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ASYMPTOTIC BEHAVIOR OF THE LOCAL TIME OF A RECURRENT RANDOM WALK[1]

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Let (S_j) be a lattice random walk, i.e. $S_j = X_1 + \cdots + X_j$ where $X_1, X_2, \cdots$ are independent random variables with values in the integer lattice and common nondegenerate distribution F, and let $L_n(x) = \sum_{j=0}^{n-1} 1_{\{x\}}(S_j)$, the local time of the random walk at x before time n. Define $G(x) = P\{|X_1| > x\}$, $K(x) = x^{-2} \int_{|y|\le x} y^2\, dF(y)$, $Q(x) = G(x) + K(x)$ for $x > 0$. Q is continuous, strictly decreasing for large x, and tends to zero. Thus a_y may be defined by $Q(a_y) = y^{-1}$ for large y and then we let $c_n = a_{n/\log\log n}$. The basic assumption is that $\limsup_{x\to\infty} G(x)/K(x) < 1$ and $EX_1 = 0$. We prove that there exist positive constants θ_1, θ_2 such that $\limsup_{n\to\infty} c_n n^{-1} L_n(x) = \theta_1$ a.s. for all x, $\limsup_{n\to\infty} \sup_x c_n n^{-1} L_n(x) = \theta_2$ a.s. Furthermore

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{|x-y|\le\delta c_n} c_n n^{-1} |L_n(x) - L_n(y)| = 0 \quad \text{a.s.}$$

One of the main tools is an estimate for the "absolute potential kernel":

$$\sum_{n=0}^{\infty} |P\{S_n = z\} - P\{S_n = z + x\}| \le C(|x| Q(|x|))^{-1}$$

assuming strong aperiodicity.

1. Introduction. Let $X_1, X_2, \cdots$ be independent, identically distributed, nondegenerate random variables taking values in the one dimensional integer lattice $\mathbb{Z}$. With $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$, $n \ge 1$, we define the local time at x before time n to be the number of visits to x before time n:

$$L_n(x) = \sum_{j=0}^{n-1} 1_{\{x\}}(S_j), \quad x \in \mathbb{Z}, \quad n \ge 1. \tag{1.1}$$

Our goal in the present paper is to find the rate of growth of the large values of $L_n(x)$ for fixed x and also for $\sup_x L_n(x)$.

In order to state the results, we must introduce a little notation. Let X be a random variable with the same distribution as X_1 and F its distribution function. For $x > 0$ define

$$G(x) = P\{|X| > x\}, \quad K(x) = x^{-2} \int_{|y|\le x} y^2\, dF(y), \tag{1.2}$$
$$Q(x) = G(x) + K(x) = E(x^{-1}|X| \wedge 1)^2.$$

Our basic assumption throughout will be

$$\limsup_{x\to\infty} \frac{G(x)}{K(x)} < 1 \quad \text{and} \quad EX = 0. \tag{A$_1$}$$

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If X is in the domain of attraction of a stable law of index α then

$$\lim_{x\to\infty}\frac{G(x)}{K(x)}=\frac{2-\alpha}{\alpha},$$

so that we are including all distributions in the domain of attraction of a stable law of index $\alpha > 1$ which have zero mean. But the class of distributions we are considering is clearly much larger than this. We should note here that if the random walk is transient, then $L_n(x)$ is bounded in n; we will only consider recurrent random walks. The first part of (A_1) implies that $E|X| < \infty$ so that the second part ($EX = 0$) is just to ensure recurrence. Those random walks which are recurrent and in the domain of attraction of a Cauchy distribution are among those excluded by (A_1). We expect their behavior to be quite different and hope to deal with it in a future paper.

The function Q defined in (1.2) is continuous and strictly decreasing for $x \geq x_0$ where

$$x_0 = \sup\{x: P\{|X| \leq x\} = 0\}.$$

Thus we can define a_y by

$$Q(a_y) = (1/y) \quad \text{for} \quad y > y_0$$

where $y_0 = 1/Q(1)$. We will make the convention that $a_y = 1$ for $y \in [0, y_0]$. Note that $a_y \uparrow \infty$. The sequence $\{a_n\}$ will play the role of the normalizing sequence for weak convergence of S_n; even though there is no weak convergence result outside the domain of attraction setting it is still the case that under (A_1) the sequence $\{a_n^{-1}S_n\}$ is tight and there is an approximate version of the local limit theorem for this sequence [7]. This result then leads to the needed probability estimates.

Now we define

$$c_n = a_{n/\log\log n} \quad \text{for} \quad n \geq 3.$$

We may now state our results. To avoid needlessly complicating the statements we assume that

$$\Sigma = \{x \in \mathbb{Z}: P\{S_n = x\} > 0 \text{ for some } n\} = \mathbb{Z}.$$

Since Σ is a subgroup of $\mathbb{Z}$ in the recurrent case (see pages 15 and 19 of [13]) this really amounts to relabeling the state space. Also recall that we are assuming (A_1).

THEOREM 1. *There exists $\theta_1 \in (0, \infty)$ such that for all $x \in \mathbb{Z}$*

$$\limsup_{n\to\infty}(c_n/n)L_n(x) = \theta_1 \quad \text{a.s.}$$

THEOREM 2. *Given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\limsup_{n\to\infty}\sup_{|x-y|\leq\delta c_n}(c_n/n)\,|L_n(x) - L_n(y)| < \varepsilon \quad \text{a.s.}$$

THEOREM 3. *There exists $\theta_2 \in (0, \infty)$ such that*

$$\limsup_{n\to\infty}\sup_x(c_n/n)L_n(x) = \theta_2 \quad \text{a.s.}$$

In this generality we do not yet know whether $\theta_1 = \theta_2$. However, for X in the domain of attraction of a stable law G we can show that $\theta_1 = \theta_2 = \theta_G$ where θ_G is the constant for the analogous problem for the stable process corresponding to G. This problem has been studied by Donsker and Varadhan [3] and they even evaluated θ_G for the symmetric stable processes. We obtain an invariance principle and some of its consequences in the domain of attraction setting in [9] using techniques and results from [3] and [8]. Theorem 2 is used in proving Theorem 3 and is also a basic tool in [9]. Theorems 1 and 3 are due to Kesten [11] in the classical setting of mean zero and finite variance for X.

The required probability estimates are in the next section. They depend heavily on the approximate local limit theorem results mentioned above that have been derived in [7] in collaboration with Philip Griffin. His ideas are also involved in the proofs of several of the lemmas. The techniques of Donsker and Varadhan [3] will also be apparent. Incidentally, these estimates show that $\{a_n n^{-1} L_n(x)\}$ is stochastically compact, i.e. tight and no subsequence converges to a degenerate limit. Darling and Kac [2] have necessary and sufficient conditions for weak convergence of $\{a_n n^{-1} L_n(x)\}$ and have shown that the limit must be a Mittag-Leffler distribution. It would be of interest to determine the class of subsequential limits of $\{a_n n^{-1} L_n(x)\}$ for random walks satisfying (A_1).

One lemma that should be mentioned here since it should have many uses is Lemma 7 which asserts that under (A_1) and an aperiodicity assumption

$$\sum_{n=0}^{\infty} |P\{S_n = z\} - P\{S_n = x + z\}| = O\left(\frac{1}{|x| Q(|x|)}\right).$$

In the stable case the bound is $|x|^{\alpha-1}$ which is of the right order for $z = 0$. For the classical case of summands having finite variance, Spitzer [13, page 354] shows the sum without absolute values behaves like a constant times $|x|$ as $|x| \to \infty$ when $z = 0$. We believe the bound to be fairly sharp for $z = 0$ in general but we have not tried to find a lower bound since we do not need it. Although Spitzer has shown [13, page 352] that the potential kernel converges for general recurrent random walk, the question of absolute convergence in general is still open.

Once the probability estimates are obtained in Section 2, the proofs of the theorems are fairly standard Borel Cantelli arguments. These appear in Section 3.

Bert Fristedt has pointed out to us that the inverse of the local time at x is an increasing random walk and therefore the results of [5] imply that there will be a result like Theorem 1 for any recurrent random walk. The advantage of Theorem 1 is that the normalization is obtained explicitly and can be easily computed from the distribution of X.

Results analogous to Theorems 1–3 can be proved by essentially the same techniques for general (i.e. non-lattice) recurrent random walks satisfying (A_1) if one replaces $L_n(x)$ by

$$L_n(x) = \sum_{j=0}^{n-1} 1_{(x-\delta, x+\delta]}(S_j) \quad x \in \mathbb{R}^1, \quad n \geq 1.$$

Of course, in the lattice case one wants $\delta \geq 1/2$.

2. Probability estimates. This section will consist of a sequence of lemmas which will lead to the desired probability estimates for the upper tail of the distribution of $n^{-1}c_nL_n(x)$. We will assume throughout that (A_1) holds. A useful consequence of (A_1) is that there exists a $\lambda > 1$ and an x_0 such that

$$x^\lambda Q(x)\downarrow \quad \text{for} \quad x \geq x_0 \tag{2.1}$$

(see Lemma 2.4 of [12]). It then follows trivially that there is a c such that

$$x^\lambda Q(x) \geq c y^\lambda Q(y) \quad \text{for} \quad \pi^{-1} \leq x \leq y. \tag{2.2}$$

It is clear that

$$x^2 Q(x)\uparrow \tag{2.3}$$

and then it is a consequence of (2.1), (2.3), and the fact that $a_y\uparrow$ that

$$\gamma^{1/\lambda} a_n \leq a_{\gamma n} \leq \gamma^{1/2} a_n \quad \text{for} \quad \gamma \leq 1, \qquad \gamma n \text{ large.} \tag{2.4}$$

We also need to discuss the periodicity of the random walk a bit. Since we have assumed that $\Sigma = \mathbb{Z}$ we know that $\varphi(u) = 1$ iff $u = 2\pi k$ for some $k \in \mathbb{Z}$ where φ is the characteristic function of X. But the symmetrized random walk may live on a subgroup $p\mathbb{Z}$ of $\mathbb{Z}$ for some integer $p > 1$ and then p is the time period of the original random walk. It is determined by $|\varphi(u)| = 1$ iff $u = 2\pi k p^{-1}$ for some $k \in \mathbb{Z}$. The periodic structure of the random walk is that $S_{np} \in p\mathbb{Z}$ for all $n \geq 0$ and for each $k \in [1, p-1]$, $\{S_{np+k}, n \geq 0\}$ will be contained in one of the $p-1$ cosets $p\mathbb{Z} + j$, $j = 1, \cdots, p-1$. For example, simple random walk has $p = 2$. For a discussion of these facts see pages 42–43 of [13]. We will use p throughout to represent this period.

First we will quote four results from [6] and [7] that provide the basis for the later lemmas. We adopt the usual practice of using c, C for constants that depend only on the distribution of X and may change from line to line. Recall that (A_1) is assumed throughout.

LEMMA 1. *(Theorem 2.10 of [6]) There is a positive c such that*

$$|\varphi(u)| \leq 1 - cQ(|u|^{-1}), \quad 0 < |u| \leq \pi p^{-1}.$$

Here φ is the characteristic function of X and Q is defined in (1.2).

LEMMA 2. *(Theorems 1 and 3 of [7]) There is a C such that*

$$P\{S_n = x\} \leq \frac{C}{a_n} \quad \text{for all} \quad x \in \mathbb{Z}, \qquad n \geq 1. \tag{2.5}$$

Furthermore, for any M, there is a c and an n_0 such that

$$P\{S_n = x\} \geq \frac{c}{a_n}$$

for all $n \geq n_0$ and $|x| \leq Ma_n$ provided x is in the coset of $p\mathbb{Z}$ visited by S_n at time n.

LEMMA 3. *(Theorems* 1 *and* 3 *of* [7]) *There is a C such that*

$$P\{\max_{1\le k\le n}|S_k| \ge Ma_n\} \le CM^{-\lambda} \quad \textit{for all} \quad n, \qquad M \ge 1,$$

where λ *is as in* (2.1).

PROOF. To put in the max, use Skorokhod's inequality.

LEMMA 4. *(Theorem* 4 *of* [7]) *There is a C such that*

$$|P\{S_n = y\} - P\{S_n = x\}| \le C\frac{|y-x|}{a_n^2} \tag{2.6}$$

for all x, y *provided* $y - x \in p\mathbb{Z}$, $n \ge 1$.

Next we need a technical lemma concerning integration.

LEMMA 5. *Let H be a monotone nondecreasing right continuous function defined on* $[0, \infty)$ *with* $H(0) = 0$. *Then*

$$\int_{(0,a]} \frac{dH(y)}{H^\alpha(y)} \le \frac{1}{1-\alpha}\{H(a)\}^{1-\alpha}, \quad 0 < \alpha < 1,$$

and

$$\int_{(a,\infty)} \frac{dH(y)}{H^\alpha(y)} \le \frac{1}{\alpha-1}\{H(a)\}^{1-\alpha}, \quad \alpha > 1.$$

PROOF. These are trivial to prove (with equality) if H is continuous and $H(\infty) = \infty$. For a general H write

$$\int_{(a,b]} \frac{dH(y)}{H^\alpha(y)} = \alpha \int_{(a,b]} \int_{H(y)}^\infty \frac{dz}{z^{\alpha+1}}\, dH(y)$$

and use Fubini. Letting $\chi(y, z) = 1\{(y, z) : z \ge H(y)\}$ we have

$$\int_{(a,b]} \frac{dH(y)}{H^\alpha(y)} = \alpha \int_{(0,\infty)} \int_{(a,b]} \chi(y, z)\, dH(y) \frac{dz}{z^{\alpha+1}}$$

$$\le \alpha \int_{H(a)}^{H(b)} (z - H(a)) \frac{dz}{z^{\alpha+1}} + \alpha \int_{H(b)}^{\infty} (H(b) - H(a)) \frac{dz}{z^{\alpha+1}}$$

$$= \frac{1}{1-\alpha}(H^{1-\alpha}(b) - H^{1-\alpha}(a)).$$

Letting $a \to 0$ gives the first result and $b \to \infty$ the second.

Now we will prove the lemmas required for our probability estimates.

LEMMA 6. *Suppose that* $b_n\downarrow$ *and* $\sum_{k=0}^{n-1} b_k \le Cnb_n$. *Then*

$$\sum_{k\in I(n,r)} \prod_{i=1}^{r} b_{k_i} \le (4Cmb_m)^r \quad \textit{for all} \quad n, r \ge 1,$$

where $m = [nr^{-1}] + 1$ *and*

$$I(n, r) = \{k = (k_1, \cdots, k_r) \in \mathbb{Z}^r : k_i \ge 0, \textstyle\sum_{i=1}^{r} k_i < n\}.$$

REMARK. We will use the lemma with $b_n = a_n^{-1}$. The second hypothesis is valid since by (2.2)

$$a_k \ge \left(c\frac{k}{n}\right)^{1/\lambda} a_n \quad \text{so that} \quad b_k \le \left(c^{-1}\frac{n}{k}\right)^{1/\lambda} b_n, \qquad k \ge 1.$$

PROOF. We first block the interval of summation for k_i into blocks of length m. For any $j_i \ge 0$, we have

$$\sum_{k_i=j_im}^{(j_i+1)m-1} b_{k_i} \le Cmb_m$$

by hypothesis. Thus, if we expand $I(n, r)$ to include all of these cubes of side m which intersect it we obtain the bound $N(n, r)(Cmb_m)^r$ where $N(n, r)$ is the number of cubes intersecting $I(n, r)$. Now if the cube with edges $j_im \le k_i < (j_i + 1)m$, $i = 1, \cdots, r$, intersects $I(n, r)$ we must have

$$\sum_{i=1}^{r} j_im < n \quad \text{so that} \quad \sum_{i=1}^{r} j_i < (n/m) < r.$$

Now the number of ways of choosing the j_i to give a sum less than r is $\binom{2r-1}{r}$ (see [4, page 38]) so that

$$N(n, r) \le \binom{2r-1}{r} \le 4^r.$$

Next we obtain the estimate for the "absolute potential kernel". The appropriate analogue for $x \notin p\mathbb{Z}$ is in the corollary that follows.

LEMMA 7. *There is a* C *such that for* $x \in p\mathbb{Z}$, $z \in \mathbb{Z}$,

$$\sum_{n=0}^{\infty} |P\{S_n = z\} - P\{S_n = x + z\}| \le \frac{C}{|x|Q(|x|)}.$$

REMARK. If one assumes in addition to (A_1) that $x^\rho Q(x)\uparrow$ for some $\rho < 2$, then this follows easily from (2.5) and (2.6).

PROOF. We are going to simplify the writing a little by assuming that $z = 0$. The general case may be deduced from this easily by waiting until the random time when $\{S_n\}$ hits the two point set $\{z, x + z\}$ and then starting over. Recall that $P\{S_n = 0\}$ and $P\{S_n = x\}$ are both zero unless $n \in p\mathbb{Z}$ so we shall assume that $n \in p\mathbb{Z}$. This means that $\varphi^n(u)$ will have period $2\pi p^{-1}$ since for such n, S_n

takes its value in $p\mathbb{Z}$. By the inversion formula,

$$P\{S_n = 0\} - P\{S_n = x\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} (1 - e^{-ixu})\varphi^n(u)\,du$$
$$= \frac{p}{2\pi}\int_{-(\pi/p)}^{(\pi/p)} (1 - e^{-ixu})\varphi^n(u)\,du$$

so that

$$|P\{S_n = 0\} - P\{S_n = x\}|$$
$$\le \frac{p}{\pi}\int_0^{(\pi/p)} (1 - \cos xu)\,|\varphi(u)|^n\,du + \frac{p}{\pi}\int_0^{(\pi/p)} |\sin xu \operatorname{Im} \varphi^n(u)|\,du. \tag{2.7}$$

We fix a $\delta \in (0, x_0^{-1})$ with x_0 as in (2.1). Then since $\max_{\delta \le u \le \pi/p} |\varphi(u)| = s < 1$, we get a trivial bound of $3s^n$ for the sum of both integrals over $[\delta, \pi/p]$. This sums to $3(1 - s)^{-1}$ which is acceptable since (2.1) implies that $|x|\,Q(|x|) \to 0$ as $|x| \to \infty$. Thus we will only be concerned with these integrals over $[0, \delta]$. For the first one, we have by Lemma 1, (2.1), and (2.3)

$$\sum_{n=0}^{\infty}\int_0^{\delta} (1 - \cos xu)\,|\varphi(u)|^n\,du$$
$$\le \int_0^{\delta} (1 - \cos xu)(1 - |\varphi(u)|)^{-1}\,du$$
$$\le c^{-1}\int_0^{|x|^{-1}} x^2u^2(Q(u^{-1}))^{-1}\,du + c^{-1}\int_{|x|^{-1}}^{\delta} 2(Q(u^{-1}))^{-1}\,du$$
$$\le c^{-1}x^2\int_{|x|}^{\infty}\frac{dv}{v^4Q(v)} + 2c^{-1}\int_{x_0}^{|x|}\frac{dv}{v^2Q(v)}$$
$$\le c^{-1}\frac{1}{Q(|x|)}\int_{|x|}^{\infty}\frac{dv}{v^2} + 2c^{-1}\frac{1}{|x|^{\lambda}Q(|x|)}\int_{x_0}^{|x|}\frac{dv}{v^{2-\lambda}}$$
$$\le c^{-1}\frac{1}{|x|\,Q(|x|)}\left(1 + \frac{2}{\lambda - 1}\right).$$

Now we must deal with the second integral in (2.7) which is more complicated. Since

$$|\varphi^n(u) - (\operatorname{Re}\varphi(u))^n| \le n\,|\operatorname{Im}\varphi(u)|\,|\varphi(u)|^{n-1} \tag{2.8}$$

we have

$$|\operatorname{Im} \varphi^n(u)| \le n |\operatorname{Im} \varphi(u)| |\varphi(u)|^{n-1}. \tag{2.9}$$

Now $EX = 0$ so that

$$\operatorname{Im} \varphi(u) = \int \sin uy \, dF(y) = \int (\sin uy - uy) \, dF(y),$$

$$|\operatorname{Im} \varphi(u)| \le \int_{|uy|\le 1} |uy|^3 \, dF(y) + 2 \int_{|uy|>1} |yu| \, dF(y). \tag{2.10}$$

Using (2.9) and Lemma 1 we now bound the second term in (2.7):

$$\begin{aligned}\sum_{n=0}^{\infty} \int_0^{\delta} |\sin xu \operatorname{Im} \varphi^n(u)| \, du &\le \int_0^{\delta} |\sin xu \operatorname{Im} \varphi(u)| \sum_{n=0}^{\infty} n |\varphi(u)|^{n-1} \, du \\ &= \int_0^{\delta} |\sin xu \operatorname{Im} \varphi(u)| (1 - |\varphi(u)|)^{-2} \, du \\ &\le c^{-2} \int_0^{\delta} |\sin xu| |\operatorname{Im} \varphi(u)| (Q(u^{-1}))^{-2} \, du.\end{aligned}$$

Now we will bound $|\sin xu|$ by $|xu| \wedge 1$, $|\operatorname{Im} \varphi(u)|$ by (2.10) and then change the order of integration. In carrying out the u integration, there will be two cases: (1) $|y| > |x|$ and (2) $|y| \le |x|$.

CASE (1). In this case $|y|^{-1} < |x|^{-1}$ and the u integral will be split into three parts. First by (2.3)

$$\begin{aligned}\int_0^{|y|^{-1}} |xu| |uy|^3 (Q(u^{-1}))^{-2} \, du &= |xy^3| \int_{|y|}^{\infty} \frac{dv}{v^6 Q^2(v)} \\ &\le \frac{|x|}{|y| Q^2(|y|)} \int_{|y|}^{\infty} \frac{dv}{v^2} = \frac{|x|}{y^2 Q^2(|y|)}.\end{aligned}$$

On the middle term we will need to use both (2.2) and (2.3); there will be no harm in assuming that $|x| \ge 1$ since the lemma is trivial for $x = 0$. Thus we have for $|x| \le v \le |y|$

$$v^4 Q^2(v) = (v^{\lambda} Q(v))^{\lambda} (v^2 Q(v))^{2-\lambda} v^{\lambda(2-\lambda)} \ge (c|y|^{\lambda} Q(|y|))^{\lambda} (x^2 Q(|x|))^{2-\lambda} v^{\lambda(2-\lambda)}$$

and then

$$\int_{|y|^{-1}}^{|x|^{-1}} |xu||uy|(Q(u^{-1}))^{-2}\,du = |xy|\int_{|x|}^{|y|}\frac{dv}{v^4Q^2(v)}$$

$$\leq (c^{\lambda}|x|^{3-2\lambda}(Q(|x|))^{2-\lambda}|y|^{\lambda^2-1}(Q(|y|))^{\lambda})^{-1}\int_{|x|}^{|y|} v^{-\lambda(2-\lambda)}\,dv$$

$$\leq c^{-\lambda}(\lambda-1)^{-2}(|x|^{3-2\lambda}(Q(|x|))^{2-\lambda}|y|^{2\lambda-2}(Q(|y|))^{\lambda})^{-1}.$$

The final term will not be needed if $|x| < \delta^{-1}$; otherwise

$$\int_{|x|^{-1}}^{\delta} |uy|(Q(u^{-1}))^{-2}\,du \leq |y|\int_{x_0}^{|x|}\frac{dv}{v^3Q^2(v)} \leq \frac{|y|}{|x|^{2\lambda}Q^2(|x|)}\int_{x_0}^{|x|}\frac{dv}{v^{3-2\lambda}}$$

$$\leq \frac{1}{2(\lambda-1)}\,\frac{|y|}{x^2Q^2(|x|)}.$$

Now we have three terms to integrate with respect to $dF(y)$ over the set $|y| > |x|$. For the first two we introduce the monotone function $H(x) = \int_{|y|\leq x} y^2\,dF(y) = x^2K(x)$. Then by Lemma 5, we have for large $|x|$

$$|x|\int_{|y|>|x|}\frac{1}{y^2Q^2(|y|)}\,dF(y) = |x|\int_{(|x|,\infty)}\frac{dH(y)}{y^4Q^2(y)} \leq |x|\int_{(|x|,\infty)}\frac{dH(y)}{H^2(y)}$$

$$\leq \frac{|x|}{H(|x|)} = \frac{1}{|x|K(|x|)} \leq C\frac{1}{|x|Q(|x|)},$$

where we have used (A_1) at the last step. Of course, this also shows that the integral converges and this is enough for small $|x|$. Similarly

$$|x|^{2\lambda-3}(Q(|x|))^{\lambda-2}\int_{|y|>|x|}\frac{dF(y)}{|y|^{2\lambda-2}(Q(|y|))^{\lambda}}$$

$$\leq |x|^{2\lambda-3}(Q(|x|))^{\lambda-2}\int_{(|x|,\infty)}\frac{dH(y)}{H^{\lambda}(y)} \leq \frac{|x|^{2\lambda-3}(Q(|x|))^{\lambda-2}}{(\lambda-1)H^{\lambda-1}(|x|)}$$

$$= \frac{|x|^{-1}(Q(|x|))^{\lambda-2}}{(\lambda-1)K^{\lambda-1}(|x|)} \leq C\frac{1}{|x|Q(|x|)}.$$

For the final term, we have using an integration by parts (see, e.g. Lemma 2.2 in

[12])

$$\frac{1}{x^2Q^2(|x|)}\int_{|y|>|x|}|y|\,dF(y)=\frac{1}{x^2Q^2(|x|)}\left(\int_{|x|}^{\infty}G(y)\,dy+|x|G(|x|)\right)$$

$$\leq\frac{1}{x^2Q^2(|x|)}\left(\int_{|x|}^{\infty}y^{\lambda}Q(y)y^{-\lambda}\,dy+|x|Q(|x|)\right)$$

$$\leq\left(\frac{1}{\lambda-1}+1\right)\frac{1}{|x|Q(|x|)}.$$

Thus we have completed the proof for the terms arising from case (1).

CASE (2). In this case $|y|^{-1}\geq|x|^{-1}$. The first integral will be

$$\int_0^{|x|^{-1}}|xu|\,|uy|^3(Q(u^{-1}))^{-2}\,du=|xy^3|\int_{|x|}^{\infty}\frac{dv}{v^6Q^2(v)}$$

$$\leq\frac{|y|^3}{|x|^3Q^2(|x|)}\int_{|x|}^{\infty}\frac{dv}{v^2}=\frac{|y|^3}{x^4Q^2(|x|)}.$$

For the middle integral we introduce $\eta=(\lambda-1)/(2-\lambda)$. Then for $x_0\leq|y|\leq v\leq|x|$ (we assume here that $\lambda<3/2$ so $0<\eta<1$)

$$v^5Q^2(v)=(v^{\lambda}Q(v))^{1+\eta}(v^2Q(v))^{1-\eta}v^2\geq(|x|^{\lambda}Q(|x|))^{1+\eta}(y^2Q(|y|))^{1-\eta}v^2$$

and then for $|y|\geq x_0$

$$\int_{|x|^{-1}}^{|y|^{-1}}|uy|^3(Q(u^{-1}))^{-2}\,du=|y|^3\int_{|y|}^{|x|}\frac{dv}{v^5Q^2(v)}$$

$$\leq(|x|^{\lambda(1+\eta)}(Q(|x|))^{1+\eta}|y|^{-2\eta}(Q(|y|))^{1-\eta})^{-1}.$$

Finally,

$$\int_{|y|^{-1}}^{\delta}|uy|(Q(u^{-1}))^{-2}\,du\leq|y|\int_{x_0}^{|y|}\frac{dv}{v^3Q^2(v)}\leq\frac{|y|}{|y|^{2\lambda}Q^2(|y|)}\int_{x_0}^{|y|}\frac{dv}{v^{3-2\lambda}}$$

$$\leq\frac{1}{2(\lambda-1)}\frac{1}{|y|Q^2(|y|)}.$$

If $|y|<x_0$, this last term will not be needed since then $|y|^{-1}>\delta$. Now we must carry out the integration with respect to $dF(y)$ over $|y| \le |x|$. The first term is

$$\int_{|y|\le|x|} \frac{|y|^3}{x^4Q^2(|x|)}\,dF(y) \le \frac{1}{|x|^3Q^2(|x|)}\int_{|y|\le|x|} y^2\,dF(y)$$

$$= \frac{K(|x|)}{|x|\,Q^2(|x|)} \le \frac{1}{|x|\,Q(|x|)}.$$

For the middle term we use the function H and Lemma 5:

$$|x|^{-\lambda(1+\eta)}(Q(|x|))^{-(1+\eta)}\int_{|y|\le|x|} |y|^{2\eta}(Q(|y|))^{-1+\eta}\,dF(y)$$

$$\le |x|^{-\lambda(1+\eta)}(Q(|x|))^{-(1+\eta)}\int_{(0,|x|]} \frac{dH(y)}{H^{1-\eta}(y)}$$

$$\le \frac{1}{\eta}\frac{H^{\eta}(|x|)}{|x|^{\lambda(1+\eta)}(Q(|x|))^{1+\eta}} \le \frac{1}{\eta}\frac{1}{|x|\,Q(|x|)}.$$

The final term is only needed if $|y| \ge x_0$. In that case we will use that for $x_0 \le y \le |x|$

$$y^3Q^2(y) = (y^{\lambda}Q(y))^{1+\eta}(y^2Q(y))^{1-\eta} \ge (|x|^{\lambda}Q(|x|))^{1+\eta}(y^2Q(y))^{1-\eta}$$

and then

$$\int_{|y|\le|x|} \frac{1}{|y|Q^2(|y|)}\,dF(y) = \int_{(0,|x|]} \frac{dH(y)}{y^3Q^2(y)}$$

$$\le |x|^{-\lambda(1+\eta)}(Q(|x|))^{-(1+\eta)}\int_{(0,|x|]} \frac{dH(y)}{H^{1-\eta}(y)}$$

$$\le \frac{1}{\eta}\frac{1}{|x|\,Q(|x|)}$$

as above. This completes the proof of the lemma.

COROLLARY. *There is a positive C such that*

$$\sum_{n=0}^{\infty} \left| \sum_{k=0}^{p-1} (P\{S_{np+k} = z\} - P\{S_{np+k} = x + z\}) \right| \le C\frac{1}{|x|\,Q(|x|)}.$$

REMARK. Note that for $x \notin p\mathbb{Z}$, if one sums the absolute differences as in Lemma 7, the series will diverge since there is no cancellation.

PROOF. We take $\ell \in [0, p-1]$ so that the random walk visits the coset containing z at times congruent to $\ell \pmod p$, and without loss of generality we assume it visits the coset containing $x+z$ at times congruent to $\ell + j \pmod p$ with $\ell + j \in [0, p-1]$ and $j \geq 0$. Then

$$\begin{aligned}
&\textstyle\sum_{n=0}^{\infty} \left| \sum_{k=0}^{p-1} (P\{S_{np+k} = z\} - P\{S_{np+k} = x+z\}) \right| \\
&\quad = \textstyle\sum_{n=0}^{\infty} | P\{S_{np+\ell} = z\} - P\{S_{np+\ell+j} = x+z\} | \\
&\quad = \textstyle\sum_{n=0}^{\infty} | \sum_w P\{S_j = w\}(P\{S_{np+\ell} = z\} - P\{S_{np+\ell} = x+z-w\}) | \\
&\quad \leq \textstyle\sum_w P\{S_j = w\} \sum_{n=0}^{\infty} | P\{S_{np+\ell} = z\} - P\{S_{np+\ell} = x+z-w\} | \\
&\quad \leq \textstyle\sum_{w \neq x} P\{S_j = w\} \dfrac{C}{|x-w| Q(|x-w|)}
\end{aligned}$$

by Lemma 7 since $P\{S_j = w\} > 0$ only if $x - w \in p\mathbb{Z}$. Now we break the sum into two parts; for $|x-w| \leq |x|$ we use the bound $|x-w| Q(|x-w|) \geq c|x| Q(|x|)$ which is a consequence of (2.2) and for $|x-w| \geq |x|$ we use $|x-w|^2 Q(|x-w|) \geq |x|^2 Q(|x|)$ which follows from (2.3). This leads to the bound

$$\textstyle\sum_{\{w: 0<|w-x| \leq |x|\}} P\{S_j = w\} \dfrac{C}{|x| Q(|x|)} + \sum_{\{w: |w-x| > |x|\}} P\{S_j = w\} \dfrac{C|x-w|}{|x|^2 Q(|x|)}.$$

The first sum clearly leads to the desired bound; for the second we use

$$\begin{aligned}
\textstyle\sum_w |x-w| P\{S_j = w\} &\leq \textstyle\sum_w |P\{S_j = w\} + \sum_w |x| P\{S_j = w\} \\
&= E|S_j| + |x| \leq pE|X| + |x| \leq C|x|.
\end{aligned}$$

Now we are ready to derive the probability estimates for the upper tail of the distribution of $c_n n^{-1} L_n(x)$. These will be given in the next five lemmas.

LEMMA 8. *There is a C such that*

$$P\left\{\frac{c_n}{n} L_n(x) \geq Ce^{\gamma}\right\} \leq \frac{e^{\gamma}}{(\log n)^{\gamma}} \quad \textit{for all} \quad x, n, \gamma.$$

PROOF. We start with

$$\begin{aligned}
E(L_n(x))^r &= E \textstyle\sum_{j_1, \ldots, j_r = 0}^{n-1} 1(S_{j_1} = x) \cdots 1(S_{j_r} = x) \\
&\leq r! E \textstyle\sum_{0 \leq j_1 \leq \cdots \leq j_r < n} 1(S_{j_1} = x) \cdots 1(S_{j_r} = x) \\
&\leq r! \textstyle\sum_{k \in I(n,r)} P\{S_{k_1} = x\} P\{S_{k_2} = 0\} \cdots P\{S_{k_r} = 0\}
\end{aligned}$$

where $I(n, r)$ is defined in Lemma 6 and we have made the substitution $k_i = j_i - j_{i-1}$. Now Lemmas 2 and 6 yield (with m as in Lemma 6)

$$E(L_n(x))^r \leq r!(C_1 m a_m^{-1})^r. \tag{2.11}$$

Now we take $r = [\log \log n]$ which makes $a_m \geq c_n$ so that

$$E\left(\frac{c_n}{n} L_n(x)\right)^r \leq r! \left(\frac{c_n}{n}\right)^r (C_1 m a_m^{-1})^r \leq C^r$$

and so

$$P\left\{\frac{c_n}{n} L_n(x) \geq Ce^{\gamma}\right\} \leq C^{-r}e^{-\gamma r}E\left(\frac{c_n}{n} L_n(x)\right)^r \leq e^{-\gamma r} \leq e^{\gamma}(\log n)^{-\gamma}.$$

LEMMA 9. *There is a C such that if* $|x - y| \leq \eta c_n$, *then with* λ *as in* (2.2),

$$P\left\{\frac{c_n}{n} |L_n(x) - L_n(y)| \geq C\eta^{(\lambda-1)/4}\right\} \leq \eta^{-(\lambda-1)/2}\left(\frac{1}{\log n}\right)^{(\lambda-1)2^{-1}\log\eta^{-1}}$$

for all n, $0 < \eta < 1$.

PROOF. We first give the proof assuming that x, y are in the same coset and then explain the modifications necessary if x and y are in different cosets at the end. Fix x and y and let $\psi(z) = 1_{\{x\}}(z) - 1_{\{y\}}(z)$. Then we have

$$E(L_n(x) - L_n(y))^{2r} = \sum_{j_1,\ldots,j_{2r}=0}^{n-1} E\psi(S_{j_1}) \cdots \psi(S_{j_{2r}}).$$

We will first estimate $E\psi(S_{j_1}) \cdots \psi(S_{j_{2r}})$. Without loss of generality, we may assume that $j_1 \leq j_2 \leq \cdots \leq j_{2r}$ and then we will let $k_i = j_i - j_{i-1}$. This leads to

$$E\psi(S_{j_1}) \cdots \psi(S_{j_{2r}}) = \sum_{z_1,\ldots,z_{2r}} \prod_{i=1}^{2r} \psi(z_i)P\{S_{k_i} = z_i - z_{i-1}\}$$

with $z_0 = 0$. Now we carry out the sum over the z's having even subscripts. This is easy since each z is only involved in two consecutive factors. We obtain

$$E\psi(S_{j_1}) \cdots \psi(S_{j_{2r}}) = \sum_{z_1,z_3,\ldots,z_{2r-1}} \prod_{i=1}^{r} \psi(z_{2i-1})P\{S_{k_1} = z_1\}\prod_{i=1}^{r} h(i, z_{2i-1}, z_{2i+i})$$

where for $1 \leq i \leq r - 1$

$$h(i, u, v) = P\{S_{k_{2i}} = x - u\}P\{S_{k_{2i+1}} = v - x\} - P\{S_{k_{2i}} = y - u\}P\{S_{k_{2i+1}} = v - y\}$$

and

$$h(r, u, v) = P\{S_{k_{2r}} = x - u\} - P\{S_{k_{2r}} = y - u\}.$$

At this point we have obtained the essential cancellation and we may put on absolute values. For $i < r$, we have by Lemma 2

$$\begin{aligned} |h(i, u, v)| &\leq P\{S_{k_{2i}} = x - u\} \, |P\{S_{k_{2i+1}} = v - x\} - P\{S_{k_{2i+1}} = v - y\}| \\ &\quad + P\{S_{k_{2i+1}} = v - y\} \, |P\{S_{k_{2i}} = x - u\} - P\{S_{k_{2i}} = y - u\}| \\ &\leq \frac{C}{a_{k_{2i}}} |P\{S_{k_{2i+1}} = v - x\} - P\{S_{k_{2i+1}} = v - y\}| \\ &\quad + \frac{C}{a_{k_{2i+1}}} |P\{S_{k_{2i}} = x - u\} - P\{S_{k_{2i}} = y - u\}|. \end{aligned}$$

Thus

$$E(L_n(x) - L_n(y))^{2r} \le (2r)! \sum_{k\in I(n,2r)} |E\psi(S_{j_1}) \cdots \psi(S_{j_{2r}})|$$
$$\le (2r)! \sum_{k\in I(n,2r)} \sum_{z_1,z_3,\dots,z_{2r-1}} \prod_{i=1}^{r} |\psi(z_{2i-1})| C a_{k_1}^{-1}$$
$$\cdot \prod_{i=1}^{r-1} \left(\frac{C}{a_{k_{2i}}} |P\{S_{k_{2i+1}} = z_{2i+1} - x\} - P\{S_{k_{2i+1}} = z_{2i+1} - y\}| \right.$$
$$\left. + \frac{C}{a_{k_{2i+1}}} |P\{S_{k_{2i}} = x - z_{2i-1}\} - P\{S_{k_{2i}} = y - z_{2i-1}\}| \right)$$
$$\cdot |P\{S_{k_{2r}} = x - z_{2r-1}\} - P\{S_{k_{2r}} = y - z_{2r-1}\}|.$$

Now we are going to sum next over $k \in I(n, 2r)$. Since $I(n, 2r)$ is defined symmetrically in terms of k_{2i}, k_{2i+1} and these are only involved in one factor in the product we may interchange k_{2i}, k_{2i+1} in the first terms of each factor so that the two terms will each contain $a_{k_{2i+1}}^{-1}$. Now we sum over the k_i's having even subscripts and increase the domain of summation by letting these go all the way to infinity while still requiring $k_1 + k_3 + \cdots + k_{2r-1} < n$. Finally, we relabel the remaining k's as $k_1, \cdots, k_r$ to simplify the notation. We obtain

$$E(L_n(x) - L_n(y))^{2r} \le (2r)! \sum_{z_1,z_3,\dots,z_{2r-1}} \prod_{i=1}^{r} |\psi(z_{2i-1})| \sum_{k\in I(n,r)} C^r \prod_{i=1}^{r} a_{k_i}^{-1}$$
$$\cdot\, 2^{r-1} C^r \left(\frac{1}{|x-y| Q(|x-y|)} \right)^r$$
$$\le (2r)! C^r (m a_m^{-1})^r (|x-y| Q(|x-y|))^{-r},$$

where we have used Lemma 6 and the fact that there are only 2^r values of $(z_1, z_3, \cdots, z_{2r-1})$ that give a non-zero summand. Now since $|x - y| \le \eta c_n$ we have by (2.2) that

$$|x-y|^{\lambda} Q(|x-y|) \ge c c_n^{\lambda} Q(c_n) \quad \text{so that}$$
$$|x-y| Q(|x-y|) \ge c \eta^{-(\lambda-1)} c_n n^{-1} \log\log n.$$

Using this in the above bound and again letting $r = [\log \log n]$ leads to

$$E\left(\frac{c_n}{n} (L_n(x) - L_n(y)) \right)^{2r} \le C^r (2r)^{2r} \left(\frac{c_n}{n} \right)^{2r} \left(\frac{m}{a_m} \right)^r \left(\frac{n}{c_n \log\log n} \right)^r \eta^{r(\lambda-1)} \le C^r \eta^{r(\lambda-1)}.$$

We now use Markov's inequality as above to complete the proof.

If x, y are in different cosets, we proceed as before except that before putting on the absolute values we must sum over all the k_i within blocks of length p. Thus, for example, if we sum the term $P\{S_{k_{2i}} = x - u\} P\{S_{k_{2i+1}} = v - x\}$ that appears in $h(i, u, v)$ over a block of length p for both k_{2i} and k_{2i+1} we will obtain only one non-zero term. Then we put on absolute values and continue as before. When we sum over k_{2i}, instead of having

$$\sum_{k_{2i}=0}^{\infty} |P\{S_{k_{2i}} = x - u\} - P\{S_{k_{2i}} = y - u\}|,$$

for example, which would diverge, we will have

$$\sum_{r_{2i}=0}^{\infty} \left| \sum_{s=0}^{p-1} (P\{S_{r_{2i}p+s} = x - u\} - P\{S_{r_{2i}p+s} = y - u\}) \right|$$

and we may use the corollary to Lemma 7 to estimate this. Since the last block in the sum may be incomplete, however, one of the non-zero terms may be missing from that block; we will bound the sum over the incomplete block by one. Thus we are led to a bound for the sum over k_{2i} of

$$1 + \frac{C}{|x-y|Q(|x-y|)} \le \frac{C_1}{|x-y|Q(|x-y|)}$$

since $uQ(u)$ is bounded above for $|u| \ge 1$ by (2.2). Since this is the same bound as in the original proof (with a different constant) the remainder of the proof is as before.

Lemma 10. *There is a C such that if $|x - y| \le \eta c_n$, then with λ as in* (2.2),

$$P\left\{\max_{n\le k\le 2n} \frac{c_k}{k} |L_k(x) - L_k(y)| \ge C\eta^{(\lambda-1)/4}\right\} \le C\eta^{-(\lambda-1)/2}\left(\frac{1}{\log n}\right)^{(\lambda-1)2^{-1}\log\eta^{-1}}$$

for all n, $0 < \eta < ½$.

Proof. The proof is as for Lévy's inequality. First choose n_0 so that the probability estimate in Lemma 9 is at most ½ for all $\eta \in (0, ½)$ if $n \ge n_0$ ($n_0 = \exp(\exp(1 + 2/(\lambda - 1)))$ will suffice) and so that $c_n n^{-1}\downarrow$ for $n \ge n_0$ (to see this, $xQ(x)\downarrow$ for large x by (2.1); use this with $x = c_n$). Next, with C as in Lemma 9, for $n \le k \le 2n$; let

$$\Lambda_k = \left\{\frac{c_k}{k}|L_k(x) - L_k(y)| \ge 3C\eta^{(\lambda-1)/4}\right\}, \quad \Gamma_k = \Lambda_n^c\Lambda_{n+1}^c \cdots \Lambda_{k-1}^c\Lambda_k.$$

By Lemma 9, we have with $\psi(z) = 1_{\{x\}}(z) - 1_{\{y\}}(z)$ and $k \le 2n - n_0$

$$\begin{aligned} P&\left\{\Gamma_k, \frac{c_k}{k}\left|\sum_{j=k}^{2n-1} \psi(S_j)\right| \ge C\eta^{(\lambda-1)/4}\right\} \\ &= \Sigma_\xi P\{\Gamma_k, S_k = \xi\}P\left\{\frac{c_k}{k}\left|\sum_{j=0}^{2n-k-1} \psi(S_j + \xi)\right| \ge C\eta^{(\lambda-1)/4}\right\} \\ &\le ½ \sum_\xi P\{\Gamma_k, S_k = \xi\} = ½\, P(\Gamma_k), \end{aligned}$$

since $k \ge n$ implies that $2n - k \le k$ so that $c_{2n-k}(2n - k)^{-1} \ge c_k k^{-1}$. Thus, letting $m = 2n - n_0$

$$\begin{aligned} P(\cup_{k=n}^m \Gamma_k) &= \sum_{k=n}^m P(\Gamma_k) \le 2\sum_{k=n}^m P\left\{\Gamma_k, \frac{c_k}{k}\left|\sum_{j=k}^{2n-1} \psi(S_j)\right| < C\eta^{(\lambda-1)/4}\right\} \\ &= 2P\left(\cup_{k=n}^m\left\{\Gamma_k, \frac{c_k}{k}\left|\sum_{j=k}^{2n-1} \psi(S_j)\right| < C\eta^{(\lambda-1)/4}\right\}\right) \\ &\le 2P\left\{\frac{c_n}{n}|L_{2n}(x) - L_{2n}(y)| \ge 2C\eta^{(\lambda-1)/4}\right\}. \end{aligned}$$

Finally, by (2.3)

$$c_{2n}^2 Q(c_{2n}) \geq c_n^2 Q(c_n) \quad \text{and so} \quad \frac{c_n}{n} \leq \left(2\frac{\log\log 2n}{\log\log n}\right)^{1/2} \frac{c_{2n}}{2n}$$

so that $c_n n^{-1} \leq 2c_{2n}(2n)^{-1}$ for large n. Thus we can apply Lemma 9 to obtain the required bound for $P(\cup_{k=n}^{m} \Gamma_k)$. For the remaining n_0 values of k, we simply use n_0 times the bound in Lemma 9.

LEMMA 11. *Given* $\varepsilon > 0$, *there is a* $\delta > 0$ *and a* C *such that*

$$P\left\{\max_{n\leq k\leq 2n} \sup_{|x-y|\leq \delta c_n} \frac{c_k}{k} |L_k(x) - L_k(y)| \geq \varepsilon\right\} \leq \frac{C}{\log^2 n} \quad \textit{for all} \quad n.$$

PROOF. By Lemma 3, we have

$$P\{L_{2n}(x) \neq 0 \quad \text{for some} \quad |x| \geq (\log^2 n) a_{2n}\} \leq \frac{C}{\log^{2\lambda} n}.$$

Thus we may restrict our attention to those x, y in the interval $[-\gamma_n, \gamma_n]$ where $\gamma_n = (\log^2 n) a_{2n} + c_n$ since we will take $\delta < 1$. Now we will split the interval $[-\gamma_n, \gamma_n]$ into shorter intervals of length δc_n. Then any pair x, y with $|x - y| \leq \delta c_n$, x, $y \in [-\gamma_n, \gamma_n]$, will either be in a common interval or in adjacent intervals. Since the probability to be estimated is monotone in δ, there is no loss of generality in assuming that $\delta c_n = 2^j$ for some $j \in \mathbb{Z}$. Then, if

$$i\delta c_n < x \leq (i+1)\delta c_n < y \leq (i+2)\delta c_n$$

we will use

$$|L_k(x) - L_k(y)| \leq |L_k(x) - L_k(i\delta c_n)| + |L_k((i+1)\delta c_n) - L_k(i\delta c_n)| + |L_k(y) - L_k((i+1)\delta c_n)|.$$

Therefore, it will suffice to estimate

$$P\left\{\max_{n\leq k\leq 2n} \max_{0<x\leq \delta c_n} \frac{c_k}{k} |L_k(0) - L_k(x)| \geq \varepsilon\right\}$$

and multiply by the number of short intervals since the estimate will be independent of the location of the interval. Now if $0 < x \leq \delta c_n = 2^j$ we may write $x = \sum_{i=0}^{j} \chi_i 2^i$ where each $\chi_i = 0$ or 1. Then if $x_m = \sum_{i=m}^{j} \chi_i 2^i$ we write

$$L_k(0) - L_k(x) = \sum_{m=0}^{j} (L_k(x_{m+1}) - L_k(x_m))$$

where $x_{j+1} = 0$. Now if we let x vary we see that there is only one increment possible for $m = j$ and 2^{j-m-1} possible ones that may arise for $m < j$ depending on the location of x. Thus, by Lemma 10,

$$P\left\{\max_{n\le k\le 2n}\max_{0<x\le\delta c_n}\frac{c_k}{k}|L_k(0)-L_k(x)|\ge\varepsilon\right\}$$

$$\le \sum_{m=0}^{j}2^{j-m}P\left\{\max_{n\le k\le 2n}\frac{c_k}{k}|L_k(0)-L_k(2^m)|\ge\varepsilon\,\gamma^{j-m}(1-\gamma)\right\} \tag{2.12}$$

$$\le \sum_{m=0}^{j}2^{j-m}C(2^{m-j}\delta)^{-(\lambda-1)/2}\left(\frac{1}{\log n}\right)^{(\lambda-1)2^{-1}\log(2^{j-m}\delta^{-1})}$$

if we take $\eta = 2^{m-j}\delta$ and let $\gamma = 2^{-(\lambda-1)/4}$ and choose $\delta < ½$ (which makes $\eta < ½$) and also so that $C\delta^{(\lambda-1)/4} \le \varepsilon(1-\gamma)$ where C is the constant in Lemma 10. Collecting the terms in 2^{j-m} leads to

$$2^{(j-m)(1+(\lambda-1)(1-\log n\log n)/2)}$$

so that for n sufficiently large this will give a convergent geometric series in m. Thus the probability in (2.12) is bounded by

$$C\delta^{-(\lambda-1)/2}\left(\frac{1}{\log n}\right)^{(\lambda-1)2^{-1}\log\delta^{-1}}.$$

Now δ may be chosen to give any desired positive power on the $1/\log n$ term. It remains to estimate the number of short intervals. For this we have

$$2\left(\frac{\gamma_n}{\delta c_n}+1\right)=O\left(\frac{(\log^2 n)a_{2n}}{c_n}\right),$$

and by (2.4) for large n

$$a_{2n}\le(2\log\log n)^{1/\lambda}c_n. \tag{2.13}$$

This is sufficient since we could choose δ above to make the power on the $1/\log n$ term equal to 5 in the probability estimate.

The final lemma will give a lower bound for the tail of the distribution of L_n to complement the upper bound obtained in Lemma 8.

LEMMA 12. *For any $\eta > 0$, there is a positive c such that*

$$P\left\{\frac{c_n}{n}L_n(x)\ge c\right\}\ge\left(\frac{1}{\log n}\right)^{\eta},\quad |x|\le a_n,$$

for all sufficiently large n.

PROOF. Let $m = \gamma n/\log\log n$ where γ will be chosen later but we note that there is no harm is choosing it so that m will be an integer. We will use the inequality

$$P\{Z\ge\varepsilon EZ\}\ge(1-\varepsilon)^2\frac{(EZ)^2}{EZ^2} \tag{2.14}$$

which holds for any nonnegative random variable Z (see page 6 of [10].) Now for $|z| \leq 2a_m$, we have for m sufficiently large

$$EL_m(z) = \textstyle\sum_{j=0}^{m-1} P\{S_j = z\} \geq \sum_{j=m/2}^{m-1} P\{S_j = z\} \geq cma_m^{-1}$$

by Lemma 2 and (2.4). Thus taking $\varepsilon = ½$ and using (2.11) with $r = 2$ in (2.14) leads to

$$P\left\{L_m(z) \geq c\frac{m}{a_m}\right\} \geq c \quad \text{for all} \quad |z| \leq 2a_m.$$

Now, by Lemma 3, we may choose M so that

$$P\left\{L_m(z) \geq c\frac{m}{a_m}, |S_m| \leq Ma_m\right\} \geq c \quad \text{for all} \quad |z| \leq 2a_m.$$

Using Lemma 2 again, we have

$$P\left\{L_m(z-y) \geq c\frac{m}{a_m}, |S_{2m} + y| \leq a_m\right\} \geq c \quad \text{for all} \quad |z| \leq a_m, |y| \leq a_m.$$

Iterating this i times leads to

$$P\left\{L_{2im}(z) \geq c\frac{im}{a_m}\right\} \geq c^i \quad \text{for all} \quad |z| \leq a_m.$$

Now we use Lemma 2 one more time with $\ell = [n/2]$ to obtain

$$P\{|S_\ell - x| \leq a_m\} \geq c\frac{a_m}{a_n} \quad \text{for all} \quad |x| \leq a_n.$$

Now $ca_m a_n^{-1} \geq c_1(\log\log n)^{-1/\lambda}$ as in (2.13). Thus we have

$$P\left\{L_{\ell+2im}(x) \geq c\frac{im}{a_m}\right\} \geq c_1 c^i(\log\log n)^{-1/\lambda} \quad \text{for all} \quad |x| \leq a_n. \tag{2.15}$$

We take $i = [\log\log n/4\gamma]$ so that $\ell + 2im \leq n$, and for $\gamma \geq 1$ and n large

$$\frac{im}{a_m} \geq \frac{n}{4}\left(1 - \frac{4\gamma}{\log\log n}\right)\cdot\frac{1}{c_n\gamma^{1/\lambda}} \geq c_2\gamma^{-1/\lambda}\frac{n}{c_n},$$

and since c is independent of γ we have

$$c^i = e^{i\log c} \geq e^{(\log\log n)(\log c)/4\gamma} = \log n^{(\log c)/4\gamma}.$$

Thus by choosing γ large enough so that $(\log c)/4\gamma > -\eta$, we see that (2.15) is sufficient to give the lemma.

3. The main results. We are now ready to prove the theorems stated in the introduction. The proofs are quite easy now that we have the required probability estimates.

PROOF OF THEOREM 1. Let $n_k = 2^k$ and choose $\gamma > 1$. By Lemma 8 we have

$$\limsup_k \frac{c_{n_k}}{n_k} L_{n_k}(x) \leq Ce^{\gamma}.$$

We may easily extend this result to the entire sequence by increasing the constant since $L_n(x)$ is monotone in n and since

$$c_{2n} \geq c_n \geq 2^{-1/\lambda} c_{2n} \tag{3.1}$$

for large n by (2.1).

For the lower bound we use $n_k = [\exp(k^\rho)]$ for some $\rho > 2$, and let $m_k = n_k - n_{k-1}$ and

$$\Lambda_k = \left\{ \frac{c_{m_k}}{m_k} \sum_{j=n_{k-1}}^{n_k-1} 1_{\{x\}}(S_j) \geq c \right\}$$

where c is as in Lemma 12 with $\eta = \rho^{-1}$. Then for k large

$$\begin{aligned} P(\Lambda_k) &\geq \textstyle\sum_y P\{S_{n_{k-1}} = y\} P\left\{ \frac{c_{m_k}}{m_k} L_{m_k}(x-y) \geq c \right\} \\ &\geq \left(\frac{1}{\log m_k} \right)^\eta P\{|S_{n_{k-1}} - x| \leq a_{m_k}\} \geq \frac{c}{k} \end{aligned} \tag{3.2}$$

by Lemmas 12 and 3. Next we fix $M > 1$, suppose that $k < \ell - 1$ and let

$$\Gamma_y = \Gamma_{k,\ell,y} = \left\{ \frac{c_{m_\ell}}{m_\ell} \sum_{j=n_{\ell-1}-n_k}^{n_\ell-n_k-1} 1_{\{x\}}(S_j + y) \geq c \right\}.$$

Then we have

$$\begin{aligned} &P(\Lambda_k \Lambda_\ell) \\ &= \textstyle\sum_y P(\Lambda_k, S_{n_k} = y) P(\Gamma_y) \leq \sum_{|y| \leq M^\ell a_{n_k}} P(\Lambda_k, S_{n_k} = y) P(\Gamma_y) + CM^{-\ell\lambda} \end{aligned} \tag{3.3}$$

where we have used Lemma 3 to estimate $P\{|S_{n_k}| \geq M^\ell a_{n_k}\}$. Now we choose y_1 and y_2 to maximize and minimize $P(\Gamma_y)$ for $|y| \leq M^\ell a_{n_k}$ and y in the coset for which $(S_{n_k} = y) > 0$:

$$P(\Gamma_{y_2}) \leq P(\Gamma_y) \leq P(\Gamma_{y_1}) \quad \text{for all} \quad |y| \leq M^\ell a_{n_k} \quad \text{such that} \quad P(S_{n_k} = y) > 0.$$

Then

$$\begin{aligned} &P(\Gamma_{y_1}) - P(\Gamma_{y_2}) \\ &\quad = \textstyle\sum_z (P\{S_{n_{\ell-1}-n_k} = z - y_1\} \\ &\quad\quad - P\{S_{n_{\ell-1}-n_k} = z - y_2\}) P\left\{ \frac{c_{m_\ell}}{m_\ell} L_{m_\ell}(x-z) \geq c \right\} \\ &\quad \leq \textstyle\sum_{|z| \leq 2M^\ell a_{n_{\ell-1}-n_k}} |P\{S_{n_{\ell-1}-n_k} = z - y_1\} \\ &\quad\quad - P\{S_{n_{\ell-1}-n_k} = z - y_2\}| + 2CM^{-\ell\lambda}, \end{aligned} \tag{3.4}$$

where we have used Lemma 3 and the fact that $|z| \geq 2M^{\ell}a_{n_{\ell-1}-n_k}$ implies that

$$|z - y_i| \geq 2M^{\ell}a_{n_{\ell-1}-n_k} - M^{\ell}a_{n_k} \geq M^{\ell}a_{n_{\ell-1}-n_k}$$

since $2n_k \leq 2n_{\ell-2} \leq n_{\ell-1}$ for $\ell \geq 2$. Now we estimate the sum in (3.4) by using Lemma 4. Since y_1, y_2 are in the same coset, $y_1 - y_2 \in p\mathbb{Z}$, and

$$P(\Gamma_{y_1}) - P(\Gamma_{y_2}) \leq (4M^{\ell}a_{n_{\ell-1}-n_k} + 1)C\frac{|y_1 - y_2|}{a^2_{n_{\ell-1}-n_k}} + 2CM^{-\ell\lambda} \leq CM^{2\ell}\frac{a_{n_k}}{a_{n_{\ell-1}-n_k}} + 2CM^{-\ell\lambda}.$$

Now we need to estimate $P(\Gamma_{y_2})$. We have

$$P(\Lambda_\ell) = \textstyle\sum_y P\{S_{n_k} = y\}P(\Gamma_y) \geq \sum_{|y| \leq M^{\ell}a_{n_k}} P\{S_{n_k} = y\}P(\Gamma_{y_2})$$

$$\geq P\{|S_{n_k}| \leq a_{n_k}\}P(\Gamma_{y_2}) \geq cP(\Gamma_{y_2}).$$

Putting this in the previous estimate leads to

$$P(\Gamma_{y_1}) \leq CP(\Lambda_\ell) + CM^{2\ell}\frac{a_{n_k}}{a_{n_{\ell-1}-n_k}} + CM^{-\ell\lambda}.$$

Now, recalling (3.3) we have

$$P(\Lambda_k\Lambda_\ell) \leq P(\Lambda_k)P(\Gamma_{y_1}) + CM^{-\ell\lambda} \tag{3.5}$$

$$\leq CP(\Lambda_k)P(\Lambda_\ell) + CM^{2\ell}\frac{a_{n_k}}{a_{n_{\ell-1}-n_k}}P(\Lambda_k) + CM^{-\ell\lambda}.$$

This estimate is good enough to allow us to use a generalized Borel Cantelli Lemma (see page 317 of [13]) because the first term on the right hand side causes no trouble and

$$\textstyle\sum_{\ell \geq k+2} M^{2\ell}\frac{a_{n_k}}{a_{n_{\ell-1}n_k}} \leq C\sum_{\ell \geq k+2} M^{2\ell}\frac{a_{n_k}}{a_{n_{\ell-1}}} \leq CM^{2k+4}\frac{a_{n_k}}{a_{n_{k+1}}} \leq C$$

where we have compared the series to a geometric one since

$$\frac{M^{2\ell+2}}{a_{n_\ell}} \cdot \frac{a_{n_{\ell-1}}}{M^{2\ell}} \leq M^2\left(\frac{n_{\ell-1}}{n_\ell}\right)^{1/2} \to 0$$

and for the final estimate used

$$M^{2k+4}\frac{a_{n_k}}{a_{n_{k+1}}} \leq M^{2k+4}\left(\frac{n_k}{n_{k+1}}\right)^{1/2} \leq 2M^{2k+4}\exp\{(k^{\rho} - (k+1)^{\rho})/2\}$$

$$\leq 2M^{2k+4}\exp\{-\rho k^{\rho-1}/2\} \to 0$$

since $\rho > 2$. Thus the middle term in (3.5) leads to $CP(\Lambda_k)$ when summed on ℓ while the last term will give $CM^{-\lambda(k+2)}$ which is still summable on k. Finally we simply use $P(\Lambda_k\Lambda_{k+1}) \leq P(\Lambda_k)$. Now the sum of the lower estimates for $P(\Lambda_k)$ in (3.2) diverges so we have $P(\Lambda_k \text{ i.o.}) = 1$ by the generalized Borel Cantelli lemma and the Hewitt-Savage zero-one law. To complete the proof we simply observe that

$$\frac{c_{n_k}}{n_k} L_{n_k}(x) \geq \frac{c_{m_k}}{n_k} L_{n_k}(x) \sim \frac{c_{m_k}}{m_k} L_{n_k}(x) \geq \frac{c_{m_k}}{m_k} \sum_{j=n_{k-1}}^{n_k-1} 1_{\{x\}}(S_j).$$

The lim sup is constant a.s. by the Hewitt-Savage zero-one law since $c_k/k \to 0$ as $k \to \infty$. The reason that θ_1 is independent of x is that for fixed x, y

$$\frac{L_n(y)}{L_n(x)} \to 1 \quad \text{a.s.} \tag{3.6}$$

This follows from the strong law of large numbers and the fact that the expected number of visits to y between visits to x is one (see Corollary 2 on page 49 of [1]).

PROOF OF THEOREM 2. Use $n_k = 2^k$ in Lemma 11 after dividing δ by $2^{1/\lambda}$ (recall (3.1)) and Borel Cantelli.

PROOF OF THEOREM 3. The lower bound is a consequence of Theorem 1. The upper bound will follow easily from Theorem 2. Fix $\varepsilon > 0$ and obtain the corresponding δ from Theorem 2. Choose x_n so that $\max_x L_n(x) = L_n(x_n)$. Then by Theorem 2, for large n,

$$c_n = \sum_y \frac{c_n}{n} L_n(y) \geq \sum_{\{y:|y-x_n|\leq\delta c_n\}} \frac{c_n}{n} (L_n(x_n) + L_n(y) - L_n(x_n))$$

$$\geq \left(\frac{c_n}{n} L_n(x_n) - \varepsilon\right)\delta c_n$$

or

$$\frac{c_n}{n} L_n(x_n) \leq \delta^{-1} + \varepsilon$$

which is sufficient.

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