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THE CONCAVE MAJORANT OF BROWNIAN MOTION[1]

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Let S_t be a version of the slope at time t of the concave majorant of Brownian motion on $[0, \infty)$. It is shown that the process $S = \{1/S_t : t > 0\}$ is the inverse of a pure jump process with independent nonstationary increments and that Brownian motion can be generated by the latter process and Brownian excursions between values of the process at successive jump times. As an application the limiting distribution of the L_2-norm of the slope of the concave majorant of the empirical process is derived.

1. Introduction and summary of results. Let w denote Brownian motion defined on $[0, \infty)$ and starting at the origin. For $a > 0$, let $\sigma(a)$ be the last (and, with probability one, the only) time that the maximum of $w(t) - at$ is attained, that is:

$$\sigma(a) = \sup\{t > 0 : w(t) - at = \sup_{u>0}(w(u) - au)\}. \tag{1.1}$$

Let $\tau(a) = \sigma(1/a)$, if $a > 0$, and let $\tau(0) = 0$.

In Section 2 the distribution of the process $\{\tau(a) : a \geq 0\}$ is derived and it is shown that τ is a pure jump process with independent nonstationary increments and increasing paths. In addition, the representation of the process as an integral with respect to a Poisson measure is determined and it is shown that the number of jumps of τ in an interval (a, b), $0 < a < b < \infty$, is Poisson distributed with mean $\log(b/a)$.

Let f denote the concave majorant of w (i.e. the smallest concave function $\geq w$), and let $S_t = f'(t)$, whenever f' is defined. Since f is concave, f' is meaningful except at an at most countable number of points. The process $\{1/S_t : t > 0\}$ is the inverse of the process τ, and properties of the rather complicated (non-Markovian) process of slopes S_t of the concave majorant of w are derived from properties of the simpler process τ. For example, we derive the distribution of S_t for fixed points in time. We also give a decomposition of Brownian motion in terms of jump times of the slope process and Brownian excursions in between. All these results are given in Section 2.

In Section 3 we study a particular transformation of the process τ into a pure jump process with *stationary* independent increments. This transformation is used to derive the distribution of the L_2-norm of the slope of the concave majorant of Brownian motion over particular (random) intervals. These results in turn are used to establish the asymptotic normality of the L_2-norm of the slope of the concave majorant of the empirical process (this result has first been proved by alternate methods in Groeneboom and Pyke, (1983)).

The derivation of the structure of the τ-process is based on certain results on path decomposition of downward drifting Brownian motion at its maximum (Williams, 1974, Rogers and Pitman, 1981) and in particular on the fact that the pre-maximum and post-maximum process are independent, given the value of the maximum M and the splitting time $\zeta_M = \sup\{t \geq 0 : w(t) = M\}$ (a result which holds more generally, see e.g. Millar, 1978). This derivation was suggested to me by Robert Blumenthal, and it replaces the more pedestrian approach in Groeneboom (1981).

The behavior of the slope of the concave majorant of the process $\{w(t) - f(t) : t \in \mathbb{R}\}$ at zero for two-sided Brownian motion w and particular functions f is studied in Leurgans (1982) and earlier results of this type are given in Chernoff (1964) and Prakasa Rao (1969).

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For an application of these results in statistics (in particular the estimation of densities) we refer to Barlow *et al* (1972).

2. Concave majorant of Brownian motion and Brownian excursions. The properties of the process $\{\tau(a) : \tau(a) = \sigma(1/a),\ a > 0,\ \tau(0) = 0\}$, where $\sigma(b)$ is defined by (1.1), are summarized in the following theorem.

THEOREM 2.1. *The process τ is a pure jump process with independent nonstationary increments and right-continuous increasing paths. The process has the following representation*

$$\tau(a) = \int_{0+}^{\infty} l\not{p}([0, a] \times dl), \quad a \geq 0, \tag{2.1}$$

where $\not{p}(da \times dl)$ is a Poisson measure with mean $n(da \times dl) = (a^2\sqrt{l})^{-1}\phi(\sqrt{l}/a)\,da\,dl$, for $a > 0$, $l > 0$, and $\phi(x) = (2\pi)^{-1/2}\exp(-\tfrac{1}{2}x^2)$, $x \in \mathbb{R}$.

The marginal density of $\tau(a)$ has Laplace transform

$$E\exp(-\lambda\tau(a)) = 2/(1 + (2\lambda a^2 + 1)^{1/2}), \quad a \geq 0, \lambda > 0, \tag{2.2}$$

and $\tau(b) - \tau(a)$ has Laplace transform

$$\begin{aligned} &E\exp(-\lambda(\tau(b) - \tau(a))) \\ &\qquad = (1 + (2\lambda a^2 + 1)^{1/2})/(1 + (2\lambda b^2 + 1)^{1/2}),\ b > a \geq 0, \lambda > 0. \end{aligned} \tag{2.3}$$

The number of jumps of τ in an interval (a, b), $0 < a < b < \infty$, has a Poisson distribution with mean $\log(b/a)$.

PROOF. Let $BM^0(\mu)$ denote Brownian motion with drift μ, starting at the origin and defined on $[0, \infty)$. Let B denote a $BM^0(-1/a)$ process, where $a > 0$. For $0 \leq t \leq \infty$, let $M_t = \sup_{0 \leq s < t} B_s$ and let $\rho = \sup\{t \geq 0 : B_t = M_\infty\}$. According to Corollary 2, page 580 in Rogers and Pitman (1981), the post-maximum process $\{M_\infty - B_{\rho+\mu}: u \geq 0\}$ is a $Bes^0(3, 1/a)$ process (radial part of 3-dimensional Brownian motion with drift of magnitude $1/a$ started at the origin), independent of $\{B_t, 0 \leq t \leq \rho\}$. Moreover, M_∞ is exponentially distributed with rate $2/a$.

It is seen from the definition of $\sigma(a)$, that $\{\sigma(a), a > 0\}$ has decreasing left-continuous paths, and hence τ has increasing right-continuous paths.

Let $\xi_y = \inf\{t \geq 0 : B_t = y\}$. An easy first-passage argument shows that $E\{\exp(-\lambda\xi_y) \mid \xi_y < \infty\} = \exp\{-y((1 + 2\lambda a^2)^{1/2} - 1)/a\}$, $\lambda > 0$, $y > 0$. Thus,

$$\begin{aligned} E\exp(-\lambda\tau(a)) &= E\exp(-\lambda\xi_{M_\infty}) \\ &= a^{-1}\int_0^\infty \exp\{-y((2\lambda a^2 + 1)^{1/2} - 1)\}2\exp(-2y/a)\,dy \\ &= 2/(1 + (1 + 2\lambda a^2)^{1/2}). \end{aligned}$$

Moreover, since $\tau(a)$ is independent of $\tau(b) - \tau(a)$,

$$\begin{aligned} E\exp\{-\lambda(\tau(b) - \tau(a))\}E\exp(-\lambda\tau(a)) &= E\exp(-\lambda\tau(b)) \\ &= 2/(1 + (2\lambda b^2 + 1)^{1/2}). \end{aligned}$$

Hence $E\exp\{-\lambda(\tau(b) - \tau(a))\} = \{1 + (2\lambda a^2 + 1)^{1/2}\}/\{1 + (2\lambda b^2 + 1)^{1/2}\}$. This proves (2.2) and (2.3).

Next we note that for $b > a \geq 0$,

$$E\exp\{-\lambda(\tau(b) - \tau(a))\} = \exp\left\{-\int_0^\infty (1 - e^{-\lambda l})l^{-1/2}\left(\int_a^b t^{-2}\phi(l^{+1/2}/t)\,dt\right)dl\right\},$$

and (2.1) follows (see e.g. Itô and McKean (1974), page 146). This representation shows that τ is a pure jump process, and that the number of jumps in an interval (a, b), $0 < a < b < \infty$, is distributed according to a Poisson distribution with mean

$$\int_0^\infty l^{-1/2}\left(\int_a^b t^{-2}\phi(l^{1/2}/t)\,dt\right)dl = \log(b/a). \qquad \square$$

The next corollary gives the density of $\tau(a)$ and the distribution function (df) of the increment $\tau(b) - \tau(a)$, which are obtained from Theorem 2.1 by inverting the Laplace transforms. Here and in the sequel we use the notation

$$x_+ = x, \quad \text{if } x \geq 0 \qquad (2.4)$$
$$\quad\;\; 0, \quad \text{otherwise.}$$

COROLLARY 2.1. *The density of $\tau(a)$ at $t > 0$ is given by*

$$f_a(t) = (2/a^2)E(aX/\sqrt{t} - 1)_+, \qquad (2.5)$$

where X is a standard normal random variable.

Let F be the df *degenerate at zero (i.e. $F = 1_{[0,\infty)}$) and let, for $b > a > 0$, the* df *$G_{a,b}$ be defined by the density*

$$g_{a,b}(t) = \frac{2a}{(b-a)}E\{(Y/\sqrt{t} - b^{-1})_+^{1/2}(a^{-1} - Y/\sqrt{t})_+^{1/2} - Z/\sqrt{t}\}_+^2 1_{(b^{-1},a^{-1})}(y/\sqrt{t}), \quad t > 0, \qquad (2.6)$$

where Y and Z are independent standard normal random variables. Then the random variable $\tau(b) - \tau(a)$ has the df

$$H_{a,b} = (a/b)F + (1 - a/b)G_{a,b}. \qquad (2.7)$$

We now relate the process of slopes S_t of the concave majorant of Brownian motion starting at the origin to the process τ. Since the concave majorant is a concave function, the slopes S_t are decreasing as t increases. At points where the slope of the concave majorant changes, we define S_t by $S_t = \lim_{u\uparrow t} S_u$. The concave majorant of a sample path consists almost surely of countably many straight pieces and the sample path of $\{S_t : t > 0\}$ jumps at distinct values of τ. Conversely, the set of slopes of the concave majorant is the same as the set of jump times of τ.

By the laws of the iterated logarithm for Brownian motion (Itô and McKean (1974), pages 33 and 34) we have almost surely

$$\lim_{t\downarrow 0} S_t = \infty \quad \text{and} \quad \lim_{t\to\infty} S_t = 0.$$

Finally we have the equivalence

$$1/S_t \leq a \Leftrightarrow \tau(a) \geq t. \qquad (2.8)$$

These observations become immediately clear by drawing a picture and using the properties of the process τ. Relation (2.8) yields the following corollary.

COROLLARY 2.2. *The density of the slope S_t of the concave majorant at time $t > 0$ of Brownian motion starting at the origin is given by*

$$g_t(a) = 4\{\sqrt{t}\phi(a\sqrt{t}) - at\bar{\Phi}(a\sqrt{t})\}, \quad a > 0, \qquad (2.9)$$

where ϕ is the standard normal density and $\bar{\Phi} = 1 - \Phi$, $\Phi(x) = \int_{-\infty}^x \phi(t)\,dt$.

PROOF. By (2.5) and (2.8) we have

$$P\{S_t \geq 1/a\} = P\{\tau(a) \geq t\} = (2/a^2)\int_t^\infty E\{aX/\sqrt{u} - 1\}_+\,du,$$

where X is a standard normal random variable. Hence,

$$P\{S_t \geq a\} = 2a \int_t^\infty du \int_{a\sqrt{u}}^\infty (x/\sqrt{u} - a)\phi(x)\, dx$$

and

$$\begin{aligned} g_t(a) &= -2 \int_t^\infty du \int_{a\sqrt{u}}^\infty (x/\sqrt{u} - 2a)\phi(x)\, dx \\ &= -2 \int_{a\sqrt{t}}^\infty \phi(x)\, dx \int_t^{x^2/a^2} (x/\sqrt{u} - 2a)\, du \\ &= 4 \int_{a\sqrt{t}}^\infty (x\sqrt{t} - at)\phi(x)\, dx \\ &= 4\sqrt{t}\{\phi(a\sqrt{t}) - a\sqrt{t}\,\bar{\Phi}(a\sqrt{t})\}. \quad \square \end{aligned}$$

We note that $g_t(a)$ can be written

(2.10) $$g_t(a) = 4\sqrt{t}\,\bar{\Phi}(a\sqrt{t})\{\phi(a\sqrt{t})/\bar{\Phi}(a\sqrt{t}) - a\sqrt{t}\},$$

where $\phi(a\sqrt{t})/\bar{\Phi}(a\sqrt{t})$ denotes the "failure rate" of the standard normal distribution and $a\sqrt{t}$ denotes the asymptotic failure rate (as $a\sqrt{t} \to \infty$).

We now consider the decomposition of Brownian motion into the process τ and Brownian excursions between values of τ at successive jump times. A Brownian excursion on [0, 1] with origin (0, 0) and endpoint (1, 0) (where the first coordinate denotes time and the second coordinate position) is a nonhomogeneous Markov process $\{z(t) : t \in [0, 1]\}$ with marginal densities

(2.11) $$f_{z(t)}(x) = 2x^2 \exp\{-x^2/(2t(1-t))\}/\{2\pi t^3(1-t)^3\}^{1/2}$$

and transition densities

(2.12) $$\begin{aligned} f_{z(t)|z(s)}(y \mid x) &= (n_{t-s}(y - x) - n_{t-s}(y + x)) \\ &\quad \cdot (1-s)^{3/2} y \exp\{-y^2/(2(1-t))\} \\ &\quad \cdot \{(1-t)^{3/2} x \exp\{-x^2/(2(1-s))\}\}^{-1}, \end{aligned}$$

where

(2.13) $$n_u(x) = u^{-1/2}\phi(x/u),$$

and ϕ is the standard normal density (see e.g. Itô and McKean (1976), page 76). More generally, we can consider excursions $\bar{z}$ on an interval $[a, b]$, which are obtained from the preceding ones by putting $\bar{z}(t) = \sqrt{b-a}\; z((t-a)/(b-a))$. We will show that, between successive jump times T_i and T_{i+1} of the slope process, the vertical distance of a Brownian motion sample path to the concave majorant behaves as such an excursion on the interval $[T_i, T_{i+1}]$.

We enumerate the jump times of the slope process in the following way. Let $a_0 = a_0(w) = \inf\{a > 0 : \tau(a) \geq 1\}$, where w denotes a Brownian motion sample path. Next, number the jump times of a sample path of the process τ recursively by taking a_{i+1} = jump time following a_i, $i \geq 0$, and a_{i-1} = jump time preceding a_i, $i \leq 0$. This enumeration is possible on a set of probability one, since almost surely 0 and ∞ are the only cluster points of the set of jump times of a sample path of τ. The jump times of the slope process are now given by $T_i = \tau(a_i)$, where i runs through the set of integers. In the sequel we restrict our attention to the set of probability one where the enumeration can be carried out, without further mentioning.

LEMMA 2.1. (i) *Let* $N_t^- = \sup\{T_i : T_i < t\}$ *and* $N_t^+ = \inf\{T_i : T_i \ge t\}$ *be the jump times of the slope process preceding and following t, respectively, and let* $\{w(t) : t \ge 0\}$ *be standard Brownian motion on* $[0, \infty)$ *starting at the origin. Then the conditional density of the vertical distance* $y(t)$ *of* $w(t)$ *to the concave majorant, given* $N_t^- = t_1$, $N_t^+ = t_2$, $0 < t_1 < t < t_2 < \infty$, *is*

$$f_{y(t)|N_t^-, N_t^+}(y \mid t_1, t_2) = 2cy_+^2\phi(y/\sigma), \tag{2.14}$$

where $c = \sigma^{-3} = \{(t - t_1)(t_2 - t)/(t_2 - t_1)\}^{-3/2}$.

(ii)
$$\begin{aligned} P\{T_i \in dt_i,\ i = -n, \cdots, n,\ z_i(\bar{u}_{ik}) \in dy_{ik},\ i = -n+1, \cdots n,\ k = 1, \cdots, n\} \\ = \{\textstyle\prod_{i=-n+1}^{n} f_{u_{i1},\cdots u_{im}}(y_{i1}, \cdots, y_{im})\} \\ E\{2(X_n/\sqrt{v_n})_+(X_{-n}/\sqrt{t_{-n}} - X_{-n+1}/\sqrt{v_{-n+1}})_+ \\ v_{-n+1}^{-1} \textstyle\prod_{i=-n+2}^{n} v_i^{-1} \cdot 1_{(0,\infty)}(X_{i-1}/\sqrt{v_{i-1}} - X_i/\sqrt{v_i})\} \\ \cdot dt_{-n} \cdots dt_n\, dy_{-n+1,1} \cdots dy_{nm}, \end{aligned}$$

where $v_i = t_i - t_{i-1}$, $i = -n+1, \cdots, n$; $0 < t_{-n} < \cdots < t_{-1} < 1 \le t_0 < \cdots < t_n$; $0 < u_{i1} < \cdots < u_{im} < 1$; $i = -n+1, \cdots, n$; $X_{-n}, \cdots, X_n$ *are independent standard normal random variables, and* $f_{u_{i1},\cdots,u_{im}}(y_{i1}, \cdots, y_{im})$ *is the joint density of* $(z(u_{i1}), \cdots z(u_{im}))$ *at* $(y_i, \cdots, y_{im})$, *with z the excursion process defined by* (2.11) *and* (2.12).

PROOF. *ad(i)*: Fix $h_1 > 0$ and $h_2 > 0$, where $t_1 + h_1 < t$, let $x_2 > x_1 > 0$, and let

$$M_i(h_i) = \max_{t_i \le z \le t_i + h_i}(w(z) - w(t_i)),\ i = 1, 2.$$

Then we have the following equivalence:

> $N_t^- \in (t_1, t_1 + h_1)$ and $N_t^+ \in (t_2, t_2 + h_2)$, given $w(t_1) = x_1$ and $w(t_2) = x_2$ if and only if $w(u) - bu \le x_1 + M_1(h_1) - b\sigma_1$ for $u < t_1$ and $w(u) - bu \le x_2 + M_2(h_2) - b\sigma_2$ for $u \in (t_1 + h_1, t_2)$ and $u > t_2 + h_2$,

where $b = (x_2 + M_2(h_2) - x_1 - M_1(h_1))/(\sigma_2 - \sigma_1)$ and σ_i is the location of the maximum $M_i(h_i)$. Note that b is the slope of the concave majorant of the Brownian motion sample path on the interval (σ_1, σ_2).

We now condition on the values and locations of the maxima $M_i(h_i)$, the value of $Y_i(h_i) = w(t_i + h_i) - w(t_i)$, $i = 1, 2$ and the values $w(t_1) = x_1$, $w(t_2) = x_2$ and $w(t) = x$. Denote this condition by C. Then we have

$$\begin{aligned} &P\{w(z) - bz \le x_1 - b\sigma_1 + M_1(h_1) \quad \text{for} \quad z < t_1 \mid C\} \\ &= 1 - \exp\{-2(M_1(h_1) + b(t_1 - \sigma_1))_+(x_1 + M_1(h_1) - b\sigma_1)_+/\sigma_1\}. \end{aligned} \tag{2.15}$$

Here we use some well-known inequalities for Brownian bridges and Brownian motion. For example we have

$$\begin{aligned} &P\{w(z) > az + b \quad \text{for some} \quad z \in (t_1, t_2) \mid w(t_1) = x,\ w(t_2) = y\} \\ &= \exp\{-2(at_1 + b - x)_+(at_2 + b - y)_+/(t_2 - t_1)\}, \end{aligned} \tag{2.16}$$

where $0 < t_1 < t_2 < \infty$ (see e.g. Hájek and Sidák (1967), page 183). The equality (2.16) also holds if $x = t_1 = 0$. We also have

$$\begin{aligned} &P\{w(z) > az + b \quad \text{for some} \quad z > t \mid w(t) = x\} \\ &= \exp\{-2a_+(at + b - x)_+\} \end{aligned} \tag{2.17}$$

(see e.g. Doob, 1949).

Using relations (2.16) and (2.17), we get the following equalities, similar to (2.15):

$$P\{w(z) - bz \leq x_1 + M_1(h_1) - b\sigma_1 \quad \text{for} \quad z \in (t_1 + h_1, t) | C\}$$

$$= 1 - \exp\{-2(b(t_1 + h_1) + M_1(h_1) - b\sigma_1 - Y_1(h_1))_+ (bt + x_1 + M_1(h_1) - b\sigma_1 - x)_+/(t - t_1 - h_1)\}, \tag{2.18}$$

$$P\{w(x) - bz \leq x_2 + M_2(h_2) - b\sigma_2 \quad \text{for} \quad t < z < t_2 | C\}$$

$$= 1 - \exp\{-2(bt + x_2 + M_2(h_2) - b\sigma_2 - x)_+ (bt_2 + M_2(h_2) - b\sigma_2)_+/(t - t_2)\}, \tag{2.19}$$

and finally

$$P\{w(z) - bz \leq x_2 + M_2(h_2) - b\sigma_2 \quad \text{for} \quad z > t_2 + h_2 | C\}$$
$$= 1 - \exp\{-2b_+(bt_2 + x_2 + M_2(h_2) - b\sigma_2 - x_2 - Y_2(h_2))_+\}. \tag{2.20}$$

Multiplying the right-hand sides of (2.15), (2.18), (2.19) and (2.20) and taking the expectation with respect to $M_i(h_i)$ and $Y_i(h_i)$, $i = 1, 2$, yields as the conditional probability that $N_t^- \in (t_1, t_1 + h_1)$ and $N_t^+ \in (t_2, t_2 + h_2)$, given $w(t_i) = x_i$, $i = 1, 2$, and $w(t) = x$:

$$16\{t_1(t - t_1)(t_2 - t_1)\}^{-1} a(x_1 - at_1)_+ \{a(t - t_1) - (x - x_1)\}_+ \{x_2 - x - a(t_2 - t)\}_+$$
$$\cdot E\{M_1(h_1)(M_1(h_1) - Y_1(h_1)) M_2(h_2)(M_2(h_2) - Y_2(h_2))\}$$
$$+ o(h_1 h_2), \quad \text{as} \quad h_i \downarrow 0, i = 1, 2,$$

where $a = (x_2 - x_1)/(t_2 - t_1)$. The method we used to determine this conditional probability is an extension of a method used in Chernoff (1964).

By straightforward calculation it is seen that

$$E\{M_i(t)(M_i(t) - Y_i(t))\} = \tfrac{1}{2}t.$$

Furthermore, the distribution of $w(t)$, given $w(t_i) = x_i$, $i = 1, 2$, is normal with mean $\mu = (t_2 - t_1)^{-1}\{(t_2 - t)x_1 + (t - t_1)x_2\}$ and variance $\sigma^2 = (t_2 - t)(t - t_1)/(t_2 - t_1)$. Thus,

$$P\{N_t^- \in dt_1, N_t^+ \in dt_2, w(t) \in dx | w(t_1) = x_1, w(t_2) = x_2\}$$
$$= P\{N_t^- \in dt_1, N_t^+ \in dt_2 | w(t) = x, w(t_i) = x_i, i = 1, 2\}.$$
$$\cdot \sigma^{-1}\phi((x - \mu)/\sigma)\, dx \tag{2.21}$$
$$= 4\{t_1(t - t_1)(t_2 - t)\}^{-1} a(x_1 - at_1)_+ (\mu - x)_+^2$$
$$\cdot \sigma^{-1}\phi((x - \mu)/\sigma)\, dx\, dt_1\, dt_2,$$

where ϕ is the standard normal density.

Now note that $\mu - x$ is the vertical distance of x to the line connecting (t_1, x_1) and (t_2, x_2). Hence, if $y(t)$ is the vertical distance of $w(t)$ to the concave majorant at time t, we have

$$P\{N_t^- \in dt_1, N_t^+ \in dt_2, y(t) \in dy | w(t_1) = x_1, w(t_2) = x_2\}$$
$$= 4\{t_1(t - t_1)(t_2 - t)\}^{-1} a(x_1 - at_1)_+ y_+^2 \sigma^{-1}\phi(y/\sigma)\, dy\, dt_1\, dt_2. \tag{2.22}$$

By integrating with respect to the density of $(w(t_1), w(t_2))$ we obtain

$$P\{N_t^- \in dt_1, N_t^+ \in dt_2, y(t) \in dy\}$$
$$= 4\{(t - t_1)(t_2 - t)\}^{-1}\sigma^{-1}\phi(y/\sigma) \tag{2.23}$$
$$\cdot E\{(Z_+/\sqrt{t_2 - t_1}) y_+^2 (X/\sqrt{t_1} - Z/\sqrt{t_2 - t_1})_+\}\, dt_1\, dt_2\, dy,$$

where X and Z are independent standard normal random variables. This shows that the conditional density of $y(t)$, given $N_t^- = t_1$ and $N_t^+ = t_2$, has the form

$$f_{y(t)|N_t^-,N_t^+}(y \mid t_1, t_2) = c(t, t_1, t_2) y_+^2 \phi(y/\sigma),$$

where $c(t, t_1, t_2)$ is a constant depending on t, t_1 and t_2. Integrating over y yields $c(t, t_1, t_2) = 2\sigma^{-3} = 2\{(t - t_1)(t_2 - t)/(t_2 - t_1)\}^{-3/2}$.

Part (ii) of the Lemma is proved in a completely analogous way. □

REMARK. From (2.23) it is clear that the joint density of N_t^- and N_t^+ is given by

$$f_{N_t^-,N_t^+}(t_1, t_2) = 2(t_2 - t_1)^{-3/2} E\{Z_+(X/\sqrt{t_1} - Z/\sqrt{t_2 - t_1})_+\}. \tag{2.24}$$

We will need this density in Section 3.

Lemma 2.1 immediately leads to the following result.

THEOREM 2.2. *Standard Brownian motion on* $[0, \infty)$ *can be decomposed into the process* τ *and independent Brownian excursions. More precisely, conditional on the values of the jump times of the concave majorant, the vertical distance of Brownian motion to the concave majorant is a succession of independent Brownian excursions, where the excursion between two successive values* T_i *and* T_{i+1} *of the process* τ *(i.e. successive jump times of the slope process), is distributed as* $z_u(t) = \sqrt{u}\, z((t - T_i)/u)$, *where* $u = T_{i+1} - T_i$ and *z is a standard Brownian excursion as defined by* (2.11) *and* (2.12). *The process* $\{1/S_t\colon t > 0,\ S_t$ *the slope of the concave majorant at time* $t\}$ *is the inverse of the process* τ.

REMARK. We proved Theorem 2.2 by a method which was used in Groeneboom (1981). The method has the advantage of giving explicit formulas for certain densities such as the joint density of the jump times N_t^- and N_t^+. Different proofs of Theorem 2.2 are given in Pitman (1982) and Bass (1983). Pitman's proof is based on time reversal arguments together with the path decomposition results in Williams (1974). Bass's proof relies on conditioning by means of Doob's *h*-path transforms and certain results on the decomposition of Markov processes at splitting times in Meyer, Smythe and Walsh (1972).

COROLLARY 2.3. *With probability one, a Brownian motion sample path has a local maximum at an epoch where the slope of the concave majorant changes.*

PROOF. By the construction of Theorem 2.2, the statement is equivalent with the statement $P\{z(t) > at, t \downarrow 0\} = 1$, for any $a > 0$, if $\{z(t)\colon t \in [0, 1]\}$ is a Brownian excursion. But the latter statement follows from a result of Dvoretsky and Erdös (1951), see e.g. Itô and McKean (1974), page 80, relation 6. □

3. L_2-norm over finite intervals of the slope of the concave majorant of Brownian motion, Brownian bridge and the empirical process. Let $L(a, b)$ be defined by

$$L(a, b) = \int_{(a,b]} c^{-2}\, d\tau(c), \quad 0 < a < b.$$

It is easily seen that the squared L_2-norm of the slope S_t of the concave majorant of Brownian motion over the random interval $(\tau(a), \tau(b))$ is just $L(a, b)$, i.e.

$$L(a, b) = \int_{\tau(a)}^{\tau(b)} S_t^2\, dt.$$

Now consider the process $\{f(a): a \ge a_0\}$ where $f(a) = L(e^{a_0}, e^a)$. This is a process with *stationary* independent increments. More precisely, we have the following result.

THEOREM 3.1. *Let $\{f(a): a \ge a_0\}$ be the process obtained from the process τ by putting $f(a) = L(e^{a_0}, e^a)$, $a_0 \in \mathbb{R}$. Then $\{f(a): a \ge a_0\}$ is a pure jump process with independent stationary increments and right-continuous paths. Furthermore, $f(a)$ is distributed as the random sum of N independent χ_1^2 random variables, where N has a Poisson distribution with parameter $a - a_0$.*

PROOF. The result is an immediate consequence of Theorem 2.1 and the representation (2.1) of the process τ as an integral with respect to a Poisson measure. For the Poisson measure of the process $\{f(x): x \ge a_0\}$ is obtained from the Poisson measure of the process τ by making the change of variables $(x, y) = (\log a, l/a^2)$, giving a Poisson measure with mean $n(dx \times dy) = y^{-1/2}\,\phi(\sqrt{y}),\ dx\, dy$ for $x > a_0$ and $y > 0$. The last statement of the theorem now follows, and the other statements follow immediately from the structure of the process τ. □

COROLLARY 3.1. *$L(a, b)$ is distributed as the sum of N independent χ_1^2 random variables, where N has a Poisson distribution with parameter* $\log(b/a)$, $0 \le a < b < \infty$.

We now apply Corollary 3.1 in computing the asymptotic distribution of (standardized) squared L_2-norms of slopes of the concave majorants of Brownian motion over intervals (t_1, t_2) such that $t_2/t_1 \to \infty$. These results will also give the asymptotic distribution of the squared L_2-norm of the slope of the concave majorant of the empirical process $\{U_n(t): t \in [0, 1]\} = \{\sqrt{n}(F_n(t) - t): t \in [0, 1]\}$, where F_n is the empirical distribution function of a sample of n uniform random variables on the interval [0, 1] (see Theorem 3.2). A different approach to this last result is given in Groeneboom and Pyke (1983), where also applications to statistical tests are discussed.

The next lemma shows that the asymptotic distribution of $\int_t^v S_u^2\,du$ is the same as the asymptotic distribution of $\int_{\tau(\frac{1}{t})}^{\tau(\frac{1}{v})} S_u^2\,du$, as $v/t \to \infty$.

LEMMA 3.1. *Let S_u be a version of the slope of the concave majorant of Brownian motion at u. Then,*

$$\left\{\int_t^v S_u^2\,du - \frac{1}{2}\log\left(\frac{v}{t}\right)\right\}\Big/\sqrt{\left(\frac{3}{2}\right)\log\left(\frac{v}{t}\right)} \to_d Z, \quad \text{as} \quad \frac{v}{t} \to \infty, \tag{3.1}$$
□

where Z is a standard normal random variable.

PROOF. By Corollary 3.1, $L(a, b)$ has the characteristic function $\psi(t) = (a/b)\exp\{1 - (1 - 2it)^{-1/2}\}$, $0 < a < b < \infty$. This implies

$$\left\{L(a, b) - \log\left(\frac{b}{a}\right)\right\}\Big/\sqrt{3\log\left(\frac{b}{a}\right)} \to_d Z, \quad \text{as} \quad \frac{b}{a} \to \infty, \tag{3.2}$$

where Z has the same meaning as in (3.1). Let $M > 0$. By Corollary 2.1 and formula (2.5), we have

$$P\{\tau(a) > Ma^2\} = \int_{Ma^2}^{\infty} (2/a)E(X/\sqrt{t} - 1/a)_+\,dt = \int_M^{\infty} 2E(X/\sqrt{t} - 1)_+\,dt. \tag{3.3}$$

Fix $\varepsilon > 0$. Then, by (3.3), we can choose $M > 0$, independent of a, such that

$$P\{\tau(a) > Ma^2\} + P\{\tau(a) < a^2/M\} < \varepsilon. \tag{3.4}$$

Thus, for $k > 0$, we have by Markov's inequality,

$$P\left\{\left|\int_{\tau(\sqrt{t})}^{\tau(\sqrt{v})} S_u^2\,du - \int_t^v S_t^2\,dt\right| > k\right\} < \left\{\int_{v/M}^{Mv} + \int_{t/M}^{Mt} S_t^2\,dt > k\right\} + \varepsilon$$

$$\leq k^{-1} E\left\{\int_{v/M}^{Mv} + \int_{t/M}^{Mt} S_t^2\,dt\right\} + \varepsilon. \tag{3.5}$$

Using (2.9), we find that $E\int_u^v S_t^2\,dt = \frac{1}{2}\log(v/u)$, implying that the right-hand side of (3.5) equals $(2/k)\log M + \varepsilon$. Thus, for sufficiently large $k > 0$, the right-hand side of (3.5) is smaller than 2ε, uniformly in t and v. Since $\int_{\tau(\sqrt{t})}^{\tau(\sqrt{v})} S_u^2\,du = \int_{(\sqrt{t},\sqrt{v}]} c^{-2}\,d\tau(c)$, we now obtain the result from (3.2). □

The next lemma gives the corresponding result for Brownian bridge.

LEMMA 3.2. *Let $\bar{S}_u$ be a version of the slope of the concave majorant of Brownian bridge on* $[0, 1]$ *at time* $u \in (0, 1)$. *Then the* df *of*

$$\left(\int_t^v \bar{S}_u^2\,du - \frac{1}{2}\log\{v(1-t)/(t(1-v))\}\right)\Big/\left((3/2)\log\{v(1-t)/(t(1-v))\}\right)^{1/2}$$

tends to a standard normal df, *as* $v(1-t)/(t(1-v)) \to \infty$.

PROOF. Let $\{B(t)\colon t \in [0, 1]\}$ be a Brownian bridge on $[0, 1]$, then $\{(1+t)B(t/(1+t))\colon t \geq 0\}$ represents Brownian motion on $[0, \infty)$ (Doob's transformation). Hence, there is a $1-1$ mapping of concave majorants of sample paths $t \to B(t)$, $t \in [0, 1]$, of Brownian bridge to concave majorants of Brownian motion paths $t \to (1+t)B(t/(1+t))$, $t \geq 0$. For $0 < p < q < 1$, the integral $\int_p^q \bar{S}_u^2\,du$ is almost surely a sum of the form

$$(u_1 - p)(B(u_1) - B(u_0))^2/(u_1 - u_0)^2 + \sum_{i=1}^{n-1} (B(u_{i+1}) - B(u_i))^2/(u_{i+1} - u_i)$$
$$+ (q - u_n)(B(u_{n+1}) - B(u_n))^2/(u_{n+1} - u_n)^2,$$

where the u_i's are jump times of the slope process and where $u_0 \leq p \leq u_1 < u_2 < \cdots < u_n \leq q \leq u_{n+1}$. Letting $u_i = t_i/(1+t_i)$ and $w(t) = (1+t)B(t/(1+t))$, we have,

$$(B(u_{i+1}) - B(u_i))^2/(u_{i+1} - u_i) = (w(t_{i+1}) - w(t_i))^2/(t_{i+1} - t_i) + (1+t_i)^{-1}w^2(t_i) - (1+t_{i+1})^{-1}w^2(t_{i+1}). \tag{3.6}$$

Summing over i, $1 \leq i \leq n-1$, gives

$$\int_{u_1}^{u_n} \bar{S}_u^2\,du = \int_{t_1}^{t_n} S_t^2\,dt + (1+t_1)^{-1}w^2(t_1) - (1+t_n)^{-1}w^2(t_n), \tag{3.7}$$

where S_t is the slope of the concave majorant of $t \to w(t)$, $t \geq 0$. Fix $\varepsilon > 0$. As in Lemma 2.1, let $N^-(t)$ and $N^+(t)$ be the jump times of the slope process which precede and follow t, respectively. Then, using (2.24) we find, for $M \leq 1$,

$$P\{N^+(t) - N^-(t) \geq Mt\} = \int_0^1 dt_1 \int_M^\infty 2u^{-3/2} EZ_+(X/\sqrt{t_1} - Z/\sqrt{u})_+\,du \tag{3.8}$$

and

$$P\{N^-(t) \leq t/M\} = \int_0^{1/M} dt_1 \int_{1-t_1}^\infty 2u^{-3/2} EZ_+(X/\sqrt{t_1} - Z/\sqrt{u})_+\,du. \tag{3.9}$$

Since the right-hand sides of (3.8) and (3.9) do not depend on t, we can choose $M \geq 1$, independent of t, such that

$$P\{N^+(t) - N^-(t) \geq Mt\} < \varepsilon/2 \text{ and } P\{N^-(t) \leq t/M\} < \varepsilon/2.$$

This implies that, if $r = p/(1-p)$ and $s = q/(1-q)$,

$$P\left\{\int_{t_0}^{t_1} S_t^2\,dt + \int_{t_n}^{t_{n+1}} S_t^2\,dt \geq k\right\} < (1/k)E\left\{\int_{r/M}^{Mr} + \int_{s/M}^{Ms} S_t^2\,dt\right\} + 2\varepsilon$$

$$= (2/k)\log M + 2\varepsilon < 3\varepsilon,$$

for k sufficiently large. Since we also have

$$P\{\sup_{t/M \leq u \leq Mt} w^2(u)/(1+u) \geq k_1\} < \varepsilon,$$

where k_1 is a suitably chosen constant, not depending on t, we can conclude from (3.6) and (3.7) that

$$\int_p^q \bar{S}_u^2\,du - \int_r^s S_t^2\,dt = O_p(1), \quad \text{as} \quad s/r \to \infty.$$

The result now follows from Lemma 3.1. □

REMARK 3.1. We can define a τ process for Brownian bridge and derive the distribution of the slopes of the concave majorant of Brownian bridge from the properties of this process. The process has a more complicated structure than the corresponding process for Brownian motion, however; for example, the process is no longer an independent increments process. We only give here the marginal densities of this process and of the slope process, which can be derived by methods similar to those used in Section 2. The density of $\sup\{t\colon B(t) - at = \max_{u\in[0,1]}(B(u) - au)\}$ is given by

$$\bar{f}_a(t) = 2E(X - a(t/(1-t))^{1/2})_+(X + a((1-t)/t)^{1/2})_+, \quad a \in (-\infty, \infty) \tag{3.10}$$

and the density of the slope $\bar{S}_t$ at time t is given by (see Corollary 2.2 for notation)

$$g_t(a) = \begin{cases} 4\{(t(1-t))^{1/2}\phi(a(t/(1-t))^{1/2}) - at\bar{\Phi}(a(t/(1-t))^{1/2})\}, & a \geq 0 \\ 4\{(t(1-t))^{1/2}\phi(a((1-t)/t)^{1/2}) & \\ \quad - |a|(1-t)\bar{\Phi}(|a|((1-t)/t)^{1/2})\}, & a < 0. \end{cases} \tag{3.11}$$

The densities $\bar{g}_t$, $0 < t < 1$, form a rather curious family. For each t, they have a cusp-like maximum at zero, and the first moment (which is $E\bar{S}_t$) is given by

$$E\bar{S}_t = 4(1-2t)/\{3\sqrt{2\pi t(1-t)}\}, \quad 0 < t < 1. \tag{3.12}$$

Note that, for t near zero, $E\bar{S}_t$ is approximately $\tfrac{1}{2}t^{-1/2}$.

To facilitate the comparison of the L_2-norms of the slopes of concave majorants of Brownian bridge and the empirical process, we relate this L_2-norm to another functional. This functional is given by

$$A(t, v) = \sup_{J\in\mathscr{M}} \int_{(t,v)} B(u)\,dJ(u), \quad 0 < t < v < 1, \tag{3.13}$$

where the class of functions $\mathscr{M}$ is as in the following definition.

DEFINITION 3.1. $\mathscr{M}$ is the set of real-valued nondecreasing and right-continuous step-functions J, defined on $(0, 1)$, such that $\int_0^1 J(u)\,du = 0$ and $\int_0^1 J(u)\,du = 1$.

This class of functions is also considered in Behnen (1975), Scholz (1983) and Groeneboom and Pyke (1983). We now show that $\{\int_t^{1-t} \bar{S}_u^2\,du\}^{1/2}$ behaves asymptotically as $A(t, 1-t)$, as $t \downarrow 0$.

LEMMA 3.3. *We have*

$$A(t, t-1) - \left\{\int_t^{1-t} \bar{S}_u^2\,du\right\}^{1/2} = O_p(1), \quad \text{as } t \downarrow 0. \tag{3.14}$$

PROOF. Let, for $0 < t < 1$, $C(t)$ denote the value of the concave majorant of Brownian bridge at time t. Fix $\varepsilon > 0$ and let $t \in (0, ½)$. By (3.11) and Markov's inequality, we have

$$\begin{aligned}\Pr\{C(t) \geq Mt^{1/2}\} &\leq M^{-1}t^{-1/2} \int_0^t E\,|\bar{S}_u|\,du \\ &= M^{-1}t^{-1/2}(4/3) \int_0^t \{(1-u)^{3/2}/\sqrt{2\pi u} + u^{3/2}/\sqrt{2\pi(1-u)}\}\,du < \varepsilon,\end{aligned} \tag{3.15}$$

for M sufficiently large, where M can be chosen independently from t. This also can be deduced from (2.9), by comparing slopes of concave majorants of Brownian motion and Brownian bridge and using Doob's transformation (as in the proof of Lemma 3.2). Similarly, we have for M sufficiently large,

$$\Pr\{C(1-t) \geq Mt^{1/2}\} < \varepsilon, \tag{3.16}$$

uniformly in $t \in (0, ½)$. Define, for $t \in (0, ½)$, the truncated functions $B_t = B.1_{[t,1-t]}$ and $C_t = C.1_{[t,1-t]}$, where $1_{[t,1-t]}$ is the indicator of the interval $[t, 1-t]$, B denotes a Brownian bridge sample path and C the corresponding concave majorant. The L_2-norms on $[0, 1]$ of the slopes of the concave majorants of the "clipped pieces" B_t and C_t are given by $\sup_{J\in\mathcal{M}} \int_{(0,1)} B_t\,dJ$ and $\sup_{J\in\mathcal{M}} \int_{(0,1)} C_t\,dJ$, respectively (for details on this representation, see Groeneboom and Pyke, 1983). The concave majorant of the function C_t is the same as the concave majorant of the function

$$D_t(u) = \begin{cases} C(u), & u = t \text{ and } u = 1-t, \\ B_t(u), & \text{elsewhere.}\end{cases}$$

Thus $\sup_{J\in\mathcal{M}} \int_{(0,1)} C_t\,dJ = \sup_{J\in\mathcal{M}} \int_{(0,1)} D_t\,dJ$, implying (since $-t^{1/2} \leq J(t) \leq (1-t)^{-1/2}$)

$$\begin{aligned}&\left|\sup_{J\in\mathcal{M}} \int_{(0,1)} C_t\,dJ - A(t, 1-t)\right| \\ &\quad= \left|\sup_{J\in\mathcal{M}} \int_{(0,1)} D_t\,dJ - \sup_{J\in\mathcal{M}} \int_{(0,1)} B_t\,dJ\right| \\ &\quad\leq 2t^{-1/2}\max\{|B(t) - C(t)|, |B(1-t) - C(1-t)|\}.\end{aligned}$$

By (3.15) and (3.16), the last expression is bounded in probability. The result now follows from Lemma 3.2, since

$$\sup_{J\in\mathcal{M}} \int_{(0,1)} C_t\,dJ = \left\{t^{-1}C^2(t) + t^{-1}C^2(1-t) + \int_t^{1-t} \bar{S}_u^2\,du\right\}^{1/2}. \quad \square$$

Using the preceding lemmas, we can now derive the limiting distribution of the L_2-norm of the slope of the concave majorant of the empirical process.

THEOREM 3.2. *Let* $\{U_n(t)\colon t \in [0, 1]\}$ *be the empirical process* $\{\sqrt{n}\,(F_n(t) - t)\colon t \in [0, 1]\}$, *where* F_n *is the empirical* df *of a sample of n uniform random variables on* $[0, 1]$, *and let* $S_n(t)$ *be the slope of the concave majorant of* U_n *at time t. Then*

$$\left\{\int_0^1 S_n^2(t)\,dt - \log n\right\}/\sqrt{3\log n} \to_d Z, \quad \text{as} \quad n \to \infty, \tag{3.17}$$

where Z is a standard normal random variable.

PROOF. By Lemmas 3.2 and 3.3 we only have to prove

$$\int_0^1 S_n^2(t)\,dt - \int_{1/n}^{1-1/n} \bar{S}_n^2(t)\,dt = o_p(\sqrt{\log n}), \quad \text{as} \quad n \to \infty$$

where the $\bar{S}_n$ are slopes of concave majorants of Brownian bridges such that

$$\sup_{t \in (0,1)} |U_n(t) - B_n(t)| = O_p((\log n)/\sqrt{n}),$$

as $n \to \infty$. But this is proved in Groeneboom and Pyke (1983), pages 341 and 342. □

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