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A LOCAL TIME ANALYSIS OF INTERSECTIONS OF BROWNIAN PATHS IN THE PLANE[1]

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We envision a network of N paths in the plane determined by N independent, two-dimensional Brownian motions $W_i(t)$, $t \geq 0$, $i = 1, 2, \cdots, N$. Our problem is to study the set of "confluences" z in $\mathbb{R}^2$ where all N paths meet and also the set M_0 of N-tuples of times $\mathbf{t} = (t_1, \cdots, t_N)$ at which confluences occur: $M_0 = \{\mathbf{t}: W_1(t_1) = \cdots = W_N(t_N)\}$. The random set M_0 is analyzed by constructing a convenient stochastic process X, which we call "confluent Brownian motion", for which $M_0 = X^{-1}(0)$ and using the theory of occupation densities. The problem of confluences is closely related to that of multiple points for a single process. Some of our work is motivated by Symanzik's use of Brownian local time in quantum field theory.

0. Introduction. A family of $N \geq 2$ spiders moves in the plane according to independent Brownian motions $W_i(t)$, $i = 1, \cdots, N$, each spinning a gossamer trail. Our problem is to study the set of "confluences" $z \in \mathbb{R}^2$ where all N strands meet, and also the set of N-tuples of times $t = (t_1, \cdots, t_N)$ in $\mathbb{R}_+^N$ (i.e. $t_i \geq 0$) at which confluences occur, that is $z = W_1(t_1) = \cdots = W_N(t_N)$. Notice that only the trails join at a confluence; the spiders themselves need not collide.

S. J. Taylor [18] proved that the set C of confluences has Hausdorff dimension 2, for each N, and R. Wolpert [21] showed that the corresponding set M_0 of times $t \in \mathbb{R}_+^N$ has dimension 1, though we were unable to follow his argument in a few places. Independently of this, following the suggestion by one of us (D.G.), Wolpert constructed a finite random measure on M_0 by considering M_0 as the zero set of the Gaussian random field

$$(0.1) \qquad X_t = (W_1(t_1) - W_2(t_2), \cdots, W_{N-1}(t_{N-1}) - W_N(t_N)),$$

which we will call *confluent Brownian motion* (CBM) in view of the statement of the problem. Thus $M_0 = \{t \in \mathbb{R}_+^N: X_t = 0\}$ and the measure turns out to be the local time or occupation density of X at 0.

We are going to study X_t systematically from the point of view of our paper [11], obtaining detailed information on the level sets $M_x = \{t \in \mathbb{R}_+^N: X_t = x\}$, with the above results as corollaries. What is more important is that CBM serves as a test case for the methods developed in [11]. Perhaps the most interesting aspect is that we have not been able to determine whether CBM and related random fields are locally nondeterministic, and so have had to modify the approach of [11]. The concept of local nondeterminism was introduced by Berman [3] for

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realtime Gaussian processes to account for an unremovable element of noise in the evolution of the process; it was generalized to random fields by Pitt [13], and is central to the discussion in [11], but the intuitive interpretation is less clear in the case of fields. Basically, LND says that the "increments" of a random field are "almost independent", and thus LND, when it is available, emerges as a useful computational device in handling the integrals $J_{k,\gamma}(B)$ introduced in Section 2 below. It turns out, however, that the results of [11] persist without LND: one simply attacks the integrals directly.

The problem of confluences is closely related to that of multiple points for a single process. Indeed, the classical results on multiple points of Brownian motion (e.g. Dvoretzky, et al. [6]) were obtained by looking at confluences of independent segments of the trajectory. The approach in this paper has been applied to multiple points and to confluences for other processes than Brownian motion; see Rosen [14], [15]. These problems also arise in physics; see Symanzik [17], Westwater [20], Wolpert [21].

The main results for CBM and some background material are given in Section 1, with proofs in Section 3; Section 2 contains some general results which modify and extend those of [11]; in Section 4 we consider the confluences of Brownian motions in spatial dimensions other than 2, and, finally, in Section 5 we give detailed results on the Hausdorff dimensions of the level sets of CBM.

While we were writing this paper, S. Orey pointed out to us the fundamental paper of Ehm [8], who considers occupation densities for stable sheets, and we have adapted certain of his arguments to improve some of our original results.

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1. Background and main results. We recall some terminology from [11], to which the reader is referred for details on occupation densities.

Let $X(t)$ be any Borel function of $\mathbb{R}_+^N$ with values in $\mathbb{R}^D$. The *occupation measure* of X relative to a Borel set A in $\mathbb{R}_+^N$ is defined on the Borel σ-field $\mathscr{B}_D$ of $\mathbb{R}^D$ by

$$\mu_A(B) = \lambda_N(A \cap X^{-1}(B)), \quad B \in \mathscr{B}_D, \tag{1.1}$$

where λ_N is Lebesgue measure on $\mathbb{R}^N$. If $\mu_A \ll \lambda_D$, we write $\alpha(x, A) = d\mu_A/d\lambda_D(x)$ (Radon-Nikodym derivative) and call $\alpha(x, A)$ the *occupation density* or local time on A. If there is an occupation density for each A, then we may choose $\alpha(x, A)$ to be a ***kernel,*** i.e. measurable in x and a finite measure in A. The formulation in [11] uses $[0, 1]^N$ instead of $\mathbb{R}_+^N$, but carries over with no essential changes.

The same definitions apply to each trajectory $X_t(\omega)$ of CBM, where $\omega \in \Omega$ (the underlying probability space) is held fixed, or, indeed, to any random field which is jointly measurable in (t, ω). Of course then $\mu_A(B)$ and $\alpha(x, A)$ also depend on ω. In the case of CBM, $D = 2(N - 1)$, and we assume that all the trajectories are continuous. Here are the main results for CBM.

A *rectangle* in $\mathbb{R}_+^N$ is a set $Q_{s,t} = \prod_{i=1}^N [s_i, t_i]$, $0 \le s_i < t_i < \infty$, where $s = (s_1, \cdots, s_N)$, $t = (t_1, \cdots, t_N)$. If $s = 0$, we simply write Q_t; if $Q_{s,t}$ is a cube, we write $e(Q_{s,t})$ for the (common) edge length $t_i - s_i$. Notice that all rectangles are bounded and have their edges parallel to the axes.

THEOREM 1. With probability 1, the occupation density $\alpha(x, A)$ exists for any Borel set A in $\mathbb{R}_+^N$, *and may be chosen so that* $(x, t) \mapsto \alpha(x, Q_t)$ *is jointly continuous.*

THEOREM 2. *The following local Hölder-type condition is valid for* $\alpha(x, A)$: *let* $\tau \in \mathbb{R}_+^N$ *be fixed; there exist a constant* C_1 *not depending on* τ, *and an* a.s. *finite random variable* $\varepsilon_1 = \varepsilon_1(\omega)$, *which depends on* τ, *such that, with probability* 1,

$$\alpha(x, B) \le C_1 (\lambda_N(B))^{1/N} (\lg|\lg \lambda_N(B)|)^N \tag{1.2}$$

for all $x \in \mathbb{R}^D$ *and any cube* B *with lower left corner at* τ *and* $e(B) < \varepsilon_1$.

THEOREM 3. *The following global Hölder-type condition is satisfied in any fixed, rectangle* Q *in* $\mathbb{R}_+^N$: *there exist an* a.s. *finite random variable* $\varepsilon_2 = \varepsilon_2(\omega)$ *and a constant* C_2 *such that, with probability* 1,

$$\alpha(x, B) \le C_2 (\lambda_N(B))^{1/N} |\lg \lambda_N(B)|^N \tag{1.3}$$

for all $x \in \mathbb{R}^D$ *and any cube* $B \subset Q$ *such that* $e(B) < \varepsilon_2$.

These results are along the lines of Theorems (26.1) and (27.1) of [11], but the conclusions are stronger and they do not involve local nondeterminism. The implications of such Hölder conditions for the behavior of the trajectories of X_t are explained in [11]. Ehm [8] has obtained similar results for the occupation densities of stable sheets, and we will adapt his method to prove our theorems in Section 3. In Section 5 we apply the above results to obtain the following uniform Hausdorff dimensions result for the level sets of X_t: *with probability* 1, dim $M_x = 1$ *for every* $x \in \mathbb{R}^D$.

2. Modification of the results of [11]. Our inability to determine the status of local nondeterminism (LND) for CBM is an embarrassment only partially ameliorated by the fact that the whole notion of LND can be avoided in obtaining the results of Sections 25–27 of [11]. By being careful, we will even get slightly better results. Since our goal here is to circumvent LND, we refrain from even writing down the definition; it will be found in Pitt [13] and [11].

In this section X_t *will denote a jointly* (t, ω)*-measurable Gaussian random field, not necessarily* CBM, *taking values in* $\mathbb{R}^D$, *with* $t \in \mathbb{R}_+^N$ *and* $X_0 = 0$. *It will be clear that a similar method will work for certain non-Gaussian fields, especially those for which the joint characteristic function has a simple representation, such as the stable fields.*

We use $\langle \cdot, \cdot \rangle$, $|\cdot|$ to denote, respectively, the ordinary Euclidean inner product and distance in $\mathbb{R}^D$, and V to indicate variance.

The central role in this work is played by the integral

$$(2.1)\qquad J_{k,\gamma}(B) = \int_{B^k}\int_{(\mathbb{R}^D)^k} \exp\left[-\frac{1}{2} V(\textstyle\sum_{j=1}^k \langle u^j, X_{t^j}\rangle)\right] \prod_{j=1}^k |u^j|^\gamma d\bar{u}\, d\bar{t},$$

in which $\gamma \geq 0$, $k \geq 2$ is an even integer, B is a Borel set in $\mathbb{R}_+^N$, $\bar{u} = (u^1, \cdots, u^k)$, $\bar{t} = (t^1, \cdots, t^k)$, and each $u^j \in \mathbb{R}^D$, $t^j \in \mathbb{R}_+^N$. The coordinates are then written as subscripts, e.g. $u^j = (u_1^j, \cdots, u_D^j)$.

Let $\Delta(s, t)$ be the determinant of the covariance matrix of $X_s - X_t$ and

$$(2.2)\qquad V_{k,\gamma}(B) = \int_{B^k} \prod_{j=1}^k (\Delta(t^j, t^{j-1}))^{-(1/2+\gamma)}\, d\bar{t}, \quad t^0 = 0.$$

Most of the results in this section are obtained from those of [11] by striking all mention of LND and replacing $V_{k,\gamma}$ by $J_{k,\gamma}$. Thus we need only supply details where the proofs differ substantially from those of [11].

For distinct points $t^1, \cdots, t^k$, let $p_k(\bar{t}; \bar{x})$ denote the joint density of $X_{t^1}, \cdots, X_{t^k}$, and assume that the following, called *condition* (A_k) in [11], is satisfied: *for some locally integrable function* $g_k(\bar{t})$,

$$(2.3)\qquad p_k(\bar{t}; \bar{x}) \leq g_k(\bar{t}) \quad \textit{for all } \bar{x}.$$

In this connection it is worth noting that $p_k(\bar{t}; \bar{x})$ is continuous in x and

$$p_k(\bar{t}; \bar{x}) = (2\pi)^{-kD} \int_{(\mathbb{R}^D)^k} \exp(-i\bar{u}\cdot\bar{x}) \exp\left(-\frac{1}{2} V(\textstyle\sum_{j=1}^k \langle u^j, X_{t^j}\rangle)\right) d\bar{u},$$

(see [11], page 43) so that

$$(2.4)\qquad p_k(\bar{t}; \bar{x}) \leq (2\pi)^{-kD} \int_{(\mathbb{R}^D)^k} \exp\left[-\frac{1}{2} V(\textstyle\sum_{j=1}^k \langle u^j, X_{t^j}\rangle)\right] d\bar{u}.$$

If we take $g_k(\bar{t})$ to be the right member of (2.4), then condition (A_k) follows from $J_{k,0}(B) < \infty$.

Define

$$(2.5)\qquad \alpha_0(x, B) = \liminf_{n\to\infty} \frac{1}{c_D n^{-D}} \int_B I_{(0,1/n)}(|X_s - x|)\, ds,$$

where $c_D n^{-D}$ is the volume of the ball of radius $1/n$ in $\mathbb{R}^D$, and $B \in \mathscr{B}(\mathbb{R}_+^N)$, the Borel σ-field on $\mathbb{R}_+^N$. For each B, the joint (x, ω)-measurability of $\alpha_0(x, B)$ is clear. By (21.17) of [11], a.s. X_t has an occupation density, one version of which is $\alpha_0(x, B)$; and $\alpha_0(x, B)$ is in $L_{\text{loc}}^k(dx)$ if $\lambda_N(B) < \infty$. We do not assert that $\alpha_0(x, B)$ is a kernel. As in (25.5) and (25.8) of [11] we have, for $\lambda_N(B) < \infty$ and all $\bar{x}$,

$$(2.6)\qquad \mathbb{E}[\alpha_0(x_1, B) \cdots \alpha_0(x_k, B)] = \int_{B^k} p_k(\bar{t}; \bar{x})\, d\bar{t} \leq (2\pi)^{-kD} J_{k,0}(B),$$

$$(2.7)\qquad \mathbb{E}[\alpha_0(x + w, B) - \alpha_0(x, B)]^k \leq 2(2\pi)^{-kD} |w|^{k\gamma} J_{k,\gamma}(B),$$

for any γ, $0 \leq \gamma < 1$, and $w \in \mathbb{R}^D$.

(2.8) THEOREM. *Suppose, for some rectangle* $Q = Q_t \subseteq \mathbb{R}_+^N$ *and some* k, γ *such that* $k\gamma > D$, *we have* $J_{k,\gamma}(Q) < \infty$; *then there is an occupation kernel* $\alpha(x, dt)$ *which is* a.s. *jointly continuous in the sense that* $(t, x) \mapsto \alpha(x, Q_t)$ *is continuous on* $Q \times \mathbb{R}^D$.

Remember that Q_t is the rectangle with "upper right" corner at t. It also follows that, if $J_{k,\gamma}(Q) < \infty$ for *every* Q, the joint continuity will hold on $\mathbb{R}_+^N \times \mathbb{R}^D$.

This is the analogue of part of [11], Theorem (26.1). The proof of the corresponding result of Pitt [13] has a gap, as does the proof in [11], which is also unreadable, and therefore we will give the argument in some detail. Perhaps, after going through the hands of four different authors and at least five referees, it will be correct.

Using the remark after the proof of Theorem (3.5), it is easily seen that CBM satisfies

$$\mathbb{E}(\alpha_0(x, [S, T]) - \alpha_0(x', [S', T']))^k \leq |(x, S, T) - (x', S', T')|^{k/N}$$

where $[S, T] = \{(r_1, \cdots, r_N) \mid s_i \leq r_i \leq t_i\}$, $S = (s_1, \cdots, s_N)$, $T = (t_1, \cdots, t_N)$. With such an inequality one can apply Garsia's lemma [10] in (x, s, T) and significantly shorten the proof of joint continuity. We prefer to prove the more general Theorem (2.8) in order to conform with the general framework of [11].

We may restrict attention to Q as the time set. Write $\mathcal{H}_Q$ for the field on Q generated by all boxes in Q having rational corners; thus $\mathcal{H}_Q$ consists of all finite unions of such boxes, hence is countable, and, incidentally, generates the Borel σ-field $\mathcal{B}(Q)$. Consider a fixed trajectory $X_t(\omega)$ for which $\alpha_0(x, B)$ is a version of the occupation density.

NOTE. The word "version" is used in two ways. In the last sentence it refers to the Radon-Nikodym derivative $d\mu_A/d\lambda_D$: different versions may differ on a set of λ_D-measure zero, but they all refer to a fixed ω. Later we speak about versions of stochastic processes: for each fixed value of the parameter, two versions agree for almost every ω. The intended meaning will be clear from the context.

We hold ω fixed in the ensuing discussion, noting that all of the quantities defined will be jointly (x, ω)-measurable. Let

$$\alpha_1(x, B) = \liminf_{C \downarrow x} \frac{1}{\lambda_D(C)} \int_C \alpha_0(y, B)\, dy, \quad B \in \mathcal{H}_Q, \tag{2.9}$$

where C runs through the family of cubes centered at x. The limit process here is discusssed by Saks [16], Chapter IV, Section 14, to which the reader is referred for details. Since $\alpha_0(x, B)$ is integrable, $\alpha_1(x, B)$ actually exists as a limit and is equal to $\alpha_0(x, B)$, for a.e. x. Thus $\alpha_1(x, B)$ is also a version of the occupation density relative to B and (2.9) remains valid with α_1 in place of α_0 under the integral. Moreover, for each x, there is a sequence of cubes $C_n(x)$, centered at x, for which (2.9) is a limit; the sequence $C_n(x)$ depends on B as well.

For $\delta > 0$ and L compact in $\mathbb{R}^D$ *define*

$$K(B, L; k, \delta) = \int_L \int_L \left| \frac{\alpha_1(x, B) - \alpha_1(y, B)}{|x-y|^{\gamma+D/k} |\mathrm{Lg}|x-y||^{(1+\delta)/k}} \right|^k dx\, dy,$$

where $\mathrm{Lg}(u) = \lg(u \wedge 1/3)$ for $u > 0$. By (2.7) we have

$$\mathbb{E}\, K(B, L; k, \delta) \le 2(2\pi)^{-kD} J_{k,\gamma}(B) \int_L \int_L \frac{dx\, dy}{|x-y|^D |\mathrm{Lg}|x-y||^{1+\delta}} < \infty. \tag{2.10}$$

According to Garsia's lemma [10] we have

$$\begin{aligned} &|\alpha_1(x, B) - \alpha_1(y, B)| \\ &\qquad \le 8(K(B, L; k, \delta))^{1/k} |x-y|^{\gamma - D/k} |\mathrm{Lg}|x-y||^{(1+\delta)/k} \end{aligned} \tag{2.11}$$

for a.e. pair (x, y) from L. Rewrite (2.11) in the form

$$-\phi(|x-y|) < \alpha_1(x, B) - \alpha_1(y, B) < \phi(|x-y|),$$

so $\phi(|x-y|)$ is a bounded, continuous function on $L \times L$. Let $C_n(x)$, $C_n(y)$ be the sequences of cubes described after (2.9) corresponding to x and y, respectively. If we average the last inequality over $C_n(x)$, $C_n(y)$, respectively, and let $n \to \infty$, we obtain (2.11) for every pair (x, y). Since $\mathscr{H}_Q$ is countable, there is a set of probability 1 on which $K(B, L; k, \delta) < \infty$ simultaneously for all $B \in \mathscr{H}_Q$ and all L. Thus we have a Hölder-type condition for, and so the continuity of, $\alpha_1(x, B)$ as a function of $x \in \mathbb{R}^D$. Now we want to replace α_1 with a *kernel* having the desired properties.

Let $\alpha_2(x, dt)$ be any version of the occupation kernel (cf. Section 1). As on page 47 of [11], we find $x_0 \in \mathbb{R}^D$ such that a.s., $\alpha_1(x_0, B) = \alpha_2(x_0, B)$ for all $B \in \mathscr{H}_Q$, and the measure $\alpha_2(x_0, dt)$ has no mass on any "hyperplane", i.e. a set in Q of the form $\{t: t_i = a\}$. We now observe that $\alpha_1(x, B)$ *is finitely additive on* $\mathscr{H}_Q$ *for every* $x \in \mathbb{R}^D$: first, referring to Section 25 of [11], we know that, a.s., for a.e. $x \in \mathbb{R}^D$, $\alpha_0(x, B)$ exists as a finite limit in (2.5), for every $B \in \mathscr{H}_Q$, and is clearly finitely additive. Thus $\alpha_1(x, \sum_1^n B_j) = \sum_{j=1}^n \alpha_1(x, B_j)$ for a.e. x, and then for every x, by the continuity just established above.

Take $Q = [0, 1]^N$ for simplicity in the remainder of the proof. Fix i, $1 \le i \le N$, and let

$$I_{nmi} = \left\{ s \in Q: \frac{m-1}{n} < s_i \le \frac{m}{n} \right\}, \quad 1 \le m \le n.$$

Also, let $\|f\|_L = \sup_{x \in L} |f(x)|$. We will now show that, as n runs through a suitable subsequence,

$$\textstyle\sum_{m=1}^n \|\alpha_1(\cdot, I_{nmi})\|_L^k \to 0 \quad (n = n_p \to \infty). \tag{2.12}$$

Indeed, from (2.11), a.s.

$$\|\alpha_1(\cdot, I_{nmi})\|_L \le \alpha_2(x_0, I_{nmi}) + c d_L^{\gamma - D/k} |\mathrm{Lg}\, d_L|^{(1+\delta)/k} (K(I_{nmi}, L; k, \delta))^{1/k},$$

where $d_L = \max_{x \in L} |x - x_0|$. Using the inequality $(a+b)^k \le 2^k(a^k + b^k)$ we see that the left member of (2.12) is majorized by 2^k times

$$(\max_{1\le m\le n}\alpha_2(x_0, I_{nmi}))^{k-1}\alpha_2(x_0, Q) + d_L^{k\gamma-D}|\operatorname{Lg} d_L|^{1+\delta}\textstyle\sum_{m=1}^n K(I_{nmi}, L; k, \delta). \tag{2.13}$$

Since $\alpha_2(x_0, dt)$ has no hyperplane masses, the function $f_i(u) = \alpha_2(x_0, \{s: s_i \le u\})$ is continuous for $0 \le u \le 1$, hence uniformly continuous, and so the first term in (2.13) tends to zero as $n \to \infty$. To prove (2.12) it is enough to show

$$\lim_{n\to\infty} \textstyle\sum_{m=1}^n \mathbb{E}\, K(I_{nmi}, L; k, \delta) = 0. \tag{2.14}$$

The sum in (2.14) is at most a constant times

$$\sum_{m=1}^n J_{k,\gamma}(I_{nmi}) \le \int_{\cup_{m=1}^n I_{nmi}^k} \int_{(R^D)^k} (\mathbb{R}^D)^k \exp\left[-\frac{1}{2} V(\textstyle\sum_{j=1}^k \langle u^j, X_t j\rangle)\right] \prod_{j=1}^k |u^j|^\gamma \, d\bar{u}\, d\bar{t}$$

by (2.10). But it is not hard to see that

$$\lambda_N \times \cdots \times \lambda_N(\cup_{m=1}^n I_{nmi}^k) \le n^{-(k-1)} \to 0$$

where $\lambda_N \times \cdots \times \lambda_N$ is the k-fold product of λ_N on Q^k. This implies (2.14). In effect, (2.12) says that $\alpha_1(x, \cdot)$ has no hyperplane masses; however, $\alpha_1(x, \cdot)$ is not (yet) a measure.

Define the distance between two rectangles in Q to be the maximum distance between the corresponding corners. Let $x \in L \subset \mathbb{R}^D$ and $\omega \in \Omega$ such that all of the above assertions hold. The function $\alpha_1(x, B)$ defined for rational rectangles $B \in \mathscr{H}_Q$ can be extended uniquely to a function, again denoted $\alpha_1(x, B)$, defined for all rectangles B in Q, and continuous in the sense of the distance between rectangles. In fact the symmetric difference between two overlapping rectangles is contained in $\cup_{i=1}^N I_{nm_i i}$ for some n and choice of m_i, $1 \le m_i \le n$. It is a classical result, Saks [16], Chapter III, that $\alpha_1(x, B)$ extends to a measure on $\mathscr{B}(Q)$, which we denote $\alpha(x, B)$. The joint measurability of $\alpha(x, B)$ is obtained by a monotone class argument. Thus $\alpha(x, B)$ will be the desired kernel.

Now consider $\alpha(x, Q_t)$ as a function of x and t: (i) if $t = (t_1, \cdots, t_N)$ is rational, i.e. each t_i is rational, then $\alpha(x, Q_t)$ is continuous in x; and (ii) for each x, it is a continuous function of t. This implies $\alpha(x, Q_t)$ is jointly continuous, as follows. Let $(t^n, x^n) \to (t, x) \in Q \times \mathbb{R}^D$, and $\varepsilon > 0$. Choose $s \in Q$ rational so that $s_i > t_i$ for each $i = 1, \cdots, N$, so $Q_s \supset Q_t$, and so that $\alpha(x, Q_s) < \alpha(x, Q_t) + \varepsilon$. This is possible by (ii). For sufficiently large n we will have $t_i^n < s_i$ for each i, and so $Q_{t^n} \subset Q_s$. Thus

$$\begin{aligned}\limsup_{n\to\infty}\alpha(x^n, Q_{t^n}) &\le \limsup_{n\to\infty}\alpha(X^n, Q_s)\\ &= \alpha(x, Q_s)\\ &\le \alpha(x, Q_t) + \varepsilon\end{aligned}$$

the second line being due to (i). Since ε is arbitrary, we have

$$\limsup_n \alpha(x^n, Q_{t^n}) \le \alpha(x, Q_t),$$

and a similar argument "from the inside" gives

$$\liminf_n \alpha(x^n, Q_{t^n}) \geq \alpha(x, Q_t).$$

So Theorem (2.8) is proven at last.

The next result is stated for the unit cube $T = [0, 1]^N$, but is valid for any given Q_t in $\mathbb{R}_+^N$. Since we want to apply these results to continuous fields, it is no restriction to assume that, a.s., the trajectories are bounded on compacts in $\mathbb{R}_+^N$ and, in addition, separability.

Let $\mathcal{D}_n$ be the nth "dyadic partition" of T into 2^{nN} cubes, each of measure 2^{-nN}, and put $\mathcal{D} = \cup_n \mathcal{D}_n$.

(2.15) **Theorem.** *Suppose $k\gamma > D$ and*

$$J_{k,\gamma}(B) \leq c_1(\lambda_N(B))^{1+\rho} \text{ for every } B \in \cup_{n \geq n_0} \mathcal{D}_n, \tag{2.16}$$

where c_1, $\rho > 0$ and n_0, a positive integer, may depend on k, γ; we also require (2.16) *to hold for $\gamma = 0$. Then, for any $\eta > 0$, there exist a constant $c > 0$ and a random variable $\varepsilon = \varepsilon(\omega)$ such that,* a.s., *the following global Hölder condition is satisfied*:

$$\alpha(x, B) \leq c(\lambda_N(B))^{\rho/k} |\lg \lambda_N(B)|^{(1+\eta)/k} \tag{2.17}$$

for all $x \in \mathbb{R}^D$ and all cubes B in T with $e(B) < \varepsilon$.

This is a more precise version of Theorem (27.1) of [11]; the condition $k > d/\gamma$ in that theorem should be replaced by $k > 2d/\gamma$. It is understood that $\alpha(x, B)$ is the good version obtained in (2.8). The condition involving $\gamma = 0$ is not a serious restriction; it is fulfilled in the application below. One of the main difficulties in the proof is the uniformity in x of the Hölder condition (2.17), which necessitates most of the hard work. This result is already interesting for a fixed x. The same comments apply to the proof of Theorem 2 below (Section 3).

We write c_2, c_3, etc. for positive constants whose exact values are unimportant, and ν_1, ν_2, etc. for a.s. finite, positive integer-valued random variables; and we put $g(\lambda) = \lambda^{\rho/k} |\lg(\lambda)|^{(1+\eta)/k}$, $\lambda > 0$.

Using (2.6), (2.7), and (2.16), we obtain, from Markov's inequality,

$$\mathbb{P}\{\alpha(x, B) \geq g(\lambda_N(B))\} \leq c_2 2^{-nN} n^{-(1+\eta)}, \quad B \in \mathcal{D}_n, \tag{2.18}$$

where $n \geq n_0$ henceforth, and, for any $z > 0$,

$$\mathbb{P}\{|\alpha(y_1, B) - \alpha(y_2, B)| \geq |y_1 - y_2|^\gamma g(\lambda_N(B)) z^{1/k}\} \leq c_2 2^{-nN} n^{-(1+\eta)} z^{-1}. \tag{2.19}$$

Let $G_n = \mathbb{Z}^D \cap [-\lg n, \lg n]^D$, i.e. the points of the integer lattice $\mathbb{Z}^D$ which fall in the cube of edge length $2 \lg n$, centered at the origin, in $\mathbb{R}^D$. The cardinality of G_n satisfies $\#G_n \leq c_3 (\lg n)^D$ for all large n.

Now

$$\begin{aligned}
\mathbb{P}\{\max_{x \in G_n} \alpha(x, B) \geq g(\lambda_N(B)) \text{ for some } B \in \mathcal{D}_n\} \\
&\leq \textstyle\sum_{x \in G_n} \sum_{B \in \mathcal{D}_n} \mathbb{P}\{\alpha(x, B) \geq g(\lambda_N(B))\} \\
&\leq c_2 c_3 2^{nN} (\lg n)^D 2^{-nN} n^{-(1+\eta)}, \quad \text{by (2.18)}.
\end{aligned}$$

Thus, since $(\lg n)^D n^{-(1+\eta)}$ is summable, Borel-Cantelli gives

$$\max_{x\in G_n}\alpha(x, B) \le g(\lambda_N(B)) \text{ for all } B \in \mathscr{D}_n, \quad \text{for } n \ge \nu_1, \tag{2.20}$$

for some ν_1.

Next, for positive integers, n, h, and $x \in G_n$, define

$$F(n, h, x) = \{y \in \mathbb{R}^D : y = x + \textstyle\sum_{j=1}^h 2^{-j}\varepsilon_j \quad \text{for} \quad \varepsilon_j \in \{0, 1\}^D\}.$$

As a little picture will indicate, $F(n, h, x)$ consists of the "dyadic points up to order h" in the unit cube of edge 1 with lower left corner at x. Any pair y_1, y_2 of adjacent points in $F(n, h, x)$ is said to be *linked*; more precisely, y_1 and y_2 are linked if $y_1 - y_2 = \varepsilon 2^{-h}$ for some ε in $\{0, 1\}^D$. We then have at most 2^{Dh} linked pairs, and so, by (2.19),

$$\begin{aligned}
&\mathbb{P}[\cup_{B\in D_n}\cup_{x\in G_n}\cup_{h\ge 1}\cup_{y_1,y_2\in F(n,h,x)}\{|\alpha(y_;, B) - \alpha(y_2, B)| \\
&\quad \ge |y_1 - y_2|^\gamma g(\lambda_N(B))2^{(D+\delta)h/k}\}] \\
&\qquad \le c_4 2^{nN}(\lg n)^D \textstyle\sum_{h=1}^\infty 2^{Dh}2^{-nN}n^{-(1+\eta)}2^{-(D+\delta)h} \\
&\qquad \le c_4(\lg n)^D n^{-(1+\eta)};
\end{aligned} \tag{2.21}$$

here we take $0 < \delta < k\gamma - D$. Thus, by Borel-Cantelli,

$$\begin{aligned}
|\alpha(y_1, B) - \alpha(y_2, B)| &\le c_5 2^{-h\gamma} g(\lambda_N(B))2^{(D+\delta)h/k} \\
&= c_5 g(\lambda_N(B))2^{-h(\gamma-(D+\delta)/k)}
\end{aligned} \tag{2.22}$$

for every $B \in \mathscr{D}_n$, $x \in G_n$, $h \ge 1$, and linked pair y_1, y_2 in $F(n, h, x)$, if $n \ge \nu_2$.

Since the trajectories are bounded, we have $|X_t| \le \lg n$ for all $t \in T$, $n \ge \nu_3$, a.s. Let $n \ge \nu_1, \nu_2, \nu_3$, and consider $y \in \mathbb{R}^D$. If $|y| > \lg n$, we have $\alpha(y, T) = 0$, so that (2.17) holds (for y); if $|y| \le \lg n$, we can find $x \in G_n$ such that $y = \lim_{h\to\infty} y_h$, with

$$y_h = x + \textstyle\sum_{j=1}^h 2^{-j}\varepsilon_j, \quad \varepsilon_j \in \{0, 1\}^D, \quad y_0 = x,$$

so each pair y_h, y_{h-1} is linked in $F(n, h, x)$. From (2.22) we conclude

$$|\alpha(y, B) - \alpha(x, B)| \le c_5 g(\lambda_N(B)) \textstyle\sum_{h=1}^\infty 2^{-h(\gamma-(D+\delta)/k)} = c_6 g(\lambda_N(B)), \tag{2.23}$$

and, with (2.20), we find that (2.17) holds for all $y \in \mathbb{R}^D$ and $B \in \mathscr{D}_n$ for n large, say $n \ge \nu_4$. As in [11], page 50, any cube B with $e(B) \le \varepsilon = 2^{-\nu_4}$ can be covered by at most 16^N dyadic cubes of edge length $\le 2^{-\nu_4}$, and this completes the proof.

It is also possible to formulate a local Hölder condition, but we will not do so. When (2.16) holds for a fixed $\gamma > 0$ and *all k*, a more accurate version of both the local and global Hölder conditions may be given; these are formulated for CBM in Theorems 2 and 3 in Section 1.

3. Proofs of the results in Section 1. Let X_t denote CBM, defined in (0.1). Our first task is to untangle the variance appearing in $J_{k,\gamma}(B)$ in (2.1). We will write points $u \in \mathbb{R}^D$ $(D = 2(N - 1))$ in the form $u = (u_1, u_2, \cdots, u_\nu)$ where $\nu =$

$N - 1$ and each $u_i \in \mathbb{R}^2$. Thus the u^j appearing in (2.1) are now written as $u^j = (u_1^j, \cdots, u_\nu^j)$, $u_i^j \in \mathbb{R}^2$. We also take $u_0^j = u_N^j = 0$ in what follows.

Define

$$(3.1) \qquad v_i^j = u_i^j - u_{i-1}^j, \quad 1 \leq i \leq N,$$

so $v_i^j (1 \leq i \leq \nu)$ are independent coordinates in $\mathbb{R}^D$ and

$$v_N^j = -\sum_{i=1}^{\nu} v_i^j.$$

Let $W(t) = (W_1(t_1), W_2(t_2), \cdots, W_N(t_N))$; then

$$\sum_{j=1}^{k} \langle u^j, X_{t^j} \rangle = \sum_{j=1}^{k} v^j \cdot W(t^j) = \sum_{i=1}^{N} \sum_{j=1}^{k} v_i^j \cdot W_i(t_i^j);$$

the dot products being in $\mathbb{R}^{2N}$ and $\mathbb{R}^2$, respectively. Since the W_i are independent planar Brownian motions, we have

$$(3.2) \qquad V\Big(\sum_{j=1}^{k} \langle u^j, X_{t^j} \rangle\Big) = \sum_{i=1}^{N} V\Big(\sum_{j=1}^{k} v_i^j \cdot W(t_i^j)\Big)$$

where W denotes a generic planar Brownian motion.

Let S_k denote the set of permutations of $\{1, \cdots, k\}$. If we neglect ties among the ith coordinates, there is a unique permutation $\pi_i \in S_k$ such that

$$(3.3) \qquad t_i^{\pi_i(1)} < t_i^{\pi_i(2)} < \cdots < t_i^{\pi_i(k)}.$$

In fact, the set of $\bar{t} \in (\mathbb{R}_+^N)^k$ having a tie in the ith coordinates for some i is a set of Lebesgue measure 0 and may be neglected for the present purpose. Thus, neglecting ties, $(\mathbb{R}_+^N)^k$ is partitioned into $(k!)^N$ disjoint sets $\Delta(\pi_1, \cdots, \pi_N)$, $\pi_i \in S_k$, where $\bar{t} \in \Delta(\pi_1, \cdots, \pi_N)$ iff (3.3) holds for each $i = 1, \cdots, N$; there is a set $\Delta(\pi_1, \cdots, \pi_N)$ for each choice of $(\pi_1, \cdots, \pi_N) \in S_k^N$.

If the terms on the right side of (3.2) are now rearranged to take advantage of the independent increments of Brownian motion, one obtains

$$(3.4) \qquad V\Big(\sum_{j=1}^{k} \langle u^j, X_{t^j} \rangle\Big) = \sum_{i=1}^{N} \sum_{j=1}^{k} |w_i^j|^2 \tau_i^j$$

where

$$w_i^j = \sum_{p=j}^{k} v_i^{\pi_i(p)}, \quad \tau_i^j = t_i^{\pi_i(j)} - t_i^{\pi_i(j-1)}$$

$(\pi_i(0), = 0, t_i^0 = 0)$.

The key ingredient in proving the results in Section 1 is the following estimate involving $J_{k,\gamma}(B)$.

(3.5) **Theorem.** *For $0 \leq \gamma < 1/\nu$ and $B = \prod_1^N [a_i, a_i + h]$,*

$$J_{k,\gamma}(B) \leq a^k h^{k(1-\gamma/2)} (k!)^N$$

where a is a constant independent of B.

Proof. Break up the integral for $J_{k,\gamma}(B)$ into a sum of $(k!)^N$ terms of the form

$$\int_\Delta \int_{(R^D)^k} \prod_{j=1}^k |u^j|^\gamma \prod_{i=1}^N \exp(-\textstyle\sum_{j=1}^k |w_i^j|^2 \tau_i^j)\, d\bar{u}\, d\bar{t}, \tag{3.6}$$

where $\Delta = \Delta(\pi_1, \cdots, \pi_N) \cap B^k$, $\pi_1, \cdots, \pi_N \in S_k$, the w's are functions of the u's, and the τ's are functions of the t's, as described above. Now make the change of variables $hs_i^j = t_i^j$, $h^{1/2}\bar{u} = \bar{v}$. We note that for $j = 1$, $\tau_i^1 = t_i^{\pi_i(1)}$ which is in the interval $[a_i, a_i + h]$. The integral can only get larger if we replace this interval by $[0, h]$, and, when this is done, a factor of $h^{k(1-\gamma/2)}$ comes out because of the change of variables. Changing the letters s, v back to t, u, we may consider (3.6) anew, but now with $B = [0, 1]^N$, and it suffices to show that this new version of (3.6) is dominated by a^k.

Let $\|j_i\|_N$ denote the norm in $L^N((\mathbb{R}^D)^k, \lambda_{Dk})$. We will use a generalized version of Hölder's inequality:

$$\int \prod_{i=1}^N f_i \le \prod_{i=1}^N \|f_i\|_N.$$

We apply this to

$$f_i = (\textstyle\prod_{j=1}^k |u^j|^\gamma)^{1/N} \prod_{m\ne i, 1\le m\le N} \exp(-\tfrac{1}{2} \sum_{j=1}^k |w_m^j|^2 \tau_m^j/\nu),$$

the product of which is the integrand in (3.6). The result is that (3.6) is less than or equal to

$$\int_\Delta \prod_{i=1}^N \left\| \prod_{j=1}^k |u^j|^{\gamma/N} \prod_{m\ne i} \exp\left(-\frac{1}{2} \sum_{j=1}^k |w_m^j|^2 \tau_m^j/\nu\right)\right\|_N d\bar{t}. \tag{3.7}$$

The transformation $u \to v$ on $\mathbb{R}^D$ given in (3.1) is linear, so that $|u| \le c_1 |v|$ for some constant c_1. Thus (3.7) is at most

$$c_1^{k\gamma} \int_\Delta \prod_{i=1}^N \left\| \prod_{j=1}^k |v^j|^{\gamma/N} \prod_{m\ne i} \exp\left(-\frac{1}{2} \sum_{j=1}^k |w_m^j|^2 \tau_m^j/\nu\right)\right\|_N d\bar{t}. \tag{3.8}$$

We now show that

$$\prod_{j=1}^k |v^j|^\gamma \le (15)^{k\gamma} \prod_{m\ne i} \prod_{j=1}^k (1 + |w_m^j|^2)^\gamma, \tag{3.9}$$

where $w_i^j = \sum_{p=j}^k v_i^{\pi_i(p)}$, as in (3.4). From the equation for v_N^j after (3.1) we have $v_i^j = -\sum_{1\le m\le N, m\ne i} v_m^j$, so

$$|v^j| \le \textstyle\sum_{m=1}^\nu |v_m^j| = \sum_{m\le \nu, m\ne i} |v_m^j| + |v_i^j|$$

$$\le 2 \textstyle\sum_{1\le m\le N, m\ne i} |v_m^j| \le 2 \prod_{m\ne i, 1\le m\le N} (1 + |v_m^j|);$$

the last inequality is easily derived by expanding the indicated product. Thus

$$\prod_{j=1}^k |v^j| \le 2^k \prod_{m\ne i, 1\le m\le N} \prod_{j=1}^k (1 + |v_m^j|)$$

$$= 2^k \prod_{m\ne i, 1\le m\le N} \prod_{j=1}^k (1 + |v_m^{\pi_m(j)}|)$$

$$\leq 2^k \prod_{m\neq i, 1\leq m\leq N} \prod_{j=1}^k (1 + |w_m^j| + |w_m^{j+1}|)$$
$$\leq (15)^k \prod_{m\neq i, 1\leq m\leq N} \prod_{j=1}^k (1 + |w_m^j|^2),$$

noting $v_m^{\pi_m(j)} = w_m^j - w_m^{j+1}$; to prove the last inequality it suffices to consider $a_1, \cdots, a_k \geq 0$, $a_{k+1} = 0$, and to estimate

$$\prod_{j=1}^k (1 + a_j + a_{j+1}) = 2^{-k} \quad {}_{j=1}^{k} [(1 + 2a_j) + (1 + 2a_{j+1})]$$
$$\leq 2^{-k} \sum_\beta \prod_{j=1}^k (1 + 2a_j)^{\beta_j},$$

where the sum is over all $\beta = (\beta_1, \cdots, \beta_k)$ subject to $\beta_j = 0, 1$, or 2, and $\sum_j \beta_j = k$. The number of such β's is at most 3^k. Thus the last expression is dominated by

$$3^k 2^{-k} \prod_j (1 + 2a_j)^2 \leq 3^k 2^{-k} 5^k \prod_j (1 + a_j^2),$$

whence the stated result.

Consider now the L^N-norm appearing in (3.8); if we raise it to the Nth power, the resulting expression will be at most

$$15^{k\gamma N} \int_{(R^D)^k} \prod_{m\neq i} \prod_{j=1}^k (1 + |w_m^j|^2)^\gamma \prod_{m\neq i} \exp\left(-\frac{1}{2} \sum_{j=1}^k |w_m^j|^2 \tau_m^j N/\nu\right) d\bar{u}.$$

Since the transformations $u \to v \to w$ are all linear, we may replace $d\bar{u}$ by $d\bar{w}$, at the same time getting a new constant c_2^k in front, and then the integral will be

$$c_2^k \prod_{m\neq i} \prod_{j=1}^k \int_{R^2} (1 + |z|^2)^\gamma \exp(-|z|^2 N\tau_m^j/\nu)\, dz$$

(dz; Lebesgue measure on $\mathbb{R}^2$)

$$\leq c_3^k \prod_{m\neq i} \prod_{j=1}^k (\tau_m^j)^{-1} \int_0^\infty (1 + u/\tau_m^j)^\gamma e^{-u}\, du$$
$$\leq c_4^k \prod_{m\neq i} \prod_{j=1}^k (\tau_m^j)^{-(r+1)},$$

using $(1 + u/\tau)^\gamma \leq 1 + (u/\tau)^\gamma$ in the last line.

Recall now that $\tau_m^j = t_m^{\pi_m(j)} - t_m^{\pi_m(j-1)}$; we take the Nth root above and go back and dominate (3.8) by

$$c_5^k \int_\Delta \prod_{i=1}^N \prod_{m\neq i} \prod_{j=1}^k (\tau_m^j)^{-(\gamma+1)/N} d\bar{t}$$
$$= c_5^k \int_\Delta \prod_{i=1}^N \prod_{j=1}^k (\tau_i^j)^{-(\nu/N)(\gamma+1)} d\bar{t}$$
$$= c_5^k \left(\int_{0<s_1<\cdots<s_k<1} \prod_j (s_j - s_{j-1})^{-(\nu/N)(\gamma+1)} ds \right)^N$$
$$\leq c_5^k \left(\int_0^1 u^{-((N-1)/N)(\gamma+1)} du \right)^{Nk},$$

and the integral here is finite iff $\gamma < 1/\nu$. Thus (3.5) is proven.

REMARK. If B is a rectangle of the form $\prod_{i=1}^{N}[a_i, a_i + h_i]$ (and we wait until (3.9′) before scaling out the h_i's), we obtain

$$J_{k,\gamma}(B) \le a^k(k!)^N(h_1h_2 \cdots h_N)^{(k/N)(1-\gamma\nu/2)}.$$

We now turn to properties of the occupation density of CBM. First we remark that the existence of an occupation density follows from $J_{k,0}(B) < \infty$, by (21.17) of [11]. Just for the record, the existence also follows from

$$\text{(3.10)} \qquad \int_Q \frac{1}{\Delta^{1/2}}\,dt < \infty \quad \text{for a.e. } s$$

where $\Delta = \Delta(s, t) = \det \mathrm{Cov}(X_s - X_t)$ and Q is any rectangle as is proven in (22.1) of [11]. To show that (3.10) holds, one must evaluate the determinant Δ. If $s = (s_1, \cdots, s_N)$, $t = (t_1, \cdots, t_N)$, and $\delta_i = |s_i - t_i|$, then it can be shown, by some extremely tedious computations, that

$$\Delta^{1/2} = \textstyle\sum_{i=1}^{N} \prod_{m\ne i}\delta_m;$$

then, by the geometric-arithmetic means inequality applied to the partial products $\prod_{m\ne i}\delta_m$, $\Delta^{1/2} \ge N^{1/N} \prod_{i=1}^{N} \delta_i^{(N-1)/N}$ and (3.10) follows.

Theorem 1 of Section 1 is now immediate in view of (2.8) and (3.5).

For the proofs of Theorems 2 and 3 we need some preliminary estimates. Remember, as in Section 1, all cubes have edges parallel to the axes.

(3.11) LEMMA. *Let B be a cube with lower left corner at τ; then, for any $x, y \in \mathbb{R}^D$, even integer k, and $\gamma < 1/\nu$,*

$$\text{(3.12)} \qquad \mathbb{E}[\alpha(x + X_\tau, B)]^k \le A^k(\lambda_N(B))^{k/N}(k!)^N$$

$$\text{(3.13)} \qquad \mathbb{E}[\alpha(x + X_\tau, B) - \alpha(y + X_\tau, B)]^k \le 2A^k|x - y|^{k\gamma}(\lambda_N(B))^{(k/N)(1-\gamma/2)}(k!)^N,$$

where $A = a/(2\pi)^D$ (a appears in (3.5)).

We observe that $\alpha(x + X_\tau, B)$ is the occupation density on B of the random field $Y_t = X_t - X_\tau$, which has the same law as $X_{t-\tau}$ on B, since $t_i \ge \tau_i$, $i = 1, \cdots, N$, when $t \in B$. Thus the J-integral over B corresponding to Y, given by (2.1), is the same as the J-integral for X over $[0, h]^N$, where $h = e(B)$. Thus, applying (2.6) and (2.7) to Y and estimating $J_{k,\gamma}([0, h]^N)$ by (3.5), we obtain (3.11).

NOTE. Of course (3.12) and (3.13) remain valid if we replace X_τ by zero therein; the same comment applies to (3.15) and (3.16) below.

(3.14) LEMMA. *With the notation of* (3.11), *there are absolute constants b, $c > 0$ such that, for any $z > 0$,*

$$\text{(3.15)} \qquad \mathbb{P}\{\alpha(x + X_\tau, B) \ge b(\lambda_N(B))^{1/N}z^N\} \le ce^{-2z}$$

$$\text{(3.16)} \qquad \mathbb{P}\{|\alpha(x + X_\tau, B) - \alpha(y + X_\tau, B)| \ge b|x - y|^\gamma(\lambda_N(B))^{(1/N)(1-\gamma/2)}z^N\} \le ce^{-2z}.$$

"Absolute" means not depending on τ, B, x or y.

We prove these inequalities simultaneously. Let Λ be shorthand for either of the quantities

$$\alpha(x + X_\tau, B)/(\lambda_N(B))^{1/N}$$

or

$$|\alpha(x + X_\tau, B) - \alpha(y + X_\tau, B)|/(\lambda_N(B))^{(1/N)(1-\gamma/2)}|x - y|^\gamma.$$

Then (3.11) yields, in either case, $\mathbb{E}\,\Lambda^k \le 2A^k(k!)^N$, or, using Jensen's inequality,

$$\mathbb{E}\,\Lambda^{k/N} \le 2^{1/N}A^{k/N}k! \quad (k \text{ even}).$$

If k is even, we also have

$$\mathbb{E}\,\Lambda^{(k-1)/N} \le (\mathbb{E}\,\Lambda^{k/N})^{(k-1)/k} \le 2^{1/N}A^{(k-1)/N}k!,$$

so that, for any integer $m \ge 0$, $\mathbb{E}\,\Lambda^{m/N} \le 2^{1/N}A^{m/N}(m + 1)!$.

Dividing this inequality by M^m, for any $M > A^{1/N}$, and summing on $m \ge 0$, we obtain

$$\mathbb{E}\exp(\Lambda^{1/N}/M) \le 2^{1/N}\textstyle\sum_{m=0}^\infty (m + 1)(A^{1/N}/M)^m \equiv c.$$

Chebyshev then gives

$$\mathbb{P}(\Lambda \ge M^N z^N) \le ce^{-z}, \quad z > 0;$$

now replace z by $2z$ and let $b = (2M)^N$ to finish the proof of (3.14).

Proof of Theorem 2. We use c_1, c_2, etc. for positive constants and ν_1, ν_2, etc. for positive integer-valued, a.s. finite random variables.

Let $B_n = \prod_{i=1}^N [\tau_i, \tau_i + 2^{-n}]$. The local law of the iterated logarithm says that, for a standard Brownian motion $w(t)$,

$$\limsup_{h\downarrow 0} \frac{|w(t + h) - w(t)|}{\sqrt{2h \lg\lg(1/h)}} = 1 \quad \text{a.s.}$$

for each fixed $t \ge 0$. Applying this to the components of X_t we obtain, a.s.,

$$\sup_{t\in B_n}|X_t - X_\tau| \le c_1 2^{-n/2}(\lg n)^{1/2} \quad \text{for all } n \ge \nu_1. \tag{3.17}$$

Let $\theta_n = c_1 2^{-n/2}(\lg\lg 2^{nN})^{N/\gamma}/(\lg n)^{N/\gamma}$, $n \ge 2$, and

$$G_n = \{x \in \mathbb{R}^D : |x| \le c_1 2^{-n/2}(\lg n)^{1/2},\ x = \theta_n p \quad \text{for some } p \in \mathbb{Z}^D\},$$

where $\mathbb{Z}^D$ is the integer lattice in $\mathbb{R}^D$. Thus G_n consists of those points in the lattice of step size θ_n which are contained in the ball of radius $c_1 2^{-n/2}(\lg n)^{1/2}$, centered at the origin. The cardinality of G_n satisfies

$$\# G_n \le \left(\frac{\text{diameter}}{\text{step size}} + 1\right)^D = \left(\frac{2(\lg n)^{(1/2)+(N/\gamma)}}{(\lg\lg 2^{nN})^{N/\gamma}} + 1\right)^D,$$

which is less than a positive power of $\lg n$, thus $\# G_n \le (\lg n)^\beta$ for some $\beta > 0$. By

(3.15) we have, with $g(\lambda) = \lambda^{1/N}(\lg|\lg\lambda|)^N$ for $\lambda > 0$,

$$\mathbb{P}\{\alpha(x + X_\tau, B_n) \geq bg(\lambda_N(B_n)) \text{ for some } x \in G_n\} \leq c_2 n^{-2} \# G_n,$$

and so, by Borel-Cantelli,

$$\max_{x \in G_n} \alpha(x + X_\tau, B_n) \leq bg(\lambda_N(B_n)) \text{ for all } n \geq \nu_2, \text{ a.s.} \tag{3.18}$$

Now, for integers, $n, h \geq 1$ and $x \in G_n$, define

$$F(n, h, x) = \{y \in \mathbb{R}^D: y = x + \theta_n \textstyle\sum_{j=1}^h \varepsilon_j 2^{-j} \text{ for } \varepsilon_j \in \{0, 1\}^D\}.$$

(The reader is advised to draw a picture to clarify the proof.) Define a pair of points $y_1, y_2 \in F(n, h, x)$ to be *linked* if $y_1 - y_2 = \theta_n \varepsilon 2^{-h}$ for some ε in $\{0, 1\}^D$. Then, using (3.16), we get, as in (2.21),

$$\begin{aligned} &\mathbb{P}\{|\alpha(y_1 + X_\tau, B_n) - \alpha(y_2 + X_\tau, B_n)| \\ &\quad \geq b(\lambda_N(B_n))^{(1/N)(1-\gamma/2)}|y_1 - y_2|^\gamma h^N (\lg n)^N \\ &\quad \textit{for some } x \in G_n, h \geq 1 \textit{ and some linked pair } y_1, y_2 \in F(n, h, x)\} \\ &\quad \leq \# G_n \textstyle\sum_{h=1}^\infty 2^{Dh} e^{-2h \lg n} \leq (\lg n)^p \dfrac{2^D/n^2}{1 - 2^D/n^2}. \end{aligned} \tag{3.19}$$

Since this is summable, Borel-Cantelli gives us a random variable ν_3 such that, a.s., if $n \geq \nu_3$, the event indicated in (3.19) does not occur.

Consider the set of outcomes of probability 1 for which (3.17), (3.18), and the conclusion in the last paragraph all hold. Let $n \geq \nu_1, \nu_2, \nu_3$, and $y \in \mathbb{R}^D$, $|y| < c_1 2^{-n/2}(\lg n)^{1/2}$. We represent y as $y = \lim_{h\to\infty} y_h$, where

$$y_h = x + \theta_n \textstyle\sum_{j=1}^h \varepsilon_j 2^{-j} \quad (y_0 = x) \tag{3.20}$$

for some $x \in G_n$. Then each y_{h-1}, y_h is linked, so

$$\begin{aligned} &|\alpha(y + X_\tau, B_n) - \alpha(x + X_\tau, B_n)| \\ &\quad \leq b(\lambda_N(B_n))^{(1/N)(1-\gamma/2)} D^{\gamma/2} \theta_n^\gamma (\lg n)^N \textstyle\sum_{h=1}^\infty h^N 2^{-\gamma h} \\ &\quad = c_3 g(\lambda_N(B_n)). \end{aligned} \tag{3.21}$$

To complete the proof, we again choose $n \geq \nu_1, \nu_2, \nu_3$, and observe that $\alpha(y + X_\tau, B_n) = 0$ if $|y| \geq c_1 2^{-n/2}(\lg n)^{1/2}$, and, otherwise, $\alpha(y + X_\tau, B_n) \leq (b + c_3) g(\lambda_N(B_n))$ by (3.18) and (3.21). □

PROOF OF THEOREM 3. This is quite similar to the proof of Theorem 2. It suffices to consider the case $T = [0, 1]^N$.

Let $\mathscr{D}_n$ be the family of 2^{nN} cubes obtained by successive subdivision of T, let $\theta_n = 2^{-n/2}$, and let

$$G_n = \{x \in \mathbb{R}^D: |x| \leq n, x = \theta_n p \quad \text{for some} \quad p \in \mathbb{Z}^D\}.$$

Then

$$\lambda_N(B) = 2^{-nN} \quad \text{for} \quad B \in \mathscr{D}_n \quad \text{and} \quad \#G_n = \left(\frac{2n+1}{\theta_n}\right)^D \leq c_1 n^D 2^{nD/2}.$$

Let $g_1(\lambda) = \lambda^{1/N}|\lg\lambda|^N$, $\lambda > 0$. From (3.15) we have

$$\mathbb{P}\{\max_{x\in G_n}\alpha(x, B) \geq bg_1(\lambda_N(B)) \quad \text{for some} \quad B \in \mathcal{D}_n\}$$
$$\leq c_2 n^D 2^{nD/2}\, 2^{nN}\exp(-2\lg 2^{nN})$$
$$= c_2 n^D 2^{-n}.$$

Thus, by Borel-Cantelli,

(3.22) $\quad \max_{x\in G_n}\alpha(x, B) \leq bg_1(\lambda_N(B))$ for all $B \in \mathcal{D}_n$ whenever $n \geq \nu_1$, a.s.

Next, define $F(n, h, x)$, $x \in G_n$, by the same formula as in the proof of Theorem 2 (the symbols have new meanings, of course). Using (3.16) we have

$$\mathbb{P}\{|\alpha(y_1, B) - \alpha(y_2, B)| \geq b|y_1 - y_2|^\gamma(\lambda_N(B))^{(1/N)(1-\gamma/2)}h^N(\lg 2^{nN})^N$$
$$\text{for some } B \in \mathcal{D}_n,\ x \in G_n,\ h \geq 1, \text{ and linked pair } y_1, y_2 \in F(n, h, x)\}$$
$$(3.23) \qquad \leq c_2 n^D 2^{nD/2} 2^{nN} \textstyle\sum_{h=1}^\infty 2^{Dh}\exp(-2h\lg 2^{nN})$$
$$\leq c_3 n^D 2^{-n},$$

and thus, a.s., if $n \geq \nu_2$, the event in (3.23) does not occur.

Finally, since the trajectories are a.s. bounded—indeed, they are continuous—there exists ν_3 such that $|X_t| \leq \nu_3$.

Let $n \geq \nu_1, \nu_2, \nu_3$. For any $y \in \mathbb{R}^D$, if $|y| > n$, $\alpha(y, T) = 0$, whereas if $|y| \leq n$, we can write $y = \lim_h y_h$ with y_h as in (3.20), and, arguing as in (3.22), we find

$$|\alpha(y, B) - \alpha(x, B)| \leq c_4\theta_n^\gamma g_1(\lambda_N(B))2^{n\gamma/2}\textstyle\sum_{h=1}^\infty 2^{-h\gamma}h^N, \quad B \in \mathcal{D}_n,$$
$$(3.24) \qquad = c_5 g_1(\lambda_N(B)).$$

Putting this together with (3.22) we have

$$\alpha(y, B) \leq c_6 g_1(\lambda_N(B)) \quad \text{for all} \quad B \in \mathcal{D}_n, \quad n \geq \nu_1, \nu_2, \nu_3.$$

Now *any* cube B can be covered by at most 16^N dyadic cubes of edge length at most $e(B)$, and this concludes the proof.

4. Related results for d-dimensional Brownian motions. In forming the confluent Brownian motion X_t in (0.1), we used planar Brownian motion. If, instead, we use d-dimensional Brownian motions W_t^d in (0.1), we denote the resulting confluent random field by $X_t^{N,d}$. We now show that $X_t^{N,d}$ *has an occupation density for the following combinations of N and d, and no others*:

i) $d = 1$ or 2, all $N \geq 2$
ii) $d = 3$, $N = 2$.

The case $d = 2$ has been the main concern of the previous sections, so we need not treat it further; $d = 1$ can be handled along the same lines as $d = 2$, only a few changes being needed in the proof of (3.5).

Let $d \geq 4$, $N \geq 2$, and suppose $X_t^{N,d}$ has an occupation density with positive probability. A simple Fubini argument shows that there is a point $x \in \mathbb{R}^D$ such that the level set $M_x(\omega) = \{t: X_t^{N,d}(\omega) = x\}$ is nonempty with positive probability;

here $D = d(N-1)$ is the dimension of the range space of $X_t^{N,d}$. But then, with positive probability, there is a point $t \in \mathbb{R}_+^N$ such that $W_i^d(t_i) = W_{i+1}^d(t_{i+1}) + x_i$, $i = 1, \cdots, N-1$, which is impossible [6] if $d \geq 4$, so there is no occupation density. A similar argument using [7] handles the case $d = 3$, $N \geq 3$.

This leaves only the case $d = 3$, $N = 2$. It suffices to prove the finiteness of

$$J_{k,\gamma}([0,1]^2) = \int_{[0,1]^{2k}} \int_{\mathbb{R}^{3k}} \exp\left[-\frac{1}{2} V(\textstyle\sum_{j=1}^k \langle u^j, W_1^3(t_1^j) - W_2^3(t_2^j) \rangle) \right] d\bar{u}\, d\bar{t}.$$

To simplify the notation a bit, we will write everything with subscripts, and let $t_j = (r_j, s_j)$, $j = 1, \cdots, k$, $u = (u_1, \cdots, u_k)$, $r = (r_1, \cdots, r_k)$, $s = (s_1, \cdots, s_k)$. Using the independence of W_1^3 and W_2^3, we can rewrite the above integral thus:

$$J_{k,\gamma}([0,1]^2) = \int_{[0,1]^k} \int_{[0,1]^k} \int_{\mathbb{R}^{3k}} \psi(u,r)\psi(u,s)\xi(u)\, du\, dr\, ds, \tag{4.1}$$

where $\xi(u) = \prod_1^k |u_j|^\gamma$ and $\psi(u,r) = \exp[-(\frac{1}{2}) V(\sum_{j=1}^k \langle u_j, W(r_j) \rangle)]$ with $W = W^3$. We will show this is finite for $\gamma < \frac{1}{2}$.

Using the Schwarz inequality on the innermost integral and writing J for $J_{k,\gamma}(\cdots)$ we obtain

$$J^{1/2} \leq \int_{[0,1]^k} \left(\int_{\mathbb{R}^{3k}} \psi^2(u,r)\xi(u)\, du \right)^{1/2} dr.$$

As in Section 3, we break this integral into a sum of terms corresponding to the permutations $\pi \in S_k$. Write $\pi(u) = (u_{\pi(1)}, \cdots, u_{\pi(k)})$ and similarly for $\pi(r)$, and let

$$A = \{ r \in [0,1]^k : r_1 < r_2 < \cdots < r_k \}.$$

The last integral is then the sum over $\pi \in S_k$ of

$$I_\pi = \int_{\pi(A)} \left(\int_{\mathbb{R}^{3k}} \psi^2(u,r)\xi(u)\, du \right)^{1/2} dr = \int_A \left(\int_{\mathbb{R}^{3k}} \psi^2(u,\pi(r))\xi(u)\, du \right)^{1/2} dr$$

$$= \int_A \left(\int_{\mathbb{R}^{3k}} \psi^2(\pi(u),\pi(r))\xi(\pi(u))\, du \right)^{1/2} dr = \int_A \left(\int_{\mathbb{R}^{3k}} \psi^2(u,r)\xi(u)\, du \right)^{1/2} dr,$$

since $\psi(\pi(u), \pi(r)) = \psi(u,r)$ and $\xi(\pi(u)) = \xi(u)$. Thus I_π is independent of $\pi \in S_k$, so we can write I_0 for the common value, which we want to show is finite.

Consider the transformation T of $\mathbb{R}^{3k}$ given by

$$T(v_1, \cdots, v_k) = (v_1 - v_2, v_2 - v_3, \cdots, v_k - v_{k+1}), \quad v_{k+1} \equiv 0.$$

A simple calculation shows

$$\psi(T(v), r) = \exp[-\tfrac{1}{2} V(\textstyle\sum_{j=1}^k \langle v_j, W(r_j) - W(r_{j-1}) \rangle)]$$

$$= \exp(\tfrac{1}{2} \textstyle\sum_{j=1}^k |v_j|^2 (r_j - r_{j-1}),$$

where $r \in A$ and $r_0 \equiv 0$. Make a change of variables $u = T(v)$. We obtain

$$I_0 = \int_A \left(\int_{\mathbb{R}^{3k}} \psi^2(u, r)\xi(u)\, du \right)^{1/2} dr$$

$$= \int_A \left(\int_{\mathbb{R}^{3k}} \exp(- \textstyle\sum_{j=1}^k |v_j|^2 \Delta r_j) \sum_{j=1}^k |v_j - v_{j+1}|^\gamma\, dv \right)^{1/2} dr,$$

with $\Delta r_j = r_j - r_{j-1}$. Arguing as in [11], page 44, we dominate the inner integral by a sum of terms of the form

$$\int_{\mathbb{R}^{3k}} \textstyle\prod_{j=1}^k \exp(- |v_j|^2 \Delta r_j)\, |v_j|^{\gamma\beta_j}\, dv,$$

where each $\beta_j = 0$, 1 or 2. Now, since $x^{1/2}$ is subadditive, I_0 is bounded by a finite sum of terms of the form

$$\int_A \left(\int_{\mathbb{R}^{3k}} \textstyle\prod_{j=1}^k \exp(- |v_j|^2 \Delta r_j)\, |v_j|^{\gamma\beta_j}\, dv \right)^{1/2} dr,$$

and this becomes

$$\int_A \textstyle\prod_{j=1}^k \left(\int_{\mathbb{R}^3} \exp(- |v_j|^2 \Delta r_j)\, |v_j|^{\gamma\beta_j}\, dv_j \right)^{1/2} dr$$

$$= \text{const.} \int_A \textstyle\prod_{j=1}^k \frac{1}{(\Delta r_j)^{(\gamma\beta_j+3)/4}} \left(\int_0^\infty e^{-\sigma^2} \sigma^{\gamma\beta_j+2}\, d\sigma \right)^{1/2} dr.$$

The $d\sigma$ integral is clearly finite and pulls out. After a change of variables $s_j = r_j - r_{j-1}$, the remaining integral is dominated by

$$\int_{[0,1]^k} \textstyle\prod_{j=1}^k s_j^{-(\gamma\beta_j+3)/4}\, ds,$$

which is finite for $\gamma < \frac{1}{2}$.

5. Hausdorff dimensions. Let X_t be the CBM random field in (0.1) based on independent planar Brownian motions $W_1, \cdots, W_N$, and recall that the level sets are $M_x = \{t \in \mathbb{R}_+^N: X_t = x\}$.

(5.1) THEOREM. *With probability* 1, dim $M_x = 1$ *for every* $x \in \mathbb{R}^D$, $D = 2(N-1)$.

Recall that a point $z \in \mathbb{R}^2$ is a *confluence* if $z = W_i(t_i)$, $i = 1, \cdots, N$, for some $t = (t_1, \cdots, t_N)$ in $\mathbb{R}_+^N$, whence $t \in M_0$.

(5.2) COROLLARY. *With probability* 1, *the set C of confluences has dimension* 2.

The results for C and M_0 are known, Wolpert [21]; and Ehm [8] gives a partial result along the lines of (5.1) for stable sheets. We are able to use the Hölder continuity of the trajectories of CBM (lacking for stable fields) to give the complete result. Extensions to confluences of other processes than Brownian motion, and to self-intersections of various processes, are also possible; see Rosen [14], [15].

The proof of (5.1) is based on the following results, similar to those of Tran [19] for the Brownian sheet; see also Ehm, [8], Lemma 6.2.

(5.3) THEOREM. a) $\mathbb{P}\{\alpha(0, Q_t) > 0$ *for all* $t > 0\} = 1$; b) $\mathbb{P}\{\alpha(x, \mathbb{R}_+^N) > 0$ *for all* $x\} = 1$.

Remember that $Q_t = \prod_{i=1}^N [0, t_i]$ and $t > 0$ means all $t_i > 0$. The occupation density α in (5.3) is the jointly continuous version.

Assuming (5.3) for the moment, here is the proof of (5.1). We begin by noting that X_t satisfies a Hölder condition of order δ, for any $\delta < \frac{1}{2}$. Since X: $\mathbb{R}^N \to \mathbb{R}^D$ has a jointly continuous occupation density, a theorem of Adler [2], page 230, gives $\dim M_x \le N - \delta D$ for all x and each $\delta < \frac{1}{2}$, whence $\dim M_x \le 1$ for all x. Next, combining Theorem 3 of Section 1, Theorem 8.7.4 of Adler [2], and (5.3b) above, we have $\dim M_x \ge 1$ for all x.

Corollary (5.2) follows as in Kaufman's [12] theorem. More precisely, Kaufman's lemma shows that for planar Brownian motion $W(t)$, $\mathbb{P}(F) = 1$, where F denotes the set of $W(\cdot)$ for which there exists an N such that for any disc B of radius 2^{-n}, $n \ge N$, $W^{-1}(B) \cap [0, 1]$ is contained in the union of less than n^4 intervals of the form $[k4^{-n}, (k+1)4^{-n}]$, $0 \le k \le 4^n$.

Now let $\vec{W}(t) = (W_1(t_1), \cdots, W_N(t_N))$ be a path in F^N for which $\dim(M_0 \cap [0, 1]^N) = 1$. Assume that $\dim(C) = \alpha < 2 - \varepsilon$. Then for any $\delta > 0$, we can find arbitrarily large n, and a sequence of discs B_i of radii $\le 2^{-n}$ such that $C \subset \cup_i B_i$ and

$$(*) \qquad \textstyle\sum_i (\text{radius } B_i)^{2-\varepsilon} \le \delta.$$

Define n_i by $2^{-n_i} \le \text{radius } B_i \le 2^{-(n_i-1)}$ and let $\bar{B}_i$ be the disc centered at B_i with radius $2^{-(n_i-1)}$. Then

$$M_0 \cap [0, 1]^N \subseteq \cup_i \vec{W}^{-1}(\bar{B}_i^N) \cap [0, 1]^N \subseteq \cup_i \{\text{the union of} \le n_i^{4N} \text{ cubes of the form } \textstyle\prod_{l=1}^N [k_l 4^{-n_i+1}, (k_l+1)4^{-n_i+1}]\}.$$

Let I_{ij}, $1 \le j \le n_i^{4N}$ denote these cubes. By (*) for n_i large,

$$\sum_{i,j} e(I_{ij}) \le \sum_i n_i^{4N} 4^{-n_i+1} \le 4 \sum_i (n_i^{4N} 4^{-n_i\varepsilon/2}) 4^{-n_i(1-\varepsilon/2)}$$
$$\le \sum_i 2^{-n_i(2-\varepsilon)} \le \sum_i (\text{radius } B_i)^{2-\varepsilon} \le \delta.$$

This would imply $\dim(M_0 \cap [0, 1]^N) < 1$, and this contradiction shows that we must have $\dim(C) \ge 2$, and hence exactly 2.

Turning to (5.3), we will prove (b) first, assuming the result of part (a). Let

$$a_n = \mathbb{P}\{\alpha(x, \mathbb{R}_+^N) > 0 \textit{ for all } x \textit{ such that } |x| \le 1/n\}.$$

By scaling, we have $a_n \equiv a$. Suppose $a < 1$. Then $\mathbb{P}(B_n) = 1 - a > 0$, where

(5.4) $$B_n = \{\alpha(x, \mathbb{R}_+^N) = 0 \text{ for some } x, |x| \le 1/n\}.$$

But then we would have $\mathbb{P}(\cap_n B_n) = 1 - a > 0$. However,

$$\begin{aligned} \cap_n B_n &= \{\alpha(x, \mathbb{R}_+^N) = 0 \text{ for some sequence } x = x_k \to 0\} \\ &\subset \{\alpha(x, Q_t) = 0 \text{ for some sequence } x = x_k \to 0\} (t > 0 \text{ fixed}) \\ &= \{\alpha(0, Q_t) = 0\} \quad \text{a.s.} \end{aligned}$$

since $\alpha(x, Q_t)$ is a.s. jointly continuous. Consequently, $\mathbb{P}(\cap_n B_n) = 0$, contradicting $\mathbb{P}(\cap_n B_n) = 1 - a > 0$ and (b) follows.

Here is a sketch of the proof of (a). It suffices to show that $\mathbb{P}(A) = 1$ where A is the event $\{\alpha(0, [0, 1/n]^N) > 0 \text{ for all } n \ge 1\}$. Since $\mathbb{E}\,\alpha(0, [0, 1/n]^N) > 0$ by (2.6) we have $\mathbb{P}\{\alpha(0, [0, 1/n]^N) > 0\} = b_n > 0$. Using the scaling property again, $b_n \equiv b$, and so $\mathbb{P}(A) = b > 0$. We will develop an analogue of the Blumenthal zero-one law [4], page 30. Since A is a "germ event" of positive probability, it will follow that $\mathbb{P}(A) = 1$.

We may take the probability space Ω to be the space of continuous functions from $\mathbb{R}_+^N$ to $\mathbb{R}^{2N}$; we introduce coordinate functions $W_t = (w_1(t_1), \cdots, w_N(t_N))$, $t \in \mathbb{R}_+^N$, σ-fields $\mathscr{F}_t^0 = \sigma\{W_s : s \le t\}$ ($s \le t$ means $s_i \le t_i$ for all i) and $\mathscr{F}^0 = \sigma\{W_s: s \in \mathbb{R}_+^N\}$, and translation operators $\theta_t : \Omega \to \Omega$ such that $W_s \circ \theta_t = W_{s+t}$. Let $\mathbb{P}^x$ be a (regular conditional) probability on Ω for each $x \in \mathbb{R}^{2N}$ such that the coordinates $w_1(t_1), \cdots, w_N(t_N)$ are independent planar Brownian motions; each of these has its own structure as a Markov process. Finally, we write $\mathbb{P}$ for $\mathbb{P}^0$, and we complete $\mathscr{F}^0$ under $\mathbb{P}$, obtaining a σ-field $\mathscr{F}$; then we "augment" $\mathscr{F}_t^0$ by adjoining all sets in $\mathscr{F}$ of $\mathbb{P}$-measure zero, obtaining $\mathscr{F}_t$. These are not the same completions as in [4], Chapter I, but they suffice for our purposes.

The result is a process $(W_t, \mathscr{F}, \mathscr{F}_t, \mathbb{P}^x, \theta_t)$ as in [4], except that now $t \in \mathbb{R}_+^N$. Writing $\mathscr{F}_{t+} = \cap_{\varepsilon > 0} \mathscr{F}_{t+\varepsilon}$, where $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_N)$, we have

(5.5) Lemma. *For any $\mathscr{F}^0$-measurable nonnegative random variable Y,*

$$\mathbb{E}[Y \circ \theta_t | \mathscr{F}_{t+}] = \mathbb{E}^{W_t}[Y].$$

This can be proven along the lines in [4], but we give a more direct argument. Consider a family of bounded, continuous functions ϕ_{ij} on $\mathbb{R}^2$, and times $t^1, \cdots, t^n \in \mathbb{R}_+^N$. It suffices to prove (5.5) for $Y = \prod_{i=1}^N Y_i$, where

$$Y_i = \prod_{j=1}^n \phi_{ij}(w_i(t_i^j));$$

we can use n instead of n_i by taking some of the $\phi_{ij} = 1$. To begin with, assume that each $t_i^j > 0$, and let

$$A = \cap_{i=1}^N A_i, \quad A_i \in \sigma\{w_i(s_i): s_i \le t_i + \varepsilon_i\},$$

where $0 < \varepsilon_i < t_i^j$ for all i, j. An elementary computation gives

(5.6) $$\mathbb{E}[Y \circ \theta_t; A] = \mathbb{E}[\textstyle\prod_i \mathbb{E}_i^{w_i(t_i + \varepsilon_i)}(\prod_j \phi_{ij}(w_i(t_i^j - \varepsilon_i))); A]$$

using the Markov property for each of the coordinate processes; $\mathbb{E}_i^{x_i}$ is the expectation corresponding to $\mathbb{P}_i^{x_i}$, the regular conditional probability for the ith coordinate process. The family of sets A described above is a π-system which generates $\mathscr{F}^0_{t+\varepsilon}$, whence [4], page 7, (5.6) holds for all $A \in \mathscr{F}^0_{t+\varepsilon}$, and, also, $\mathscr{F}_{t+\varepsilon}$. Now let $A \in \mathscr{F}_{t+}$; then (5.6) holds for every ε, $0 < \varepsilon < t$, and, letting $\varepsilon \downarrow 0$, we obtain

$$(5.7) \quad \mathbb{E}[Y\circ\theta_t; A] = \mathbb{E}[\textstyle\prod_i \mathbb{E}_i^{w_i(t_i)}(\prod_j \phi_{ij}(w_i(t_i^j))); A] = \mathbb{E}[\mathbb{E}^{W_t}\prod_{ij}\phi_{ij}(w_i(t_i^j)); A].$$

If some of the $t_i^j = 0$, we can let $t_i^j \downarrow 0$ in (5.7) to get a valid equation, so that (5.7) now holds for all $t^j \in \mathbb{R}_+^N$, and (5.5) is proven.

We now mimic the proof of (8.13) in [4], page 43, but with our interpretation of $\mathscr{F}_t^0$ and $\mathscr{F}_t$. For Y as above, we have

$$Y = \textstyle\prod_{i,j: t_i^j \le t_i} \phi_{ij}(w_i(t_i^j)) \times \prod_{i,j: t_i^j > t_i} \phi_{ij}(w_i(t_i^j)).$$

Taking the conditional expectation given $\mathscr{F}^0_{t+}$, the first factor "pulls across" and the second plays the role of $G\circ\theta_t$ in [4]. The conclusion is

$$\mathbb{E}[Y\,|\,\mathscr{F}^0_{t+}] = \mathbb{E}[Y\,|\,\mathscr{F}^0_t],$$

and, as in [4], (8.12), page 42, that

(5.8) LEMMA. $\mathscr{F}_{t+} = \mathscr{F}_t$.

These are the necessary ingredients for the Blumenthal zero-one law [4], page 30: $\mathbb{P}(A) = 0$ or 1 for $A \in \mathscr{F}_0$. It remains to prove that $A \in \mathscr{F}_0$, where

$$A = \{\alpha(0, [0, 1/n]^N) > 0 \text{ for all } n\}.$$

Suppose $f\colon \mathbb{R}^{2N} \to \mathbb{R}$ is continuous. Approximating by Riemann sums on the event where W is continuous we see that $\int_{Q_t} f(W_s)\, ds$ is $\mathscr{F}_t$-measurable, and this extends to the case where f is only Borel measurable. Now, a.s.,

$$\alpha(0, Q_t) = \lim_{\delta\downarrow 0} \frac{1}{2\delta} \int_{Q_t} I_{(0,\delta)}(|X_s|)\, ds,$$

so $\alpha(0, Q_t)$ is $\mathscr{F}_t$-measurable, and $A \in \mathscr{F}_{0+} = \mathscr{F}_0$.

REFERENCES

[1] ADLER, R. (1978). The uniform dimension of the level sets of a Brownian sheet. *Ann. Probab.* **6** 509–515.

[2] ADLER, R. (1981). *The Geometry of Random Fields.* Wiley, New York.

[3] BERMAN, S. (1973). Local nondeterminism and local times of Gaussian processes. *Indiana Univ. Math. J.* **23** 69–94.

[4] BLUMENTHAL, R. and GETOOR, R. (1968). *Markov Processes and Potential Theory.* Academic, New York.

[5] CUZICK, J. (1983). Multiple points of a Gaussian vector field. Preprint.

[6] DVORETZKY, A., ERDÖS, P., and KAKUTANI, S. (1950). Double points of paths of Brownian motion in n-space. *Acta Sci. Math. Szeged.* **12** 75–81.

[7] DVORETZKY, A., ERDÖS, P., KAKUTANI, S., and TAYLOR, S. J. (1957). Triple points of Brownian paths in 3-space. *Proc. Camb. Phil. Soc.* **53** 856–862.

[8] EHM, W. (1981). Sample function properties of multiparameter stable processes. *Z. Wahrsch. verw. Gebiete* **56** 195–228.

[9] FRISTEDT, B. (1967). An extension of a theorem of S.J. Taylor concerning the multiple points of the symmetric stable process. *Z. Wahrsch. verw. Gebiete* **9** 62–64.

[10] GARSIA, A. (1971). Continuity properties of Gaussian processes with multidimensional time parameter. *Proc. 6th Berk. Symp. Math. Statist. Probab. II* 369–374.

[11] GEMAN, D. and HOROWITZ, J. (1980). Occupation densities. *Ann. Probab.* **8** 1–67.

[12] KAUFMAN, R. (1969). Une propriété métrique du mouvement brownien. *C.R. Acad. Sci. Paris* **268** A727–728.

[13] PITT, L. (1978). Local times for Gaussian vector fields. *Indiana Univ. Math. J.* **27** 309–330.

[14] ROSEN, J. (1984). Self-intersections of random fields. *Ann. Probab.* **12** 108–119.

[15] ROSEN, J. (1983). A local time approach to the self-intersections of Brownian paths in space. *Comm. Math. Physics*, to appear.

[16] SAKS, S. (1964). *Theory of the Integral.* Dover, New York.

[17] SYMANZIK, K. (1969). Euclidean quantum field theory. In *Local Quantum Theory* (R. Jost, ed.). Academic, New York.

[18] TAYLOR, S.J. (1966). Multiple points for the sample paths of the symmetric stable process. *Z. Wahrsch. verw. Gebiete* **5** 247–264.

[19] TRAN, L. (1976). On a problem posed by Orey and Pruitt related to the range of the N-parameter Wiener process in $\mathbb{R}^d$. *Z. Wahrsch. verw. Gebiete* **37** 27–33.

[20] WESTWATER, M.J. (1980). On Edwards' model for long polymer chains. *Comm. Math. Phys.* **72** 131–174.

[21] WOLPERT, R. (1978) Wiener path intersections and local time. *J. Functional Anal.* **30** 329–340.

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