ASYMPTOTIC NORMALITY, STRONG MIXING AND SPECTRAL DENSITY ESTIMATES[1]

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Asymptotic normality is proven for spectral density estimates assuming strong mixing and a limited number of moment conditions for the process analyzed. The result holds for a large class of processes that are not linear and does not require the existence of all moments.

1. Introduction. Strong mixing as a sufficient condition (usually together with some auxiliary moment conditions) for classical limit theorems was initially proposed and discussed in the period from the mid 50's to the early 60's (see Rosenblatt 1956, 1961; Kolmogorov and Rozanov, 1960; and Ibragimov, 1962). There had already been some investigation of the asymptotic properties of spectral density estimates. But from that time to the present day, most results on the asymptotic normality of spectral density estimates assume either a very special structure for the process (a linear process in Anderson, 1971, and Hannan, 1970) or else existence of all moments (as in Brillinger, 1975). Approximability by multilinear schemes in independent, identically distributed random variables has been considered in Rosenblatt, 1959. Here we shall show that a direct application of a central limit theorem using strong mixing together with a limited number of moment conditions will imply asymptotic normality for a large class of spectral density estimates.

2. Strong mixing and a triangular central limit theorem. Let $X = \{X_n\}$ be a strictly stationary process. Let $\mathscr{B}_n = \mathscr{B}(X_k, k \leq n)$ be the σ-field generated by the random variables. $X_k, k \leq n$ and $\mathscr{F}_m = \mathscr{B}(X_k, k \geq m)$ the σ-field generated by $X_k, k \geq m$. The process $X = \{X_n\}$ is said to be *strongly mixing* if

$$(2.1) \qquad \sup_{B \in \mathscr{A}_0, F \in \mathscr{F}_n} | P(BF) - P(B)P(F) | = \alpha(n) \rightarrow 0$$

as $n \rightarrow \infty$. The following central limit theorem can be established by a big block-small block argument just as in Rosenblatt (1959, 1961). This result will be assumed. It will be shown that the result can be applied to the case of spectral density estimates.

THEOREM 1. *Let* $\{Y_j^{(n)}, j = \cdots, -1, 0, 1, \cdots\}$, $EY_j^{(n)} \equiv 0$, $n = 1, 2, \cdots$ *be a sequence of strictly stationary processes defined on the probability space of the*

strongly mixing stationary process $X = \{X_n\}$. Let $Y_j^{(n)}$ be measurable with respect to $\mathscr{F}_{j-c(n)} \cap \mathscr{B}_{j+c(n)}$ where $c(n) = o(n)$, $c(n) \uparrow \infty$ as $n \to \infty$. Set

$$(2.2) \qquad\qquad h_n(b - a) = E \,|\, \textstyle\sum_{j=a}^b Y_j^{(n)} |^2.$$

Assume that for any two sequences $s(n)$, $m(n)$ with $c(n) = o(m(n))$, $m(n) \leq n$ and $s(n)/m(n) \to 0$ one has

$$(2.3) \qquad\qquad \frac{h_n(m(n))}{h_n(s(n))} \to \infty.$$

Let $F_{n,m}(x)$ be the distribution function of

$$\textstyle\sum_{k=1}^m \frac{Y_k^{(n)}}{\{(n/m)h_n(m)\}^{1/2}}$$

with $m = m(n)$ and $c(n) = o(m(n))$, $m(n) = o(n)$. Assume that

$$(2.4) \qquad\qquad \frac{n}{m(n)} \int_{|x|>\eta} x^2 \, dF_{n,m(n)}(x) \to 0$$

as $n \to \infty$ for each $\eta > 0$. There are then sequences $k(n)$, $p(n) \to \infty$ as $n \to \infty$ with $k(n)p(n) \cong n$ such that

$$(2.5) \qquad\qquad \textstyle\sum_{j=1}^n Y_j(n)/\sqrt{k(n)h_n(p(n))}$$

is asymptotically normally distributed with mean zero and variance one. Also if $k(n)h_n(p(n)) \cong h_n(n)$ the normalization in (2.5) can be replaced by $\sqrt{h_n(n)}$.

Condition (2.4) is like a Lindeberg condition. A corresponding result with this condition replaced by a stronger Liapounov condition is valid.

COROLLARY 1. *The conclusion of Theorem 1 still holds if all assumptions are the same except for condition (2.4) and condition (2.4) is replaced by the following condition (2.4)'. Let*

$$(2.4)' \qquad\qquad \{h_n(m)\}^{-(2+\delta)/2} E \,|\, \textstyle\sum_{k=1}^m Y_k^{(n)} |^{2+\delta} = O(1)$$

for $m = m(n)$ and some $\delta > 0$.

Because conditions in terms of cumulants are used a brief discussion relating to them is given. Let $Z = (Z_j, j = 1, \cdots, k)$ be a random k-vector with $t = (t_j, j = 1, \cdots, k)$ a k-vector of coefficients. The characteristic function of the random vector Z is

$$\varphi(t) = E \exp(it \cdot Z).$$

Let $m = (m_1, \cdots, m_k)$, $|m| = \sum m_i$. If moments up to order s exist there is a Taylor expansion

$$\varphi(t) = \textstyle\sum_{|m| \leq s} \frac{i^{|m|}\mu^{(m)}}{m!} \, t^m + o(|t|^s)$$

with

$$t^m = \prod_{j=1}^{k} t_j^{m_j}, \quad m! = \prod_{j=1}^{k} m_j!,$$

and the moments

$$\mu^{(m)} = \mu^{(m_1, \cdots, m_k)} = E\left(\prod_{j=1}^{k} Z_j^{m_j}\right).$$

There is a corresponding Taylor expansion for $\log \varphi(t)$

$$\log \varphi(t) = \sum_{|m| \leq s} \frac{i^{|m|} c^{(m)}}{m!} t^m + o(|t|^s)$$

with the coefficients $c^{(m)}$ the cumulants of Z

$$c^{(m)} = c^{(m_1, \cdots, m_k)} = \text{cum}(Z_1^{m_1}, \cdots, Z_k^{m_k}).$$

There are identities linking the moments $\mu^{(m)}$, $|m| \leq s$, up to order s with the cumulants $c^{(m)}$, $|m| \leq s$, up to order s. In fact, the moments up to order s exist if and only if the cumulants up to order s exist.

At times one will deal with random variables $W_1, \cdots, W_J$ given in terms of doubly indexed random variables

$$U_{k,\ell}, \quad \ell = 1, \cdots, \ell_k, \quad k = 1, \cdots, J$$

by

$$W_k = \prod_{\ell=1}^{\ell_k} U_{k,\ell}.$$

The object is to carry out the computation of the cumulant

$$\text{cum}(W_1, \cdots, W_J)$$

in terms of the cumulants of the $U_{k,\ell}$. Such a computation can be carried out in the following manner. Lay out the following array of the $U_{k,\ell}$'s in J rows

$$\begin{matrix} U_{1,1} & U_{1,2} & \cdots & U_{1,\ell_1} \\ U_{2,1} & U_{2,2} & \cdots & U_{2,\ell_2} \\ \vdots & \vdots & & \vdots \\ U_{J,1} & U_{J,2} & \cdots & U_{J,\ell_J}. \end{matrix}$$

A partition of this array into sets $\nu_1, \cdots, \nu_r$ is called decomposable if the union of the elements in a nonvacuous proper subcollection of sets of the partition yields exactly the elements in a nonvacuous proper subset of the rows of the table. A partition is indecomposable if it is not decomposable. One can show that

$$\text{cum}(W_1, \cdots, W_J) = \sum \prod_{s=1}^{r} \text{cum}(U_{j,\ell}, (j, \ell) \in \nu_s)$$

where the sum on the right is extended over all indecomposable partitions of the table given above (see Leonov and Shiryaev, 1959).

Let us note that if kth order moments of the process $X = \{X_n\}$ are finite, that the cumulants

$$\text{cum}(X_j, X_{j+s_1}, \cdots, X_{j+s_{k-1}}) = r^{(k)}_{s_1, \cdots, s_{k-1}}$$

are well-defined. Summability conditions on cumulants such as

(2.6) $\sum_{s_1,\cdots,s_{k-1}} |r^{(k)}_{s_1,\cdots,s_{k-1}}| < \infty$

are convenient to make at times. In Brillinger (1975) and Brillinger and Rosenblatt (1967) assumptions of the form

$$\sum_{s_1,\cdots,s_{k-1}} (1 + |s_j|) |r^{(k)}_{s_1,\cdots,s_{k-1}}| < \infty,$$

$j = 1, \cdots, k - 1$, are made for all $k \geq 2$. We shall only have to assume a limited number of the conditions (2.6) together with strong mixing.

Most central limit theorems for dependent processes making use of mixing or mixingale conditions make an assumption on the rate of decay of the mixing coefficient (see Hall and Heyde, 1980). Notice that this is not the case for the limit theorems in this paper. The conditions are basically summability conditions on cumulants. In fact, we shall later give examples of processes that satisfy the assumptions of the theorems but still have a slow rate of decay of the mixing coefficient.

3. Spectral density estimates.
The assumption (2.6) for $k = 2$ is just summability of the covariances

$$\sum_k |r_k| < \infty, \quad r_k = \text{cov}(x_j, x_{j+k}).$$

The spectral density $f(\lambda)$ of the process is then continuous with

$$f(\lambda) = 1/2\pi \sum_k r_k e^{-ik\lambda}.$$

Covariance estimates $r^{(n)}_k$ in terms of a sequence of observations, $X_1, \cdots, X_n$, $EX_k \equiv 0$, on X are given by

$$r^{(n)}_k = 1/n \sum_{j=1}^{n-k} X_j X_{j+k}, \quad 0 \leq k \leq n,$$

with $r^{(n)}_{-k} = r^{(n)}_k$. A large class of spectral density estimates $f_n(\lambda)$ are of the form

$$f_n(\lambda) = 1/2\pi \sum_{k=-n+1}^{n-1} r^{(n)}_k w^{(n)}_k \cos k\lambda$$

with

$$w^{(n)}_k = a(kb_n), \quad a(0) = 1,$$

with $a(x)$ *continuous at zero,* bounded and satisfying

$$a(x) = a(-x).$$

Here $b_n \to 0$ with $n^{-1} = o(b_n)$ as $n \to \infty$. The following proposition amounts to a simple remark on the asymptotic unbiasedness of a class of estimates of the form specified above.

PROPOSITION. *Let $X = (X_k)$ be a weakly stationary process with $EX_k \equiv 0$, $\sum |r_k| < \infty$. Assume that*

$$f_n(\lambda) = 1/2\pi \sum_{k=-n+1}^{n-1} r^{(n)}_k a(kb_n) \cos k\lambda.$$

Let $a(x)$ be bounded, continuous at zero with $a(0) = 1$ and symmetric about zero.

Then

$$Ef_n(\lambda) - f(\lambda) \to 0$$

as $b_n \to 0$, $n \to \infty$.

Notice that

$$(3.1) \quad Ef_n(\lambda) - f(\lambda) = \frac{1}{2\pi} \sum_{|k| \le n} r_k \left(\frac{n - |k|}{n} \right) [a(kb_n) - 1] \cos k\lambda$$

$$- \frac{1}{2\pi} \sum_{|k| \le n} \frac{|k|}{n} r_k \cos k\lambda - \frac{1}{2\pi} \sum_{|k| > n} r_k \cos k\lambda.$$

The absolute summability of r_k implies that the second and third sums on the right of (3.1) tend to zero as $n \to \infty$. The assumptions on $a(\cdot)$ together with the summability of r_k imply that the first sum on the right of (3.1) tends to zero as $b_n \to \infty$ and $n \to \infty$.

The following lemma will be useful in deriving some results on the asymptotic distribution of spectral density estimates.

LEMMA. *Let*

$$g(u) = 1/2\pi \sum g_k e^{-iku}, \quad \sum |g_k| < \infty.$$

Further assume that $a(\cdot)$ is piecewise continuous, continuous at 0 with $a(0) = 1$, symmetric and such that $a(x) = O(|x|^{-1/2-\varepsilon})$ for some $\varepsilon > 0$ as $|x| \to \infty$. Then if

$$W_n(u) = 1/2\pi \sum w_k^{(n)} e^{-iku}$$

with $w_k^{(n)} = a(kb_n)$ it follows that

$$(3.2) \quad b_n \int_{-\pi}^{\pi} W_n^2(u + \lambda) g(u) \, du \to g(\lambda) \int W^2(u) \, du$$

as $b_n \to 0$ where

$$W(u) = \frac{1}{2\pi} \int a(u) e^{-iu\alpha} \, du$$

while

$$(3.3) \quad \int_{-\pi}^{\pi} W_n(u + \lambda) W_n(u + \mu) g(u) \, du = o(b_n^{-1})$$

as $b_n \to 0$ if $\lambda \ne \mu$, $|\lambda - \mu| < 2\pi$.

Notice that

$$b_n \int_{-\pi}^{\pi} W_n(u + \lambda) W_n(u + \mu) g(u) \, du$$

$$= \frac{1}{2\pi} \sum_\ell g_\ell e^{-i\ell u} \frac{1}{2\pi} \sum_k b_n a(kb_n) a((k - \ell)b_n) e^{ik(\mu - \lambda)}.$$

Clearly

(3.4) $| \sum_k b_n a(kb_n) a((k - \ell)b_n) | \leq \sum b_n | a(kb_n) |^2$

and the right-hand side of (3.4) tends to

$$\int | a(u) |^2 \, du$$

as $b_n \to 0$. Also, if $\mu = \lambda$

$$\sum_k b_n a(kb_n) a((k - \ell)b_n) \to \int | a(u) |^2 \, du$$

as $b_n \to 0$ for each fixed ℓ. This yields the limiting relationship (3.2). Let us now consider the case in which $\lambda \neq \mu$, $| \lambda - \mu | < 2\pi$. Relation (3.3) will follow if we can show that

(3.5) $\sum_k b_n a(kb_n) a((k - \ell)b_n) e^{ik(\mu - \lambda)} \to 0$

as $b_n \to 0$ for each fixed ℓ. Given $\varepsilon > 0$ there is an $L(\varepsilon)$ such that

$$\sum_{b_n | k | > L(\varepsilon)} b_n | a(kb_n) |^2 < \varepsilon.$$

The function $a(x)$ can be uniformly approximated arbitrarily well by step functions over the interval $| x | \leq L(\varepsilon)$. It is therefore enough to show that (3.5) holds for a step function $a(x)$ of finite support. In fact, this will follow if one can prove (3.5) for $a(x)$ the indicator function of a finite interval, say the interval (α_1, α_2), $\alpha_1 < \alpha_2$. Then

$$\sum_k a(kb_n) a((k - \ell)b_n) e^{ik(\mu - \lambda)} = \sum_{\alpha_1 < kb_n < \alpha_2} e^{ik(\mu - \lambda)} b_n + O(b_n)$$

$$= e^{i\alpha_1 c_n(\mu - \lambda)} \sum_{0 < kb_n < \alpha_2 - \alpha_1} e^{ik(\mu - \lambda)} b_n + O(b_n)$$

$$= b_n e^{i\alpha_1 c_n(\mu - \lambda)} \frac{e^{i(\alpha_2 - \alpha_1) c_n(\mu - \lambda)} - 1}{e^{i(\mu - \lambda)} - 1} + O(b_n)$$

with $c_n = b_n^{-1}$.

Let

$$\eta(\lambda) = \begin{cases} 1 & \text{if} \quad \lambda = k\pi, \quad k \text{ integer} \\ 0 & \text{otherwise.} \end{cases}$$

We shall prove the following theorem.

THEOREM 2. *Let $X = \{X_n\}$ be a strictly stationary strongly mixing process with $EX_j \equiv 0$. Assume that the cumulant functions (2.6) of order two and four are summable. Further, let the spectral density estimate $f_n(\lambda)$ have weights $w_k^{(n)}$ defined in terms of a function $a(\cdot)$ that is piecewise continuous, continuous at zero with $a(0) = 1$, symmetric about zero and is such that $xa(x)$ is bounded. Let*

$$Y_u^{(n)}(\lambda) = \sum_{k=-c(n)}^{c(n)} X_u X_{u+k} w_k^{(n)} \cos k\lambda$$

with $w_k^{(n)} = a(kb(n))$ and $c(n) = \alpha b_n^{-1}$ for all sufficiently large fixed α. Set

$$Z_n(\lambda) = \sum_{u=1}^m \frac{Y_u^{(n)}(\lambda)}{\{nb_n^{-1}\}^{1/2}}$$

with $m = m(n)$ and $c(n) = o(m(n))$, $m(n) = o(n)$. Consider the distribution function $F_{n,m}(x)$ of $Z_n(\lambda)$ and assume that

$$(3.6) \qquad \frac{n}{m(n)} \int_{|x|>\eta} x^2 \, dF_{n,m(n)}(x) \to 0$$

as $n \to \infty$, $b_n \to 0$, $nb_n \to \infty$ for each $\eta > 0$. Then $f_n(\lambda) - Ef_n(\lambda)$ is asymptotically normally distributed with mean zero and variance

$$(3.6)' \qquad \frac{2\pi(1 + \eta(\lambda))}{nb_n} f^2(\lambda) \int W^2(\alpha) \, d\alpha.$$

Let us first consider the case in which $a(\cdot)$ has finite support. We first consider replacing $f_n(\lambda)$ by

$$\tilde{f}_n(\lambda) = \frac{1}{2\pi} \sum_{k=-c(n)}^{c(n)} \frac{1}{n} \sum_{j=1}^n x_j x_{j+k} w_k^{(n)} \cos k\lambda.$$

Here the support of the function $a(\cdot)$ has been taken as the interval $[-1, 1]$ to simplify notation. The argument goes through in exactly the same way for any other interval. Notice that

$$(3.7) \qquad 2\pi n \, \tilde{f}_n(\lambda) = \sum_{u=1}^n Y_u^{(n)}$$

where

$$Y_u^{(n)} = \sum_{k=-c(n)}^{c(n)} X_u X_{u+k} w_k^{(n)} \cos k\lambda.$$

Thus, the expression (3.7) is of the type discussed in Theorem 1. Now,

$$(3.8) \qquad \sigma^2(f_n(\lambda) - \tilde{f}_n(\lambda)) \le \frac{1}{\pi^2 n^2} \sigma^2 \left[\sum_{k=1}^{c(n)} \sum_{j=n-k}^n X_j X_{j+k} w_k^{(n)} \cos k\lambda \right]$$

and the right-hand side of (3.8) is less than

$$\frac{1}{\pi^2 n^2} \sum_{k,k'=1}^{c(n)} \sum_{j=n-k}^n \sum_{j'=n-k'}^n$$

$$\cdot \left\{ |r_{j-j'} r_{j-j'+k-k'}| + |r_{j-j'-k'} r_{j-j'+k}| + |r_{k,j'-j,j'+k'-j}^{(4)}| \right\} |w_k^{(n)} w_{k'}^{(n)}|$$

$$\le \frac{1}{\pi^2 n^2} \sum_{k,k'=1}^{c(n)} \sum_s \min(k', k) \left\{ |r_s| |r_{s+k'-k}| + |r_{s-k'} r_{s+k}| + |r_{k,s,s+k'}^{(4)}| \right\}$$

$$\cdot |w_k^{(n)}| |w_{k'}^{(n)}|.$$

The assumptions on the weights $w_k^{(n)}$ and the function $a(\cdot)$ imply that the weights are bounded and

$$|k| \, |w_k^{(n)}| \le L b_n^{-1}$$

for all k and some constant L. This together with the summability of (2.6) for

$k = 2, 4$ imply that

$$\sigma^2(f_n(\lambda) - \tilde{f}_n(\lambda)) = O((nb_n)^{-2})$$

$$= o((nb_n)^{-1}).$$

As we shall see this is of smaller order of magnitude than $\sigma^2(\tilde{f}_n(\lambda))$. Let us now consider

$$\sigma^2(\sum_{u=1}^m Y_u^{(n)}) = h_n(m(n))$$

(3.9)
$$= \sum_{u,u'=1}^m \sum_{k,k'=-c(n)}^{c(n)} \{r_{u-u'}r_{u+k-u'-k'} + r_{u'-u+k'}r_{u'-u-k} + r^{(4)}_{k,u'-u,u'-u+k'}\}$$
$$\cdot \cos k\lambda \cos k'\lambda\, w_k^{(n)} w_{k'}^{(n)}$$

$$= (1) + (2) + (3)$$

where $m = m(n) \le n$ with $b_n^{-1} = o(m(n))$ as $n \to \infty$. The first term on the right of (3.9) is

$$(1) = \int \int_{-\pi}^{\pi} \frac{\sin^2(m/2)(\alpha - \beta)}{\sin^2(1/2)(\alpha - \beta)} f(\alpha)f(\beta) |\sum_{v=-c(n)}^{c(n)} w_v^{(n)} \cos v\lambda\, e^{iv\beta}|^2\, d\alpha\, d\beta.$$

The sequence of weight functions $W_n(u)$ have the property

$$(3.10) \qquad \max_{|u| \le Am^{-1}} \left| \frac{\int_{-\pi}^{\pi} W_n(v)W_n(v+u)\, dv}{\int_{-\pi}^{\pi} W_n^2(u)\, du} - 1 \right| \to 0$$

as $m = m(n) \to \infty$ for each $A > 0$ since $c(n) = b_n^{-1} = o(m(n))$. An argument like that given on page 176 of Rosenblatt (1974) implies that

$$(3.11) \quad (1) = 2\pi m(1 + o(1)) \int_{-\pi}^{\pi} f^2(\beta) \left| \frac{1}{2} W_n(\beta + \lambda) + \frac{1}{2} W_n(\beta - \lambda) \right|^2\, d\beta$$

holds. The lemma then yields

$$(3.12) \qquad (1) = \pi m(1 + \eta(\lambda) + o(1))f^2(\lambda) \int_{-\pi}^{\pi} W_n^2(v)\, dv.$$

The second term on the right of (3.9) is

$$(2) = \int \int_{-\pi}^{\pi} \frac{\sin^2 (m/2)(\alpha - \beta)}{\sin^2 \frac{1}{2}(\alpha - \beta)} f(\alpha)f(\beta)\left(\frac{1}{2} W_n(\alpha + \lambda) + \frac{1}{2} W_n(\alpha - \lambda) \right)$$
$$\cdot \left(\frac{1}{2} W_n(\beta + \lambda) + \frac{1}{2} W_n(\beta - \lambda) \right) d\alpha\, d\beta.$$

Property (3.10) of the family of weight functions as well as the continuity of f imply that relation (3.11) holds for (2) as well as (1). This implies that we get estimate (3.12) valid for (2) as well as (1). A direct estimate shows that

$$|(3)| \le m \sum_{k,s,k'} |r^{(4)}_{k,s,k'}|$$

and this is of smaller order than (3.12) under the assumptions made. Notice that

$$\int_{-\pi}^{\pi} W_n(u)^2 \, du = (1 + o(1))b_n^{-1} \int W^2(u) \, du.$$

The estimates of (1), (2) and (3) imply that $\sigma^2(\tilde{f}_n(\lambda))$ and therefore $\sigma^2(f_n(\lambda))$ have the same asymptotic behavior as (3.1) as $n \to \infty$. This is obtained by setting $m(n) = n$. Also notice that if $m(n) = o(n)$ and $k(n)m(n) = n$, that then

$$k(n)h_n(m(n)) \cong h_n(n)$$

because of the summability of the cumulant functions for $k = 2$ and $k = 4$.

Assumption (3.6) and Theorem 1 now imply the conclusion of this theorem in the case of a function $a(\cdot)$ with finite support.

We now indicate how the argument can be modified so as to take care of a function $a(\cdot)$ that is not of finite support. One can replace $f_n(\lambda)$ by

$$\hat{f}_n(\lambda) = 1/2\pi \sum_{|k| \le c(n)} r_k^{(n)} w_k^{(n)} \cos k\lambda$$

with error term

$$g_n(\lambda) = 1/2\pi \sum_{c(n) < |k| \le n} r_k^{(n)} w_k^{(n)} \cos k\lambda$$

and

$$c(n) = Lb_n^{-1}$$

where L is fixed but sufficiently large. The asymptotic normality of $\hat{f}_n(\lambda)$ has already been determined by the argument given since it corresponds to a function $a(\cdot)$ with finite support. We just have to estimate the variance of the error $g_n(\lambda)$ and show that it is sufficiently small if L is large. That will establish the asymptotic normality of $f_n(\lambda)$ for $a(\cdot)$ not of bounded support. Now

$$\sigma^2(g_n(\lambda)) = \frac{1}{4\pi^2 n^2} \sum_{|k|,|k'| \ge c(n)} \sum_{u,u'=1}^{n} \{r_{u-u'} r_{u-u'+k-k'} + r_{u-u'-k'} r_{u-u'+k}$$

$$+ r_{k,u'-u,u'-u+k'}^{(4)} \} w_k^{(n)} w_{k'}^{(n)} \cos k\lambda \cos k'\lambda$$

and this is bounded by

(3.13)
$$\frac{1}{4\pi^2 n} \sum_{|k|,|k'| \ge c(n)} \sum_{s} \{ |r_s| \, |r_{s+k-k'}| + |r_{s-k'} r_{s+k}|$$

$$+ |r_{k,s,s+k'}^{(4)}| \} \, |w_k^{(n)}| \, |w_{k'}^{(n)}|.$$

Notice that the terms $|w_k^{(n)}| = |a(kb_n)|$ are bounded by an absolute constant. The sum in (3.13) involving fourth cumulants is $O(1/n)$. The other terms in (3.13) are less than or equal to

(3.14)
$$\frac{1}{4\pi^2 n} \sum_{|k|,|k'| \ge c(n)} 2 |r_{k-k'}|^{(2*)} |w_k^{(n)}| \, |w_{k'}^{(n)}|$$

where $|r_k|^{(2*)}$ represents the convolution of the sequence $|r_k|$ with itself. Now

(3.15)
$$|kw_k^{(n)}| = |ka(kb_n)| \le Mb_n^{-1}$$

with M an absolute constant. This implies that the sum (3.14) is bounded by

$$\frac{1}{n} \sum_s |r_s|^{(2^*)} M^2 b_n^{-2} \sum_{|k| \geq c(n)} \{|k|(|k|+|s|)\}^{-1} \leq \frac{M^2}{n} L^{-1} b_n b_n^{-2} \sum_s |r_s|^{(2^*)}$$

because of (3.15). The variance $\sigma^2(g_n(\lambda))$ is bounded by an absolute constant multiplied by $L^{-1}(b_n n)^{-1}$ and this is small compared to $(b_n n)^{-1}$ if L is large.

Joint asymptotic normality and asymptotic independence of spectral density estimates at distinct frequencies λ, $0 \leq \lambda \leq \pi$, is dealt with in the following corollary.

COROLLARY 2. *Let all the assumptions of Theorem 2 be satisfied with the exception of (3.6). This is modified as follows. Consider*

$$(3.16) \qquad\qquad \sum_{j=1}^s \alpha_j Z_n(\lambda_j)$$

with the λ_j distinct frequencies in $[0, \pi]$. Let $F_{n,m(n)}(x)$ be the distribution function of (3.16). Let

$$(3.17) \qquad\qquad \frac{n}{m(n)} \int_{|x|>\eta} x^2 \, dF_{n,m(n)}(x) \to 0$$

as $n \to \infty$, $b_n \to 0$, $nb_n \to \infty$ for all values of the α_j's and each $\eta > 0$. It then follows that $(nb_n)^{1/2}\{f_n(\lambda_j) - Ef_n(\lambda_j)\}$ are jointly asymptotically independent and normally distributed with variances

$$2\pi(1 + \eta(\lambda_j)) f^2(\lambda_j) \int W^2(\alpha) \, d\alpha, \, j = 1, \cdots, s.$$

The conditions of Corollary 2 imply that every linear combination

$$(nb_n)^{1/2} \sum_{j=1}^s \alpha_j \{f_n(\lambda_j) - Ef_n(\lambda_j)\}$$

is asymptotically normal. That implies that the spectral estimates $(nb_n)^{1/2}\{f_n(\lambda_j) - Ef_n(\lambda_j)\}$, $j = 1, \cdots, s$, are jointly asymptotically normal. The asymptotic orthogonality and hence asymptotic independence follows from (3.3) of the lemma.

The conditions of the form (3.17) are sometimes difficult to verify directly. The following corollary replaces these by stronger cumulant conditions that are sometimes easier to implement and interpret.

COROLLARY 3. *Let the assumptions of the corollary be satisfied except for (3.17). These are replaced by summability of the cumulant functions up to order eight. The conclusion of the Corollary then holds.*

It is enough to show this in the case of a single spectral estimate. The argument for several estimates is the same except for the additional notation.

In order to apply Corollary 2 in this situation we shall verify condition (3.17) by showing that

$$\sigma^{-4} \left(\sum_{u=1}^m Y_u^{(n)}\right) E \left|\sum_{u=1}^m (Y_u^{(n)} - EY_u^{(n)})\right|^2 = O(1)$$

where $Y_u^{(n)} = Y_u^{(n)}(\lambda)$, for $m = m(n)$ as $n \to \infty$. Now

$$E \mid \sum_{u=1}^m (Y_u^{(n)} - EY_u^{(n)}) \mid^4 = \sigma^4 (\sum_{u=1}^m Y_u^{(n)}) + \mathrm{cum}_4 (\sum_{u=1}^m Y_u^{(n)}).$$

Because of the multilinear character of the cumulant

$$\mathrm{cum}_4 (\sum_{u=1}^m Y_u^{(n)}) = \sum_{u_1, u_2, u_3, u_4 = 1}^m \mathrm{cum}_4 (Y_{u_1}^{(n)}, \ Y_{u_2}^{(n)}, \ Y_{u_3}^{(n)}, \ Y_{u_4}^{(n)})$$

and

$$\mathrm{cum}_4 (Y_{u_1}^{(n)}, \ Y_{u_2}^{(n)}, \ Y_{u_3}^{(n)}, \ Y_{u_4}^{(n)})$$

$$= \left(\prod_{i=1}^4 w_{k_i} \cos k_i \lambda \right) \sum_\nu \mathrm{cum}(X_s, s \in \nu_1) \cdots \mathrm{cum}(X_s, s \in \nu_p)$$

where the summation over $\nu = \nu_1 \cup \cdots \cup \nu_p$ is over all indecomposable partitions (see the discussion in Section 2) of the following table

$$x_{u_1} x_{u_1 + k_1}$$
$$x_{u_2} x_{u_2 + k_2}$$
$$x_{u_3} x_{u_3 + k_3}$$
$$x_{u_4} x_{u_4 + k_4}.$$

One of the several indecomposable partitions consisting entirely of pairs leads to the sum

(3.18)
$$\sum_{k_i} \sum_{u_i} r_{u_1 - u_2} r_{u_3 - u_4} r_{u_3 - u_1 + k_3 - k_1} r_{u_4 - u_2 + k_4 - k_2}$$
$$w_{k_1}^{(n)} \cos k_1 \lambda w_{k_2}^{(n)} \cos k_2 \lambda \ w_{k_3}^{(n)} \cos k_3 \lambda \ w_{k_4}^{(n)} \cos k_4 \lambda.$$

Introduce $a = u_1 - u_2$, $b = u_3 - u_4$, $\alpha = k_3 - k_1$, $\beta = k_4 - k_2$. Expression (3.18) is bounded by

$$\sum_{a,b,u_1,u_3,k_1,k_2,\alpha,\beta} \mid r_a \mid \ \mid r_b \mid \ \mid r_{u_3 - u_1 + \alpha} \mid \ \mid r_{u_3 - u_1 - b + \alpha + \beta} \mid \ \mid w_{k_1}^{(n)} \mid \ \mid w_{k_1 + \alpha}^{(n)} \mid \ \mid w_{k_2}^{(n)} \mid \ \mid w_{k_2 + \beta}^{(n)} \mid.$$

If one first sums over k_1, k_2 the bound

$$b_n^{-2} \sum_{a,b,u_1,u_3,\alpha,\beta} \mid r_a \mid \ \mid r_b \mid \ \mid r_{u_3 - u_1 + \alpha} \mid \ \mid r_{u_3 - u_1 - b + \alpha + \beta} \mid$$

is obtained. A sum over α, β yields the bound

$$b_n^{-2} \sum_{a,b,u_1,u_3} \mid r_a \mid \ \mid r_b \mid.$$

The sum over a, b and then u_1, u_3 gives us the final bound

$$b_n^{-2} m^2.$$

All the other indecomposable partitions consisting entirely of pairs lead to the same bound. Indecomposable partitions which do not consist entirely of pairs yield smaller bounds. An example is given by the partition leading to the sum

(3.19)
$$\sum_{k_i} \sum_{u_i} r_{u_2 - u_1, u_3 - u_1, u_4 - u_1}^{(4)} r_{k_2 - k_1 + u_2 - u_1} r_{k_4 - k_3 + u_4 - u_3} \prod_{i=1}^4 (w_{k_i}^{(n)} \cos k_i \lambda).$$

Introduce $u_2 - u_1 = a$, $u_3 - u_1 = b$, $u_4 - u_1 = c$, $\alpha = k_2 - k_1$, $\beta = k_4 - k_3$. Expression (3.19) is then bounded by

$$\sum_{u_1, a, b, c, \alpha, \beta, k_1, k_2} \mid r_{a,b,c}^{(4)} \mid \ \mid r_{\alpha + a} \mid \ \mid r_{\beta + b} \mid \ \mid w_{k_1}^{(n)} \mid \ \mid w_{k_1 + \alpha}^{(n)} \mid \ \mid w_{k_2}^{(n)} \mid \ \mid w_{k_2 + \beta}^{(n)} \mid.$$

Summing first over k_1, k_2 leads to the bound

$$b_n^{-2} \sum_{u_1,a,b,c,\alpha,\beta} |r_{a,b,c}^{(4)}| \; |r_{\alpha+a}| \; |r_{\beta+b}|.$$

Then sum over α, β and then over a, b, c to obtain

$$b_n^{-2} \sum_{u_1} 1 = m b_n^{-2}.$$

These estimates imply that (3.17) is valid so that Corollary 2 can be applied to obtain the asymptotic normality of such spectral density estimates.

A little reflection shows one that Corollary 3 is still valid if one replaces strict stationarity by stationarity up to eighth order.

If one has a vector-valued strongly mixing strictly stationary process, comparable conditions lead to corresponding results on the asymptotic behavior of cross-spectral estimates (see Brillinger, 1975, and Rosenblatt, 1959). These ideas can be adapted to get results for higher order spectral estimates.

4. Additional remarks. Kolmogorov and Rozanov (1960) have shown a sufficient condition for a Gaussian stationary process to be strongly mixing is that it have a continuous positive spectral density function. In their book, Ibragimov and Rozanov (1978) give an example of a strongly mixing Gaussian process with an unbounded spectral density function. This example is constructed by making use of a result of Helson and Sarason (1967). Non-Gaussian strongly mixing processes can be constructed by taking instantaneous or bandlimited functions of Gaussian strongly mixing processes.

Let $\varphi(z)$ be the standard Gaussian density function and

$$H_v(z) = (-1)^v \frac{(d^v/dz^v)\varphi(z)}{\varphi(z)},$$

$v = 0, \cdots$, the Hermite polynomials. Let $Y_1, \cdots, Y_m$ be jointly Gaussian random variables with means zero, variances one, and covariance matrix

$$\rho = (\rho_{ij}; \, i, j = 1, \cdots, m).$$

Slepian (1972) has shown that the moment

(4.1)
$$E\{H_{s_1}(Y_1) \cdots H_{s_m}(Y_m)\} = s_1! \cdots s_m! \sum{}' \frac{\rho^v}{v!}$$

$$= s_1! \cdots s_m! \sum{}' \prod_{i<j} \frac{\rho_{ij}^{v_{ij}}}{v_{ij}!}$$

where the primed summation is over all $m \times m$ symmetric matrices $v = (v_{ij}; \, i, j = 1, \cdots, m)$ with nonnegative integer entries such that

$$s_k = \sum_{j \neq k} v_{jk}, \quad k = 1, \cdots, m.$$

This result leads directly to the following result.

COROLLARY 4. *Let $X = (X_k)$ be a strictly stationary Gaussian sequence with mean zero and summable covariance function. Assume that $\psi(x)$ is a function with*

a Fourier-Hermite expansion that has nonnegative coefficients. Further let

$$E \, | \, \psi(x) \, |^{2k} < \infty$$

for some positive integer k. Then the cumulant sequences up to order $2k$ are summable.

It is easy to generate functions ψ with nonnegative Fourier-Hermite coefficients such that all moments

$$E \, | \, \psi(x) \, |^{2m}$$

are not finite. Consider the identity

$$e^{tx}e^{-t^2/2} = \sum_{k=0}^{\infty} H_k(x)t^k/k!.$$

For any positive c

$$\psi(x) = \left(\frac{2\pi}{1+c}\right)^{1/2} \exp\left\{\frac{x^2}{2(1+c)}\right\} = \int_{-\infty}^{\infty} e^{tx}e^{-(1+c)t^2/2}\, dt$$

$$= \sum_{k=0}^{\infty} \frac{H_k(x)}{k!} \int_{-\infty}^{\infty} t^k e^{-ct^2/2}\, dt = \sum_{k=0}^{\infty} \frac{H_{2k}(x)}{2^k k!}\, c^{-k-1/2}.$$

With c sufficiently large the $(2k)$th moment of $\psi(x)$ will be finite but not higher order moments. If $X = (X_n)$ is a stationary Gaussian sequence with summable covariance sequence and positive spectral density, it will be strongly mixing. The non-Gaussian process $\psi(X_n)$ will be strongly mixing and have cumulant sequences summable up to order $2k$. If $k = 4$ the assumptions of Corollary 3 will be satisfied.

We now show how one can construct examples satisfying the assumptions of Corollary 3 but have a slow rate of decay of the mixing coefficient. A result of Ibragimov and Rozanov (1978, page 181) states that the mixing coefficient of a stationary Gaussian sequence

$$\alpha(n) = O(n^{-\beta})$$

for a fixed β, $0 < \beta < 1$, if and only if the process has a spectral density of the form

$$f(\lambda) = | \, P(e^{i\lambda}) \, |^2 w(\lambda)$$

where P is a polynomial with zeros on $| \, z \, | = 1$ and w is strictly positive and satisfies a Hölder condition of order β. One can show that if $\{Y_n\}$, $EY_n \equiv 0$, is a stationary Gaussian sequence with covariances

$$r_n = | \, n \, |^{-1-\beta}$$

for $| \, n \, | \geq 1$, its spectral density is of the form

$$f(\lambda) = c \, | \, \lambda \, |^{\beta} + g(\lambda)$$

where $c > 0$ is a constant and $g(\lambda)$ an infinitely differentiable function for $| \, \lambda \, | < \pi$ (see Zygmund, 1968, vol. 2, page 135). Notice that for β in this range

$$\sum | \, r_n | < \infty$$

but not

$$\sum |n| \, |r_n| < \infty.$$

By choosing β sufficiently close to zero, we can effect as slow an inverse polynomial rate of decay of $\alpha(n)$ as may be desired. Consider the process

$$X_n = Y_n^{2k-1}, \quad n = 1, 2, \cdots,$$

for k some integer greater than two. By (4.1) it is clear that all cumulant functions are absolutely summable. Corollary 3 is therefore applicable to the process $\{X_n\}$. However, $\{X_n\}$ has exactly the same mixing coefficient sequence as the process $\{Y_n\}$.

One can ask whether absolute summability of cumulants up to fourth order is enough to get the conclusion of Corollary 3. The examples constructed by Herrndorf (1983), indicating that moment conditions up to second order are not enough for a standard version of a central limit theorem, suggest that one may need more than summability of cumulants up to fourth order in the context of spectral density estimates.

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