The Annals of Probability
1984, Vol. 12, No. 3, 907–911

THE FINITE MEAN LIL BOUNDS ARE SHARP

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Let $X, X_1, X_2, \cdots$ be i.i.d. nonconstant mean zero random variables and put $S_n = X_1 + \cdots + X_n$. Let $K(y) > 0$ satisfy $yE\{|X/K(y)|^2 \wedge |X/K(y)|\} = 1$ (for $y > 0$). Then let

$$a_n = (\log \log n)K(n/\log \log n)$$

and

$$L = \lim \sup_{n\to\infty} S_n/a_n.$$

It is known that L is finite iff $P(X_n > a_n \text{ i.o.}) = 0$. When $L < \infty$, it is also known that $1 \le L \le 1.5$ and that it is possible for L to equal one. In this paper we construct an example for which $L = 1.5$.

Let $X, X_1, X_2, \cdots$ be independent identically distributed (i.i.d.) nonconstant zero-mean random variables. Let $S_n = X_1 + \cdots + X_n$. For each $y > 0$ there exists a unique positive real $K(y)$ such that

$$(1) \quad \frac{y}{v^2} EX^2 I(|X| \le v) + \frac{y}{v} E|X| I(|X| > v) \begin{cases} < 1 & \text{if } \; v > K(y) \\ = 1 & \text{if } \; v = K(y) \\ > 1 & \text{if } \; 0 < v < K(y). \end{cases}$$

Equivalently, $K(y)$ is the unique positive real satisfying

$$(2) \qquad yE\left\{\left(\frac{X}{K(y)}\right)^2 \wedge \left(\frac{|X|}{K(y)}\right)\right\} = 1.$$

Let $a_n = (\log \log n)K(n/\log \log n)$ and $L = \lim \sup_{n\to\infty} S_n/a_n$.

In Klass (1976, 1977) it was shown that $L < \infty$ iff $P(X_n > a_n \text{ i.o.}) = 0$. Moreover, when $L < \infty$, $1 \le L \le 1.5$. The lower bound $L = 1$ is achieved (for example) by X-distributions whose positive part is bounded above and whose negative part is regularly varying of index 1 (i.e., $yP(X < -y)$ is slowly varying as $y \to \infty$). It was not known whether the upper-bound $L = 1.5$ could be achieved. In fact, since $L = \sqrt{2}$ when X has finite variance, one could easily have suspected (and indeed in Klass (1977) it was conjectured) that when finite, L is always between one and $\sqrt{2}$. In this paper we show by example that the upper-bound of 1.5 can be attained. It is achieved by an X-distribution whose partial sums S_n are excessive due to the dual contributions of both conditional drift and conditional variance terms.

We require use of a lemma on tail probabilities. It is a triangular array version of one to be found in Chow and Teicher (1978, page 341). The proof is based on

Received March 1983; revised September 1983.

[1] This research was supported in part by National Science Foundation Grant No. MCS-80-64022.

AMS 1980 *subject classifications*. Primary 60F15; secondary, 60G50.

Key words and phrases. Generalized or universal law of the iterated logarithm, sums of i.i.d. random variables.

the idea that a sum of i.i.d. variables can be abnormally large due to many runs of "good luck," i.e., due to many successive blocks of moderately sized summands.

LEMMA. *For each* $k \ge 1$ *let* $\bar{X}_{k1}, \cdots, \bar{X}_{kn_k}$ *be i.i.d. random variables with* $n_k \to \infty$. *Let* $u_k \to \infty$ *in such a way that* $n_k/u_k^2 \to \infty$. *Assume that for any* $n_k/u_k^2 \le j_k \le n_k$,

$$(3) \qquad \mathscr{L}\left(\frac{\bar{X}_{k1} + \cdots + \bar{X}_{kj_k}}{\sqrt{j_k/n_k}}\right) \to N(0, 1).$$

Then for each $\varepsilon > 0$ $\exists \delta_\varepsilon > 0$ *and* $k_\varepsilon < \infty$ *such that for* $k \ge k_\varepsilon$

$$(4) \qquad P(\bar{X}_{k1} + \cdots + \bar{X}_{kn_k} > (1 - \varepsilon)u_k) \ge \exp(-(1 - \delta_\varepsilon)u_k^2/2).$$

PROOF. Fix $0 < \varepsilon \ll 1$ and let $Z \sim N(0, 1)$. Take $y \gg 1$ such that

$$P(Z > (1 - \varepsilon/2)\sqrt{y}) \ge e^{-(1-\varepsilon/2)y/2}.$$

Split up the X_{ki}'s for $1 \le i \le n_k$ into $r_k = [u_k^2/y]$ consecutive blocks of lengths $\ell_{k1}, \cdots, \ell_{kr_k}$ where $\ell_{k1} + \cdots + \ell_{kr_k} = n_k$ and $\ell_{ki}r_k/n_k \to 1$ as $k \to \infty$, uniformly in i. Now let $\bar{S}_{k1}, \cdots, \bar{S}_{kr_k}$ satisfy $\bar{S}_{k1} + \cdots + \bar{S}_{kj} = \bar{X}_{k1} + \cdots + \bar{X}_{k\sum_{i=1}^{j}\ell_{ki}}$. Then

$$P(\bar{X}_{k1} + \cdots + \bar{X}_{kn_k} > (1 - \varepsilon)u_k)$$

$$\ge P(\cap_{j=1}^{r_k} \{\bar{S}_{kj} > (1 - \varepsilon)u_k/r_k\})$$

$$= \prod_{j=1}^{r_k} P\left(\frac{\bar{X}_{k1} + \cdots + \bar{X}_{k\ell_{kj}}}{\sqrt{\ell_{kj}/n_k}} > \frac{(1 - \varepsilon)u_k\sqrt{n_k/\ell_{kj}}}{r_k}\right)$$

(by indep and the fact that for each k, $\bar{X}_{k1}, \cdots, \bar{X}_{kn_k}$ are i.i.d.)

$$\ge \prod_{j=1}^{r_k} P(Z > (1 - \varepsilon/2)\sqrt{y}) \quad \text{for } k \text{ large}$$

(by (3) and the fact that $yr_k/u_k^2 \to 1$)

$$\ge \exp(-r_k(1 - \varepsilon/2)y/2)$$

$$\ge \exp(-(1 - \varepsilon/4)u_k^2/2) \quad \text{for } k \text{ large.} \quad \square$$

THE EXAMPLE. Let X be bounded above, have mean zero, and negative part X^- with an absolutely continuous density of the form

$$(5) \qquad f_{X^-}(x)I(x > 0) = \sum_{k=k_0}^{\infty} x^{-2}c_k I(x \in (\alpha_k, \beta_k)),$$

where, among other conditions that will not be specified, β_k/α_k, α_{k+1}/β_k, $c_{k+1}\beta_{k+1}/c_k\beta_k$, and $c_k \log(\beta_k/\alpha_k)/c_{k+1}\log(\beta_{k+1}/\alpha_{k+1})$ all tend to infinity as $k \to \infty$. For our purposes it suffices to take

(i) $\alpha_k = \exp(k \log\log k)$
(ii) $\beta_k = \alpha_k\sqrt{\log k}$
(iii) $c_k = \exp\{-(k/2)(\log\log k + 2\log\log\log k - \log 4)\}$.

(All logarithms are taken to the base e.)

As usual, we define $K(\cdot)$ as in (2), letting $a_n = (\log\log n)K(n/\log\log n)$ and $L \equiv \limsup_{n\to\infty} S_n/a_n$. Since X^+ is bounded above, $L \le 1.5$. Fix $0 < \varepsilon \ll 1$. We need but prove that $P(S_n > (1-\varepsilon)1.5\, a_n \text{ i.o.}) = 1$. By virtue of the argument used in Section 3 of Klass (1977), it suffices to prove that

$$\sum_{k\ge k_0} P(S_{n_k} \ge (1-\varepsilon)1.5a_{n_k}) = \infty \tag{6}$$

for some integers $1 \le n_1 < n_2 < \cdots$ such that $n_{k+1}/n_k \to \infty$.

Clearly, (6) holds if

$$\liminf_{k\to\infty} kP(S_{n_k} > (1-\varepsilon)1.5a_{n_k}) > 0. \tag{7}$$

We concentrate on establishing (7). Set

$$n_k = [(\log k)(\log\log k)^{-1}\exp\{k(3/2\log\log k + \log\log\log k - \log 2)\}], \tag{8}$$

where $[x]$ denotes the largest integer not exceeding x.

It is routine to check that $n_{k+1}/n_k \to \infty$, $\log\log n_k \sim \log k$, and

$$(n_k/\log k)P(X^- > \alpha_k) \to 0.$$

Also, we have

$$(n_k/\log\log n_k)EX^-I(X^- > \alpha_k) \sim (n_k/\log k)c_k\log(\beta_k/\alpha_k) \sim 2^{-1}\alpha_k. \tag{9}$$

And

$$(n_k/\log\log n_k)EX^2I(|X| \le \alpha_k) \sim (n_k/\log k)\beta_{k-1}c_{k-1} \sim 2^{-1}\alpha_k^2. \tag{10}$$

In view of (9) and (10), $(n_k/\log\log n_k)E\{(X/\alpha_k)^2 \wedge |X/\alpha_k|\} \sim 1$. Hence there exists $\varepsilon_k \to 0$ such that $\alpha_k = K((1+\varepsilon_k)n_k/\log\log n_k)$. Since $y^{-1}K(y) \searrow$ and $K(y) \nearrow$ we see that for $|\varepsilon| < 1$, $(1-|\varepsilon|)K(y) \le K((1-|\varepsilon|)y) \le K((1+\varepsilon)y) \le K((1+|\varepsilon|)y) \le (1+|\varepsilon|)K(y)$. It therefore follows that

$$\alpha_k \sim K(n_k/\log\log n_k). \tag{11}$$

When S_{n_k} is large it is likely that no X_j for $1 \le j \le n_k$ is terribly negative. Thus we (somewhat loosely) associate the event $\{S_{n_k} > (1-\varepsilon)1.5a_{n_k}\}$ with the event $A_k \cap \{\sum_{j=1}^{n_k} X_{kj} > (1-\varepsilon)1.5a_{n_k}\}$, where $A_k = \bigcap_{j=1}^{n_k}\{X_j > -\alpha_k\}$ and $X_{kj} = (X_j \mid X_j > -\alpha_k)(=(X_j \mid X_j > -\beta_{k-1}))$. Observe that $(S_{n_k} \mid A_k) = \sum_{j=1}^{n_k} X_{kj}$. The X_{kj}'s have positive drift. Since $EX = 0$ this drift equals

$$EX_{k1} = EX^-I(X^- > \alpha_k)/P(X \ge -\alpha_k).$$

Letting γ_k satisfy $n_kEX_{k1} = \gamma_k a_{n_k}$, it is clear from (9) and (11) that

$$\gamma_k \to 1/2 \quad (\text{i.e., } n_kEX_{k1} \sim 2^{-1}a_{n_k}). \tag{12}$$

When A_k occurs, excessively large values of S_{n_k} are influenced chiefly by the drift component $(n_kEX_{k1} = \gamma_k a_{n_k})$ and by the fluctuation component $(\sum_{j=1}^{n_k}\tilde{X}_{kj})$ of

$$\sum_{j=1}^{n_k} X_{kj} = \gamma_k a_{n_k} + \sum_{j=1}^{n_k}(X_{kj} - EX_{kj}) \equiv \gamma_k a_{n_k} + \sum_{j=1}^{n_k}\tilde{X}_{kj}.$$

We now treat the fluctuation component. Observe that it has variance $n_kE\tilde{X}_{k1}^2 \sim a_{n_k}^2/2\log k$.

Let $\bar{X}_{kj} = \tilde{X}_{kj}\sqrt{n_k E\tilde{X}_{kj}^2}$. For $n_k/3 \log k \le j_k \le n_k$,

$$j_k E\left|\frac{\bar{X}_{k1}}{\sqrt{j_k/n_k}}\right|^3 \Big/ \left(j_k E\,\frac{\bar{X}_{k1}^2}{j_k/n_k}\right)^{3/2}$$

$$\sim \frac{j_k^{-1/2} E|X|^3 I(|X| \le \beta_{k-1})}{(EX^2 I(|X| \le \beta_{k-1}))^{3/2}} \le \frac{\beta_{k-1}}{(j_k EX^2 I(|X| \le \beta_{k-1}))^{1/2}}$$

$$\le (3 \log k)^{1/2} \beta_{k-1}(n_k EX^2 I(|X| \le \beta_{k-1}))^{-1/2}$$

$$= (3 \log k)^{1/2} \beta_{k-1}\left((\log\log n_k)\,\frac{\alpha_k^2}{2}\,(1 + o(1))\right)^{-1/2}$$

$$= (1 + o(1))\sqrt{\frac{6}{\log k}} \to 0.$$

Invoke the Berry-Esseen Theorem to conclude that

$$\mathscr{L}\left(\frac{\bar{X}_{k1} + \cdots + \bar{X}_{kj_k}}{\sqrt{j_k/n_k}}\right) \to N(0, 1).$$

By the lemma, it follows that there exists $\delta_\varepsilon > 0$ such that for k large

$$P\left(\textstyle\sum_{j=1}^{n_k} X_{kj} < (1 - \varepsilon)1.5a_{n_k}\right)$$

$$\ge P\left(\textstyle\sum_{j=1}^{n_k} \tilde{X}_{kj} > (1 - \varepsilon)a_{n_k}\right) \quad (\text{since } n_k EX_{k1} \sim 2^{-1}a_{n_k})$$

$$= P\left(\frac{\sum_{j=1}^{n_k} \tilde{X}_{kj}}{\sqrt{n_k E\tilde{X}_{k1}^2}} > \frac{(1 - \varepsilon)a_{n_k}}{\sqrt{n_k E\tilde{X}_{k1}^2}}\right)$$

$$\ge \exp\left\{\frac{-2^{-1}(1 - \delta_\varepsilon)a_{n_k}^2}{n_k E\tilde{X}_{k1}^2}\right\} = \exp\{-(1 + o(1))(1 - \delta_\varepsilon)\log k\}.$$

Lower-bounding,

$$P(S_{n_k} > (1 - \varepsilon)1.5a_{n_k})$$

$$\ge P\left(\textstyle\bigcap_{j=1}^{n_k} \{X_j > -\alpha_k\}\right) P\left(\textstyle\sum_{j=1}^{n_k} X_{kj} > (1 - \varepsilon)1.5a_{n_k}\right)$$

$$= (\exp\{-(1 + o(1))n_k P(X^- \ge \alpha_k)\}) P\left(\textstyle\sum_{j=1}^{n_k} \tilde{X}_{kj} > ((1 - \varepsilon)1.5 - \gamma_k)a_{n_k}\right)$$

$$> \exp\{-o(\log k) - (1 + o(1))(1 - \delta_\varepsilon)\log k\}$$

$$> k^{-1} \quad \text{for } k \text{ large},$$

which establishes (7) and therefore also (6).

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