REMARK 2.4. One can easily verify that if $T_n$ satisfies Conditions (A)–(D) of Theorem 2.1 then $T_n' = (T_n - E(T_n))$ also satisfies the same four conditions. Hence one can assume that $E(T_n) = 0$, although this assumption is not really needed in the proof of Theorem 2.1.

REMARK 2.5. Condition (B) is really a condition on the sequence $\{m_n\}$. This is trivially satisfied if $m_n$ is equal to $E(T_n)/n$; however, in practice we would like to choose $m_n$ so that $m_n - E(T_n)/n$ tends to a limit.

REMARK 2.6. Condition (C) is easily verified if $G_n(t)$ is increasing with $|t|$, otherwise it seems to be a rather difficult condition to check. The following lemma, which holds for any sequence of real valued functions $f(t)$, $n \geq 1$, provides an easily verifiable sufficient condition. In Example 3.1 we will be verifying this sufficient condition instead of Condition (C).

LEMMA 2.7. *Let $f_n(t)$ be a sequence of continuous real valued functions such that $f_n(0) = 0$ and zero is the unique minimum of $f_n$ for all $n \geq 1$. Assume that the following conditions hold for all $n \geq 1$.*

(i) *There exists $\eta_1 > 0$ such that $f_n(t)$ is increasing on $(0, \eta_1)$ and decreasing on $(-\eta_1, 0)$.*

(ii) *There exists $\varepsilon > 0$ such that $\inf_{|t| \geq \eta_1} f_n(t) > \varepsilon$.*

(iii) *There exists $0 < \eta < \eta_1$ such that $\sup_{|t| \leq \eta} f_n(t) < \varepsilon$.*

*Then for any $0 < \delta < \eta$*

$$\inf_{|t| \geq \delta} f_n(t) = \min[f_n(\delta), f_n(-\delta)] \quad \text{for} \quad n \geq 1. \tag{2.3}$$

PROOF. Let $0 < \delta < \eta$ be fixed. Conditions (ii) and (iii) imply that

$$\inf_{|t| \geq \eta_1} f_n(t) > \varepsilon > f_n(\delta).$$

Using Condition (i) we obtain

$$\inf_{|t| \geq \delta} f_n(t) = \inf_{\eta_1 \geq |t| \geq \delta} f_n(t) = \min[f_n(\delta), f_n(-\delta)]. \quad \square$$

Lemma 2.7 provides a simple way to verify Condition (C) of Theorem 2.1. Let $f_n(t) = \text{Real}(G_n(t))$. Note that $f_n(0) = 0$ and 0 is the unique minimum for $f_n$ for all $n \geq 1$ since $T_n$ is nonlattice valued. If $f_n$ satisfies conditions (i), (ii) and (iii) of Lemma 2.7, then $T_n$ will satisfy Condition (C) of Theorem 2.1.

REMARK 2.8. When $T_n$ is the sum of i.i.d. random variables with c.g.f. $\psi$ then $\text{Real}(\psi_n(\tau + it)) = \text{Real}(\psi(\tau + it))$ for all $n$ and $G_n(t)$ does not depend on $n$. Also $f(t) = \text{Real } G_n(t) = \psi(\tau) - \text{Real}(\psi(\tau + it))$. Since $\psi$ is the c.g.f. of a nonlattice valued random variable, $f(t)$ has a unique minimum at $t = 0$ and it satisfies all the three assumptions of Lemma 2.7. Thus Condition (C) holds automatically when $T_n$ is the sum of i.i.d. nonlattice random variables.

REMARK 2.9. Condition (D) of Theorem 2.1 not only guarantees the existence of the density function of $T_n$, but also permits the use of the inversion formula to get an expression for the p.d.f. of $T_n$. It is also used to show that the term $I_{n1}$ appearing in the proof of Theorem 2.1 goes exponentially fast to 0 (see (2.15)).

We will need the following lemma in the proof of Theorem 2.1.

LEMMA 2.10. *Assume that Conditions* (A), (B) *and* (C) *of Theorem* 2.1 *are satisfied. Recall that* $G_n(t) = G_{n,\tau_n}(t) = [\psi_n(\tau_n) + itm_n - \psi_n(\tau_n + it)]$. *There exists* $\delta_1$ *less than* $\eta$ *such that for* $0 < \delta < \delta_1$,

$$\inf_{|t| \geq \delta} \text{Real}(G_n(t)) \geq \alpha\delta^2/4 \quad \textit{for all} \quad n \geq 1. \tag{2.4}$$

PROOF. For $z \in \mathbb{C}$ and $r > 0$, define $c(z, r) = \{w \in \mathbb{C}: |z - w| = r\}$. Since $\psi_n$ is analytic in $\Omega$ and $|\tau_n| < a_1$, by Cauchy's theorem for derivatives we have

$$\psi_n^{(k)}(\tau_n) = \frac{k!}{2\pi i} \int_{c(\tau_n, a-a_1)} \frac{\psi_n(w)}{(w - \tau_n)^{k+1}}\, dw \quad \text{for all} \quad k \geq 1. \tag{2.5}$$

Using Condition (A), we obtain

$$\begin{aligned} |\psi_n^{(k)}(\tau_n)| &\leq \frac{k!}{2\pi} \sup_{w \in c(\tau_n, a-a_1)} |\psi_n(w)| \int_{c(\tau_n, a-a_1)} \frac{1}{|w - \tau_n|^{k+1}}\, dw \\ &\leq \frac{k!\beta}{(a - a_1)^k} \quad \text{for} \quad k \geq 1. \end{aligned} \tag{2.6}$$

Choose a positive number $\delta_1 < \min(\eta, (a - a_1)/2)$ such that

$$\delta_1 \beta[1 + 2\delta_1/(a - a_1)] \leq \alpha(a - a_1)^3/4.$$

Since $\psi_n$ is analytic in $\Omega$ and $|\tau_n| < a$, the following expansion is valid for all $n \geq 1$ and $|t| < \delta_1$,

$$\begin{aligned} \psi_n(\tau_n + it) &= \psi_n(\tau_n) + it\psi_n'(\tau_n) + ((it)^2/2!)\, \psi_n''(\tau_n) \\ &\quad + ((it)^3/3!)\, \psi_n'''(\tau_n) + R_n(\tau_n + it), \end{aligned} \tag{2.7}$$

where

$$R_n(\tau_n + it) = \frac{(it)^4}{2\pi i} \int_{c(\tau_n, a-a_1)} \frac{\psi_n(w)}{(w - \tau_n)^4 (w - \tau_n - it)}\, dw.$$

An upper bound for the modulus of the remainder term $R_n$ can be obtained as follows:

$$
\begin{aligned}
&|R_n(\tau_n + it)| \\
(2.8)\qquad &\le \frac{t^4}{2\pi} \sup_{w\in c(\tau_n, a-a_1)} |\psi_n(w)| \int_{c(\tau_n, a-a_1)} \frac{1}{|w-\tau_n|^4 |w-\tau_n-it|}\,dw \\
&\le \frac{2\beta t^4}{(a-a_1)^4},
\end{aligned}
$$

since $|w - \tau_n - it| \ge |w - \tau_n| - |t| = (a - a_1) - |t| \ge (a - a_1)/2$. Noting that $\psi_n'(\tau_n) = m_n$, we get from (2.7) that for $|t| \le \delta_1$

$$
(2.9)\qquad G_n(t) = \frac{t^2\psi_n''(\tau_n)}{2} + \frac{it^3\psi_n'''(\tau_n)}{3!} - R_n(\tau_n + it),
$$

so that

$$
\begin{aligned}
\left| \frac{G_n(t)}{t^2} - \frac{\psi_n''(\tau_n)}{2} \right| &\le \frac{|t|\,|\psi_n'''(\tau_n)|}{3!} + \frac{|R_n(\tau_n + it)|}{t^2} \\
&\le \frac{|t|\beta}{(a-a_1)^3} + \frac{2\beta t^2}{(a-a_1)^4} \quad \text{[from (2.6) and (2.8)]} \\
&\le \frac{\alpha}{4}
\end{aligned}
$$

because of our choice of $\delta_1$. Since $\psi_n''(\tau_n) \ge \alpha$, it follows that

$$
(2.10)\qquad \mathrm{Real}(G_n(t)) \ge \frac{\alpha t^2}{4} \quad \text{for all} \quad n.
$$

Since $\delta_1 < \eta$, it follows from Condition (C) that, for $\delta < \delta_1$, $\inf_{|t|\ge\delta}\mathrm{Real}(G_n(t)) = \min[\mathrm{Real}\,(G_n(\delta)), \mathrm{Real}(G_n(-\delta))] \ge \alpha\delta^2/4$. This completes the proof of Lemma 2.10. □

PROOF OF THEOREM 2.1. Let $F_n$ be the d.f. of $T_n$. For $\tau \in I$, define the conjugate distribution $H_n$ by

$$
(2.11)\qquad dH_n(x) = \frac{e^{\tau x}}{\phi_n(\tau)}\,dF_n(x).
$$

The c.f. of the d.f. $H_n$, which is given by $\phi_n(\tau + it)/\phi_n(\tau)$, is absolutely integrable in view of Condition (D). Thus the p.d.f. of $H_n$ exists and from the inversion formula, it is given by

$$
\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\phi_n(\tau + it)}{\phi_n(\tau)}\right]e^{-itx}\,dt.
$$

The p.d.f. of $F_n$ therefore is given by $(1/2\pi)\int_{-\infty}^{\infty}\phi_n(\tau + it)e^{-(\tau+it)x}\,dt$.

Thus the p.d.f. of $T_n/n$ is given by

$$k_n(x) = \frac{n}{2\pi} \int_{-\infty}^{\infty} \phi_n(\tau + it) e^{-n(\tau+it)x}\, dt.$$

The above expression is valid for any $\tau \in I_1$. Since we are interested only in $k_n(m_n)$, we substitute $x = m_n$ and $\tau = \tau_n$ in the above and arrive at the relation below, which is the starting point of the analysis of the error terms:

$$\begin{aligned} k_n(m_n) &= \frac{n}{2\pi} \int_{-\infty}^{\infty} \phi_n(\tau_n + it)\exp(-n(\tau_n + it)m_n)\, dt \\ &= \left[\frac{n}{2\pi\psi_n''(\tau_n)}\right]^{1/2} \exp(-n\gamma_n(m_n))I_n, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} I_n &= \left[\frac{n\psi_n''(\tau_n)}{2\pi}\right]^{1/2} \int_{-\infty}^{\infty} \exp(n[\gamma_n(m_n) - (\tau_n + it)m_n])\phi_n(\tau_n + it)\, dt \\ &= \left[\frac{n\psi_n''(\tau_n)}{2\pi}\right]^{1/2} \int_{-\infty}^{\infty} \exp(n[\psi_n(\tau_n + it) - \psi_n(\tau_n) - itm_n])\, dt \\ &= \left[\frac{n\psi_n''(\tau_n)}{2\pi}\right]^{1/2} \left[\int_{|t|\geq n^{-\lambda}} \exp(-nG_n(t))\, dt + \int_{|t|<n^{-\lambda}} \exp(-nG_n(t))\, dt\right] \\ &= I_{n1} + I_{n2} \quad \text{(say)}, \end{aligned} \tag{2.13}$$

wherein we have used the fact that $\gamma_{(n)}(m_n) = \tau_n m_n - \psi_n(\tau_n)$, and $\lambda$ is chosen to be a constant such that $\frac{1}{3} < \lambda < \frac{1}{2}$. The proof is completed by showing that $I_{n1}$ goes exponentially fast to zero and $I_{n2} = I + O(1/n)$, as $n$ goes to $\infty$. First consider the term $I_{n1}$. By Lemma 2.10 we can find an $N$ such that for $n \geq N$ we have

$$\inf_{|t|\geq n^{-\lambda}} \text{Real}(G_n(t)) \geq \frac{\alpha n^{-2\lambda}}{4}. \tag{2.14}$$

For $n \geq N$,

$$\begin{aligned} |I_{n1}| &\leq \left[\frac{n\psi_n''(\tau_n)}{2\pi}\right]^{1/2} \int_{|t|\geq n^{-\lambda}} |\exp(-nG_n(t))|\, dt \\ &\leq \left[\frac{n\psi_n''(\tau_n)}{2\pi}\right]^{1/2} \sup_{|t|\geq n^{-\lambda}} |\exp(-(n - \ell)G_n(t))| \int |\exp(-\ell G_n(t))|\, dt \\ &\leq \left[\frac{n\psi_n''(\tau_n)}{2\pi}\right]^{1/2} \sup_{|t|\geq n^{-\lambda}} |\exp(-(n - \ell)G_n(t))| \int \left|\frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)}\right|^{\ell/n} dt \\ &= O(n^{p+1/2})\exp[-(n - \ell)\inf_{|t|\geq n^{-\lambda}}\{\text{Real}(G_n(t))\}] \\ &= O(n^{p+1/2})\exp[-(n - \ell)\alpha/4n^{2\lambda}] \quad (\text{by } (2.14)) \\ &= O(n^{p+1/2})\exp[-\alpha n^{1-2\lambda}/4], \end{aligned} \tag{2.15}$$

which goes exponentially fast to zero since $\frac{1}{3} < \lambda < \frac{1}{2}$. Our next step is to show that $I_{n2} = 1 + O(1/n)$. Recall that

$$I_{n2} = \left[\frac{n\psi_n''(\tau_n)}{2\pi}\right]^{1/2} \int_{|t|<n^{-\lambda}} \exp(-nG_n(t))\, dt$$
$$= \left[\frac{\psi_n''(\tau_n)}{2\pi}\right]^{1/2} \int_{|s|<n^{1/2-\lambda}} \exp(-nG_n(s/\sqrt{n}))\, ds.$$

For $|s| < n^{1/2-\lambda}$, $s/\sqrt{n}$ goes to zero uniformly in $s$ as $n \to \infty$. Hence we can use (2.9) to get

$$I_{n2} = \left[\frac{\psi_n''(\tau_n)}{2\pi}\right]^{1/2} \int_{|s|<n^{1/2-\lambda}} \exp\left(n\left[-\frac{s^2}{2n}\psi_n''(\tau_n) - \frac{is^3}{6n\sqrt{n}}\psi_n'''(\tau_n) + R_n\left(\tau_n + i\frac{s}{\sqrt{n}}\right)\right]\right) ds$$

$$(2.16)\quad = \left[\frac{\psi_n''(\tau_n)}{2\pi}\right]^{1/2} \int_{|s|<n^{1/2-\lambda}} \exp\left(-\frac{s^2}{2}\psi_n''(\tau_n)\right)\left[\exp\left(-\frac{is^3}{6\sqrt{n}}\psi_n'''(\tau_n) + nR_n\left(\tau_n + i\frac{s}{\sqrt{n}}\right)\right)\right] ds$$

$$= \left[\frac{\psi_n''(\tau_n)}{2\pi}\right]^{1/2} \int_{|s|<n^{1/2-\lambda}} \exp\left(-\frac{s^2}{2}\psi_n''(\tau_n)\right)\left[1 - \frac{is^3}{6\sqrt{n}}\psi_n'''(\tau_n) + nR_n\left(\tau_n + i\frac{s}{\sqrt{n}}\right) + L_n(s)\right] ds,$$

where

$$(2.17)\quad L_n(s) = [e^{z_n} - 1 - z_n] \quad\text{and}\quad z_n = \left[-\frac{is^3}{6\sqrt{n}}\psi_n'''(\tau_n) + nR_n\left(\tau_n + \frac{is}{\sqrt{n}}\right)\right].$$

The right-hand side of (2.16) can be written as the sum of four integrals. The first integral equals $1 - 2\Phi(-n^{1/2-\lambda}\sqrt{\psi_n''(\tau_n)})$ which by Mill's ratio (e.g. see Feller (1968) page 175, (1.8)) is $1 + o(1/n)$ and the second integral is zero. Thus we get

$$(2.18)\quad \begin{aligned} I_{n2} = 1 + o\left(\frac{1}{n}\right) &+ n\left[\frac{\psi_n''(\tau_n)}{2\pi}\right]^{1/2} \int_{|s|<n^{1/2-\lambda}} \exp\left(-\frac{s^2}{2}\psi_n''(\tau_n)\right) R_n\left(\tau_n + \frac{is}{\sqrt{n}}\right) ds \\ &+ \left[\frac{\psi_n''(\tau_n)}{2\pi}\right]^{1/2} \int_{|s|<n^{1/2-\lambda}} \exp\left(-\frac{s^2}{2}\psi_n''(\tau_n)\right) L_n(s)\, ds. \end{aligned}$$

We now show that the last two terms on the r.h.s. are $O(1/n)$. Consider the third term. The second inequality below follows from (2.8) since $s/\sqrt{n}$ goes uniformly to zero for $|s| \le n^{1/2-\lambda}$.

$$\left| n\left[\frac{\psi_n''(\tau_n)}{2\pi}\right]^{1/2} \int_{|s|<n^{1/2-\lambda}} \exp\left(-\frac{s^2}{2}\psi_n''(\tau_n)\right) R_n\left(\tau_n + i\frac{s}{\sqrt{n}}\right) ds \right|$$
$$\le n\left[\frac{\psi_n''(\tau_n)}{2\pi}\right]^{1/2} \int_{|s|<n^{1/2-\lambda}} \exp\left(-\frac{s^2}{2}\psi_n''(\tau_n)\right) \left| R_n\left(\tau_n + i\frac{s}{\sqrt{n}}\right) \right| ds$$
$$\le \left[\frac{\psi_n''(\tau_n)}{2\pi}\right]^{1/2} \frac{2\beta}{n(a - a_1)^4} \int_{|s|<n^{1/2-\lambda}} s^4 \exp\left(-\frac{s^2}{2}\psi_n''(\tau_n)\right) ds = O\left(\frac{1}{n}\right).$$

Thus,

$$(2.19)\quad n\left[\frac{\psi_n''(\tau_n)}{2\pi}\right]^{1/2} \int_{|s|<n^{1/2-\lambda}} \exp\left(-\frac{s^2}{2}\psi_n''(\tau_n)\right) R_n\left(\tau_n + \frac{is}{\sqrt{n}}\right) ds = O\left(\frac{1}{n}\right).$$

We now get an upper bound for $L_n(s)$ which will be used to show that the fourth integral on the r.h.s. of (2.16) is $O(1/n)$. For $|s| < n^{1/2-\lambda}$, since $1/3 < \lambda < 1/2$, $|s|^3/\sqrt{n}$ and $s^4/n$ converges to zero uniformly in $s$ as $n \to \infty$. Hence there exists $N_1$, not depending on $s$, such that for $n \ge N_1$ the following inequalities are valid. The second inequality follows from (2.6) and (2.8). Recall that

$$
\begin{aligned}
|z_n| &= \left| -\frac{is^3}{6\sqrt{n}} \psi_n'''(\tau_n) + nR_n\left(\tau_n + i\frac{s}{\sqrt{n}}\right) \right| \\
&\le \frac{|s|^3}{6\sqrt{n}} |\psi_n'''(\tau_n)| + n \left| R_n\left(\tau_n + i\frac{s}{\sqrt{n}}\right) \right| \qquad (2.20)\\
&\le \frac{\beta |s|^3}{\sqrt{n}(a-a_1)^3} + \frac{2\beta s^4}{n(a-a_1)^4} < \frac{1}{2}.
\end{aligned}
$$

It is easy to check that $|z| < 1/2$ implies $|e^z - 1 - z| < 2e|z|^2$. Thus for $n \ge N_1$ and $|s| \le n^{1/2-\lambda}$ we get

$$
\begin{aligned}
|L_n(s)| &\le 2e\left[\frac{\beta |s|^3}{\sqrt{n}(a-a_1)^3} + \frac{2\beta s^4}{n(a-a_1)^4}\right]^2 \qquad (2.21)\\
&= \frac{M}{n}\left[|s|^3 + \frac{2s^4}{\sqrt{n}(a-a_1)}\right]^2,
\end{aligned}
$$

where $M = 2e\beta^2/(a-a_1)^6$. Therefore for $n \ge N_1$,

$$
\begin{aligned}
&\left| \left[\frac{\psi_n''(\tau_n)}{2\pi}\right]^{1/2} \int_{|s| \le n^{1/2-\lambda}} \exp\left(-\frac{s^2}{2}\psi_n''(\tau_n)\right) L_n(s)\, ds \right| \\
&\le \frac{1}{\sqrt{\pi}} \frac{\sqrt{\beta}}{(a-a_1)} \int_{|s| \le n^{1/2-\lambda}} \exp\left(-\frac{\alpha s^2}{2}\right) |L_n(s)|\, ds \quad \text{[using (2.6)]} \qquad (2.22)\\
&\le \frac{M}{n} \frac{\sqrt{\beta}}{\sqrt{\pi}(a-a_1)} \int_{|s| \le n^{1/2-\lambda}} \exp\left(-\frac{\alpha s^2}{2}\right)\left[|s|^3 + \frac{2s^4}{\sqrt{n}(a-a_1)}\right]^2 ds \\
&= O\left(\frac{1}{n}\right).
\end{aligned}
$$

From (2.18), (2.19) and (2.22) we get that $I_n = 1 + O(1/n)$ and the proof is complete. □

REMARK 2.11. In the above proof, we only need the weaker condition that the second derivative of $\psi_n$ at the point $\tau_n$ is bounded below by $\alpha$ for all $n \ge 1$. The stronger condition $\psi_n''(\tau) \ge \alpha$ for $\tau \in I_1$ and $n \ge 1$ will be used to obtain further refinements of the expression (2.1).

COROLLARY 2.12. *Let $\{T_n, n \ge 1\}$ be a sequence of random variables satisfying the conditions of Theorem 2.1. Let $m$ be a real number such that for each $n \ge 1$ there exists $\xi_n \in I_1$ satisfying $\psi_n'(\xi_n) = m$. Suppose that $m_n \to m$ and $n^\delta |m_n - m|$*

$> 1$ *for fixed* $0 < \delta < 1$. *Then*

$$k_n(m_n) = \left[\frac{n}{2\pi\psi_n''(\xi_n)}\right]^{1/2} \exp(-n\gamma_n(m_n))[1 + O(|m_n - m|)]. \tag{2.23}$$

PROOF. The mean value theorem for real valued functions yields

$$|m_n - m| = |\psi_n'(\tau_n) - \psi_n'(\xi_n)| = |\tau_n - \xi_n||\psi_n''(\lambda_n)| \geq |\tau_n - \xi_n|\alpha, \tag{2.24}$$

where $\lambda_n \in I_1$ for each $n \geq 1$. Thus $|\tau_n - \xi_n| = O(|m_n - m|)$. By Condition (B), a version of (2.6) with $\delta_n$ instead of $\tau_n$ and the mean value theorem, we obtain

$$\begin{aligned} \left|\frac{\sqrt{\psi_n''(\xi_n)} - \sqrt{\psi_n''(\tau_n)}}{\sqrt{\psi_n''(\tau_n)}}\right| &= \left|\frac{(\xi_n - \tau_n)\psi_n'''(\delta_n)}{2\sqrt{\psi_n''(\tau_n)\psi_n''(\delta_n)}}\right| \quad [\text{for some } \delta_n \text{ in } I_1] \\ &\leq |\xi_n - \tau_n| \frac{3\beta}{(a - a_1)^3\alpha} = O(|m_n - m|). \end{aligned} \tag{2.25}$$

For Theorem 2.1 we obtain

$$\begin{aligned} k_n(m_n) &= \left[\frac{n}{2\pi\psi_n''(\tau_n)}\right]^{1/2} \exp(-n\gamma_n(m_n))\left[1 + O\left(\frac{1}{n}\right)\right] \\ &= \left[\frac{n}{2\pi\psi_n''(\xi_n)}\right]^{1/2} \exp(-n\gamma_n(m_n))\left[1 + \frac{\sqrt{\psi_n''(\xi_n)} - \sqrt{\psi_n''(\tau_n)}}{\sqrt{\psi_n''(\tau_n)}}\right]\left[1 + O\left(\frac{1}{n}\right)\right] \\ &= \left[\frac{n}{2\pi\psi_n''(\xi_n)}\right]^{1/2} \exp(-n\gamma_n(m_n))[1 + O(|m_n - m|)]\left[1 + O\left(\frac{1}{n}\right)\right] \\ &= \left[\frac{n}{2\pi\psi_n''(\xi_n)}\right]^{1/2} \exp(-n\gamma_n(m_n))[1 + O(|m_n - m|)], \end{aligned} \tag{2.26}$$

since $n^\delta|m_n - m| > 1$. □

REMARK 2.13. Theorem 2.1 is still true if Condition (C) is replaced by the weaker condition below:

(C1) There exists $\eta > 0$ and $0 < k \leq 1$ such that for any $0 < \delta < \eta$,

$$\inf_{|t|\geq\delta} \mathrm{Real}(G_n(t)) \geq k \min[\mathrm{Real}(G_n(\delta)), \mathrm{Real}(G_n(-\delta))] \quad \text{for all} \quad n \geq 1.$$

Condition (C) was used mainly in the inequalities in relation (2.15). It can be easily checked that Condition (C1) will provide a similar inequality.

REMARK 2.14. We can omit Condition (C) in Theorem 2.1 and obtain the same conclusion (2.1) if there exists functions $H_n(t)$ satisfying the following two properties.

(i) $nH_n(\pm n^{-\lambda}) \to \infty$ as $n \to \infty$, for some $1/3 < \lambda < 1/2$.
(ii) There exists $\eta > 0$ such that $0 < \delta < \eta$

$$\inf_{|t|\geq\delta} \mathrm{Real}(G_n(t)) \geq H_n(\pm\delta) \quad \text{for all} \quad n \geq 1.$$

The only modification in the proof is in the inequalities (2.15) where we will use (i) and (ii). We will use this remark in Example 3.2.

PROOF OF THEOREM 2.2. The proof of this theorem parallels the proof of Theorem 2.1 and the only major change is that the range of integration is from $-\pi/h_n$ to $\pi/h_n$ instead of the whole real line.

Let $P_n(k) = \Pr(T_n = a_n + kh_n)$. Then by definition,

$$\phi_n(z) = \textstyle\sum_{k=-\infty}^{\infty} P_n(k)\exp(z(a_n + kh_n)). \tag{2.27}$$

Multiplying both sides by $\exp(-z(a_n + k_n h_n))$ and integrating from $\tau_n - i\pi/h_n$ to $\tau_n + i\pi/h_n$ along the imaginary axis, we obtain

$$P_n(k_n) = \frac{|h_n|}{2\pi}\int_{-\pi/h_n}^{\pi/h_n} \phi_n(\tau_n + it)\exp(-(\tau_n + it)(a_n + k_n h_n))\,dt. \tag{2.28}$$

Therefore,

$$\begin{aligned}\frac{\sqrt{n}}{|h_n|}P_n(k_n) &= \frac{\sqrt{n}}{2\pi}\int_{-\pi/h_n}^{\pi/h_n} \phi_n(\tau_n + it)\exp(-(\tau_n + it)nm_n)\,dt\\ &= \left[\frac{1}{2\pi\psi_n''(\tau_n)}\right]^{1/2}\exp(-n\gamma_n(m_n))I_n,\end{aligned} \tag{2.29}$$

where

$$\begin{aligned}I_n &= \left[\frac{n\psi_n''(\tau_n)}{2\pi}\right]^{1/2}\int_{-\pi/h_n}^{\pi/h_n} \exp(n\gamma_n(m_n))\phi_n(\tau_n + it)\exp(-(\tau_n + it)nm_n)\,dt\\ &= \left[\frac{n\psi_n''(\tau_n)}{2\pi}\right]^{1/2}\int_{-\pi/h_n}^{\pi/h_n} \exp(-nG_n(t))\,dt\end{aligned}$$

and $G_n(t) = [\psi_n(\tau_n) + itm_n - \psi_n(\tau_n + it)]$. Imitating the proof of Theorem 2.1 we can show that

$$I_n = 1 + O(1/n). \tag{2.30}$$

We thus obtain

$$\begin{aligned}\frac{\sqrt{n}}{|h_n|}\Pr\left(\frac{T_n}{n} = m_n\right) &= \frac{\sqrt{n}}{|h_n|}P_n(k_n)\\ &= \left[\frac{1}{2\pi\psi_n''(\tau_n)}\right]^{1/2}\exp(-n\gamma_n(m_n))\left[1 + O\left(\frac{1}{n}\right)\right]. \quad \square\end{aligned} \tag{2.31}$$

REMARK 2.15. Let $m$ be a real number such that there exists $\xi_n \in I_1$ and $\psi_n'(\xi_n) = m$. Under the further assumption that $m_n \to m$ and $n^\delta |m_n - m| > 1$

for some $0 < \delta < 1$ we can show that

$$\frac{\sqrt{n}}{|h_n|} \Pr\left(\frac{T_n}{n} = m_n\right) = \left[\frac{1}{2\pi\psi_n''(\xi_n)}\right]^{1/2} \exp(-n\gamma_n(m_n))[1 + O(|m_n - m|)]. \tag{2.32}$$

REMARK 2.16. Condition (D′) of Theorem 2.2 can be replaced by a stronger but more easily verifiable Condition (D1):

(D1) There exists $p > 0$ such that $1/|h_n| = O(n^p)$.

**3. Applications.** In this section we present two examples to illustrate the theorems of Section 2. These applications concern local limit theorems of random variables $T_n$ which are not sums of i.i.d. random variables. It is usually condition (C) that is difficult to verify. These two examples illustrate some techniques used to do this.

In another paper, Chaganty and Sethuraman (1982), we assume that $T_n$ satisfies the conditions of the theorems of Section 2, and study the asymptotic distributions of other random variables whose distributions involve the m.g.f. of $T_n$. Such random variables are akin to those that arise in statistical mechanics, and our asymptotic distributions are Gaussian for a range of a parameter and become non-Gaussian at a special value of the parameter. This has been interpreted to qualitatively describe phase transitions.

EXAMPLE 3.1. *Wilcoxon signed-rank test.* Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables with common absolutely continuous d.f., $F$, which is symmetric about the median $m$. Arrange $|X_1|, |X_2|, \cdots, |X_n|$ in increasing order of magnitude and assign ranks $1, 2, \cdots, n$. Let

$$z_i = \begin{cases} 1 & \text{if the } X_j \text{ having rank } i \text{ is positive.} \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$U_n = \text{sum of the ranks of positive } X_j\text{'s} = \textstyle\sum_{i=1}^n iz_i.$$

The statistic $U_n$ is known as the Wilcoxon statistic and is used to test the hypothesis

$$H_0: m = 0 \quad \text{vs} \quad H_1: m \neq 0.$$

Let $T_n = U_n/n$. The m.g.f. of $T_n$, $\phi_n(z)$, under the null hypothesis is given by

$$\phi_n(z) = \textstyle\prod_{k=1}^n [(e^{kz/n} + 1)/2], \tag{3.1}$$

and

$$\psi_n(z) = (1/n) \textstyle\sum_{k=1}^n \log[(e^{kz/n} + 1)/2], \quad z \in \mathbb{C}. \tag{3.2}$$

Here $\{T_n, n \geq 1\}$ is a sequence of lattice random variables with $a_n = 0$, $h_n = 1/n$. The range of $\psi_n'(\tau)$, $-\infty < \tau < \infty$, contains the open interval (0, ½) for $n \geq 1$. Thus if $\{m_n, n \geq 1\}$ is a sequence of real numbers contained in a proper subinterval of (0, ½), we can find a constant $a_1 > 0$ and $\tau_n \in (-a_1, a_1)$ such that $\psi_n'(\tau_n) = m_n$ for all $n \geq 1$. Let us now check all the conditions of Theorem 2.2.

CONDITION (A). Let $a = 2a_1$, then it is easy to check that there exists $\beta > 0$ such that $|z| < a$ implies $|\psi_n(z)| < \beta$.

CONDITION (B). The existence of $\{\tau_n, n \geq 1\}$ is already discussed above. Straightforward calculations show that $\psi_n''(\tau)$ is bounded below by a positive number for $|\tau| < a$.

CONDITION (C′). Let

$$\begin{aligned} f_n(t) &= \mathrm{Real}(G_n(t)) = \mathrm{Real}(\psi_n(\tau_n) - \psi_n(\tau_n + it)) \\ &= -\frac{1}{2n} \textstyle\sum_{k=1}^n \log\left[1 - \dfrac{2(1 - \cos(kt/n))}{[\exp(k\tau_n/2n) + \exp(-k\tau_n/2n)]^2}\right]. \end{aligned} \tag{3.3}$$

Note that $f_n(0) = 0$ for all $n$. Condition (C′) is verified by showing that $f_n$ satisfies the three assumptions of Lemma 2.7 for all $n \geq n_0$.

(i) Take $\eta_1 = \pi/2$. Since cosine is decreasing with $|t|$ in the interval $(-\eta_1, \eta_1)$, $f_n(t)$ is increasing with $|t|$ in that interval for all $n$.

(ii) Since $1/h_n = n$, all we need to show is that there exist $\varepsilon > 0$ and $n_0 \geq 1$ such that

$$\inf_{\pi/2 \leq |t| \leq n\pi} f_n(t) > \varepsilon \quad \text{for} \quad n \geq n_0. \tag{3.4}$$

We will show this with $\varepsilon = \varepsilon'/4e^a$, where $\varepsilon' = 1 - 8/3\pi$. From (3.3) we have

$$\begin{aligned} -f_n(t) &= \frac{1}{2n} \textstyle\sum_{k=1}^n \log\left[1 - \dfrac{2(1 - \cos(kt/n))}{[\exp(k\tau_n/2n) + \exp(-k\tau_n/2n)]^2}\right] \\ &\leq -\frac{1}{n} \textstyle\sum_{k=1}^n \left[\dfrac{(1 - \cos(kt/n))}{[\exp(k\tau_n/2n) + \exp(-k\tau_n/2n)]^2}\right] \\ &\leq -\frac{1}{n} \textstyle\sum_{k=1}^n \left[\dfrac{(1 - \cos(kt/n))}{4e^a}\right] \quad [\text{since } |\tau_n| < a] \\ &\leq -\frac{1}{4e^a} + \frac{1}{4ne^a} \textstyle\sum_{k=1}^n \cos\left(\dfrac{kt}{n}\right) \\ &= -\frac{1}{4e^a} + \frac{1}{4e^a}\left[\dfrac{\sin((n+1)t/n) + \sin(t)}{2n \sin(t/n)} - \dfrac{1}{2n}\right]. \end{aligned} \tag{3.5}$$

Substituting $t = ns$ in (3.5), we can see that (3.4) is verified once we show that there exists $n_0 \geq 1$ such that

$$\sup_{\pi/2n \leq |s| \leq \pi} \frac{\sin((n+1)s) + \sin(ns)}{2n \sin(s)} < 1 - \varepsilon' \quad \text{for all} \quad n \geq n_0. \tag{3.6}$$

There exists $n_1$ such that $n \sin(\pi/2n) > 3\pi/8$ for all $n \geq n_1$. Thus for $n \geq n_1$

$$\begin{aligned} &\sup_{\pi/2n \leq |s| \leq \pi - \pi/2n} \left| \frac{\sin((n+1)s) + \sin(ns)}{2n \sin(s)} \right| \\ &\qquad \leq \sup_{\pi/2n \leq |s| \leq \pi - \pi/2n} \left| \frac{1}{n \sin(s)} \right| \leq \frac{1}{n \sin(\pi/2n)} \leq \frac{8}{3\pi} = 1 - \varepsilon'. \end{aligned} \tag{3.7}$$

Now,

$$\begin{aligned} &\sup_{\pi - \pi/2n \leq |s| \leq \pi} \left| \frac{\sin((n+1)s) + \sin(ns)}{2n \sin(s)} \right| \\ &\qquad = \sup_{0 \leq |\theta| \leq \pi/2n} \left| \frac{\sin((n+1)(\pi - \theta)) + \sin(n(\pi - \theta))}{2n \sin(\pi - \theta)} \right| \\ &\qquad = \sup_{0 \leq |\theta| \leq \pi/2n} \left| \frac{\sin((n+1)\theta) - \sin(n\theta)}{2n \sin(\theta)} \right| \\ &\qquad = \sup_{0 \leq |\theta| \leq \pi/2n} \left| \frac{\cos((2n+1)\theta/2)\sin(\theta/2)}{n \sin(\theta)} \right|, \end{aligned} \tag{3.8}$$

which goes to 0 as $n \to \infty$. Thus there exists $n_2$ such that $n \geq n_2$ implies that the r.h.s. of (3.8) is less than $1 - \varepsilon'$. Then $n_0 = \max(n_1, n_2)$ satisfies (3.6) and the proof is complete.

(iii) Let $\varepsilon$ be as in (ii). We want to show that there exists $\eta < \pi/2$ such that for all $n \geq 1$

$$\sup_{0 \leq |t| \leq \eta} f_n(t) < \varepsilon. \tag{3.9}$$

Since $\log(1 - x) > -x/(1 - x)$ for $0 < x < 1$, we get

$$\begin{aligned} -f_n(t) &= \frac{1}{2n} \textstyle\sum_{k=1}^{n} \log\left[1 - \frac{2(1 - \cos(kt/n))}{[\exp(k\tau_n/2n) + \exp(-k\tau_n/2n)]^2}\right] \\ &\geq -\frac{1}{2n} \textstyle\sum_{k=1}^{n} \left[\frac{2(1 - \cos(kt/n))}{[\exp(k\tau_n/2n) + \exp(-k\tau_n/2n)]^2 - 2(1 - \cos(kt/n))}\right] \\ &\geq -\frac{1}{2n} \textstyle\sum_{k=1}^{n} \left[\frac{(1 - \cos(kt/n))}{(1 + \cos(kt/n))}\right] \end{aligned} \tag{3.10}$$

since $e^x + e^{-x} \geq 2$. Let $\delta = 4\varepsilon/(1 + 2\varepsilon)$. Choose $\eta$ such that $\eta < \pi/2$ and $\cos(\eta) >$

$1 - \delta$. From (3.10) it follows that, for $|t| < \eta$,

$$f_n(t) \le \frac{1}{2n} \sum_{k=1}^{n} \left[\frac{(1 - \cos(kt/n))}{(1 + \cos(kt/n))}\right] \le \frac{\delta}{2(2-\delta)} = \varepsilon.$$

This establishes (3.9).

CONDITION (D′). Since $1/h_n = n$, this condition is trivially satisfied for $p = 1$.

Thus from Theorem 2.2 we can get an asymptotic expression for $\Pr(T_n = nm_m)$. In this example $T_n$ is the sum of $n$ independent but not identically distributed random variables. □

EXAMPLE 3.2. *Kendall's distribution-free test for independence.* Let $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$ have a common bivariate distribution with continuous c.d.f., $F(x, y)$, and marginals $G(x)$ and $H(y)$. We wish to test the hypothesis $H_0$: $F(x, y) = G(x)H(y)$ for all $x, y$.

Define

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Let

$$V_{ij} = \operatorname{sgn}(X_i - X_j)\operatorname{sgn}(Y_i - Y_j), \quad 1 \le i < j \le n.$$

Let

$$Q_n = \sum_{1 \le i < j \le n} (1 - V_{ij})/2 \quad \text{and} \quad W_n = 1 - 4Q_n/(n(n-1)).$$

Suppose the ranks of $Y$'s are arranged in the natural order $1, 2, \cdots, n$ and let the corresponding ranks of $X$'s be $x_1, x_2, \cdots, x_n$. $Q_n$ measures the extent of departure of the $x$'s from the natural order $(1, 2, 3, \cdots, n)$ by counting the number of inversions among them. The statistic $W_n$ was proposed by Kendall (1938) as a nonparametric test statistic for testing the null hypothesis $H_0$.

Let $T_n = nW_n = n - 4Q_n/(n-1)$. It is clear that $T_n$ is a lattice random variable with $a_n = n$ and $h_n = 4/(n-1)$. The m.g.f. of $T_n$ under the assumption of independence of $X$ and $Y$ is given by (Kendall and Stuart, 1979, pages 505–506),

$$\phi_n(z) = \frac{e^{nz}}{n!} \prod_{k=1}^{n} \left[\frac{e^{-4kz/(n-1)} - 1}{e^{-4z/(n-1)} - 1}\right]. \tag{3.11}$$

Thus

$$\begin{aligned} \psi_n(z) &= z - \frac{1}{n} \log n! + \frac{1}{n} \sum_{k=1}^{n} \log\left[\frac{1 - e^{-4kz/(n-1)}}{1 - e^{-4z/(n-1)}}\right] \\ &= z - \frac{1}{n} \sum_{k=1}^{n} [\log k - \log(1 - e^{-4kz/(n-1)}) + \log(1 - e^{-4z/(n-1)})]. \end{aligned} \tag{3.12}$$

The range of $\psi_n'(\tau)$ for $-\infty < \tau < \infty$, is the interval $(-1, 1)$ and the random variable $T_n/n$ takes values in this interval. Thus if $\{m_n, n \geq 1\}$ is a sequence of real numbers contained in $(-M, M)$, $0 < M < 1$, then for sufficiently large $a_1$, we can find $\{\tau_n, n \geq 1\}$ such that $\psi_n'(\tau_n) = m_n$ with $|\tau_n| < a_1$. For simplicity let us choose $m_n = 0$. Then $\tau_n = 0$ for all $n \geq 1$ and we will verify the conditions of the Theorem 2.2. The verification for general sequence $m_n$ is similar but a little more tedious.

CONDITION (A). Let $a = 2a_1$. It follows from (3.12) that

$$|z| < a \quad \text{implies} \quad |\psi_n(z)| < 2[1 + a + \log(1 + e^{4a}) + \log(4a)] = \beta.$$

CONDITION (B). By Remark 2.11 we only need to verify that $\psi_n''(\tau_n) \geq \alpha$ for $n \geq 2$. Since $\tau_n = 0$ we have

$$\begin{aligned}\psi_n''(0) &= \mathrm{Var}(T_n)/n = 2(2n+5)/9(n-1)\\ &\geq \alpha \quad \text{for all} \quad n \geq 2, \quad \text{with} \quad \alpha = 2/9.\end{aligned}$$

CONDITION (C′). Let

$$H_n(t) = n^{\mu-1}\mathrm{Real}[G_{n^\mu}((n^\mu - 1)t/(n-1))], \tag{3.13}$$

where $\mu = 17/18$. Let $\lambda = 7/18$. Note that

$$1/3 < \lambda < 1/2, \quad 1/3 < \mu < 1, \quad 0 < 3\mu - 2\lambda - 2 = 1/18. \tag{3.14}$$

We will now verify conditions (i) and (ii) of Remark (2.14) and thus establish Condition (C′). Since $T_n$ is lattice valued with span $h_n = 4/(n-1)$, we need only verify

$$nH_n(\pm n^{-\lambda}) \to \infty \tag{3.15}$$

and that there exists $\eta > 0$ such that for $0 < \delta < \eta$,

$$\inf_{\delta \leq |t| \leq (n-1)\pi/4} \mathrm{Real}(G_n(t)) \geq H_n(\pm\delta). \tag{3.16}$$

Using Lemma 2.10, we find that

$$nH_n(\pm n^{-\lambda}) = n^\mu \mathrm{Real}\left[G_{n^\mu}\left((n^\mu - 1)\frac{\pm n^{-\lambda}}{n-1}\right)\right] \geq \frac{\alpha n^{3\mu-2\lambda-2}}{8},$$

which goes to $\infty$ as $n \to \infty$, in view of (3.14). This establishes (3.15). Let $\eta < \pi/4$ and $0 < \delta < \eta$. Since $\tau_n = 0$,

$$\begin{aligned}\mathrm{Real}(G_n(t)) &= -\mathrm{Real}(\psi_n(it)) = -\frac{1}{n}\sum_{k=1}^n \log\left[\left|\frac{\sin(2kt/(n-1))}{k\sin(2t/(n-1))}\right|\right]\\ &= -\sum_{k=1}^n \log\left[\left|\frac{\sin(2kt/(n-1))}{k\sin(2t/(n-1))}\right|\right]^{1/n}.\end{aligned} \tag{3.17}$$

Thus

$$\inf_{\delta(n-1)^{1-\mu}\le|t|\le(n-1)\pi/4}\text{Real}(G_n(t))$$
$$= \inf_{\delta(n-1)^{-\mu}\le|t|\le\pi/4}\text{Real}(G_n((n-1)t))$$
$$= \inf_{\delta(n-1)^{-\mu}\le|t|\le\pi/4}\left[-\log\left[\prod_{k=1}^{n}\left|\frac{\sin 2kt}{k\sin 2t}\right|\right]^{1/n}\right]$$
$$\ge -\log\left[\prod_{k=1}^{n}\left|\frac{1}{k\sin(2\delta(n-1)^{-\mu})}\right|\right]^{1/n}.$$

Using the arithmetic-geometric mean inequality and the fact $|\sin x| > |x|/2$ for small values of $|x|$, we get

$$\text{(3.18)}\qquad \begin{aligned}\inf_{\delta(n-1)^{1-\mu}\le|t|\le(n-1)\pi/4}\text{Real}(G_n(t)) &\ge -2\log\left[\frac{(n-1)^{\mu}}{2\delta n}\sum_{k=1}^{n}\frac{1}{k}\right]\\ &\sim -2\log\left[\frac{\log n}{2\delta n^{1-\mu}}\right],\end{aligned}$$

which goes to $\infty$. We will now use the easily established facts that $\log(|\sin t/t|) \sim -t^2/6$ as $|t|\to 0$ in (3.19) and that $|\sin(ks)/(k\sin s)|$ is less than 1 and decreasing for $s$ in $[0,\varepsilon]$ if $k\varepsilon<\pi$ to obtain the inequality (3.20). Recall

$$\text{(3.19)}\qquad \begin{aligned}H_n(\pm\delta) &= -\frac{1}{n}\sum_{k=1}^{n^{\mu}}\log\left[\left|\frac{\sin(2k\delta/(n-1))}{k\sin(2\delta/n-1))}\right|\right]\\ &\sim -\frac{1}{n}\sum_{k=1}^{n^{\mu}}\left[\frac{4k^2\delta^2}{6(n-1)^2}-\frac{4\delta^2}{6(n-1)^2}\right] = O(n^{3\mu-3})\quad\text{as}\quad n\to\infty.\end{aligned}$$

Also,

$$\text{(3.20)}\qquad \begin{aligned}\inf_{\delta\le|t|\le\delta(n-1)^{1-\mu}}\text{Real}(G_n(t)) &\ge -\frac{1}{n}\sum_{k=1}^{n^{\mu}}\log\left[\left|\frac{\sin(2k\delta/(n-1))}{k\sin(2\delta/(n-1))}\right|\right]\\ &= H_n(\pm\delta).\end{aligned}$$

Relations (3.18), (3.19) and (3.20) establish (3.16).

This completes the verification of condition (C′).

CONDITION (D1). Since $h_n = 4(n-1)$, clearly $1/h_n = O(n^p)$ for $p=1$. Thus we have verified all the conditions of Theorem 2.2. Since $E(T_n) = 0 = m_n$, by (2.31) we get

$$\text{(3.21)}\qquad \Pr\left(\frac{T_n}{n}=0\right) = \frac{6}{\sqrt{\pi n(2n+5)(n-1)}}\left[1+O\left(\frac{1}{n}\right)\right]\quad\text{as } n\to\infty$$

through the subsequence of $\{n\}$ for which 0 is the range of $T_n/n$. □

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