The Annals of Probability
1983, Vol. 11, No. 4, 962–969

THE CLASS OF LIMIT LAWS FOR STOCHASTICALLY COMPACT NORMED SUMS

BY WILLIAM E. PRUITT[1]

University of Minnesota

Khintchine showed that every infinitely divisible law can be obtained as the limit of a subsequence of normed sums of independent, identically distributed random variables. Here we restrict the summands to be in a class which makes the normed sums stochastically compact, i.e. so that every subsequence has a further subsequence which converges to a nondegenerate limit. A nice analytic condition for stochastic compactness was obtained by Feller. Our result is an analogous characterization of the class of limit laws of subsequences of stochastically compact normed sums. One consequence is that they have C^∞ densities.

1. Introduction. Let $X_1, X_2, \cdots$ be independent, identically distributed nondegenerate random variables taking values in $\mathbb{R}^1$ and $S_n = X_1 + \cdots + X_n$. Let X be a random variable with the same distribution as X_1, F its distribution function, and for $x > 0$ define

$$G(x) = P\{|X| > x\}, \quad K(x) = x^{-2} \int_{|y| \le x} y^2 \, dF(y),$$

$$Q(x) = G(x) + K(x) = E(x^{-1}|X| \wedge 1)^2.$$

We introduce a class of distributions

$$\mathscr{F} = \left\{ F: \limsup_{x\to\infty} \frac{G(x)}{K(x)} < \infty \right\}.$$

For X in the domain of attraction of a stable law of index α,

$$\lim_{x\to\infty} \frac{G(x)}{K(x)} = \frac{2-\alpha}{\alpha}$$

so that $\mathscr{F}$ is much larger than the class of laws attracted to some stable law. Feller [1] showed that $F \in \mathscr{F}$ is necessary and sufficient for there to exist sequences $\{\gamma_n\}$, $\{\delta_n\}$ such that the sequence $\{(S_n - \delta_n)/\gamma_n\}$ is stochastically compact, i.e. that every subsequence has a further subsequence which converges weakly to a nondegenerate limit. Of course these limit laws are necessarily infinitely divisible and with no restrictions on F, Khintchine proved that every infinitely divisible law is a possible limit [2, page 184]. But in [3], in the course of obtaining a lower bound for the distribution of S_n, we noticed that for $F \in \mathscr{F}$ these limit laws for subsequences necessarily have C^∞ densities. This then raised the question: what is the class of all possible limit laws for these subsequences? It is the purpose of this paper to answer this question.

The defining property of the class will be given in terms of the Lévy measure of the infinitely divisible law that is a candidate for inclusion in the class. An infinitely divisible law has characteristic function $\varphi(u) = \exp\{\psi(u)\}$ where

$$\psi(u) = iua - \frac{1}{2}\sigma^2 u^2 + \int \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) d\nu(x)$$

Received October 1982.
[1] Research supported by NSF MCS 78-01168.
AMS 1980 *subject classification*: 60F05.
Key words and phrases: Infinitely divisible laws, weak convergence, class $\mathscr{L}$.

where ν is a (possibly infinite) measure on $\mathbb{R}^1\setminus\{0\}$ satisfying

$$\int \frac{x^2}{1+x^2}\,d\nu(x) < \infty.$$

For convenience we will write ν instead of (ν, σ^2, a). For $x > 0$, define

$$G^\nu(x) = \nu\{y : |y| > x\},\ K^\nu(x) = x^{-2}\left(\sigma^2 + \int_{|y|\le x} y^2\,d\nu(y)\right)$$

$$Q^\nu(x) = G^\nu(x) + K^\nu(x).$$

Then we may state our result:

THEOREM. *Let*

$$\mathscr{H} = \{H : \exists F \in \mathscr{F}, \{\delta_n\}, \{\gamma_n\}, \text{ and } \{n_k\} \ni (S_{n_k} - \delta_{n_k})/\gamma_{n_k} \Rightarrow H\}.$$

Then if H has Lévy measure ν, $H \in \mathscr{H}$ if and only if there exists a $C > 0$ such that

(1) $$G^\nu(x) \le CK^\nu(x) \text{ for all } x \in (0, \infty).$$

REMARK 1. If one requires that the $\{\delta_n\}$, $\{\gamma_n\}$ that make the sequence $\{(S_n - \delta_n)/\gamma_n\}$ stochastically compact are to be used in the definition of $\mathscr{H}$, the class remains the same except that the degenerate laws are excluded.

REMARK 2. Since $G^\nu(x)\downarrow$ and $x^2K^\nu(x)\uparrow$, the condition (1) is equivalent to the two conditions

(2) $$\limsup_{x\to 0} \frac{G^\nu(x)}{K^\nu(x)} < \infty \quad \text{and} \quad \limsup_{x\to\infty} \frac{G^\nu(x)}{K^\nu(x)} < \infty$$

provided that we exclude the degenerate laws so that the ratio G^ν/K^ν is well defined. Since $\int_{|x|\le 1} x^2\,d\nu(x) < \infty$ implies that $\lim_{x\to 0} x^2G^\nu(x) = 0$, the first condition in (2) is always satisfied when $\sigma^2 > 0$. Thus, for example, while the Poisson distribution is not in $\mathscr{H}$, the convolution of the Poisson and the normal is in $\mathscr{H}$.

Finally, it seems natural to ask how the class $\mathscr{H}$ compares with the class $\mathscr{L}$ which consists of all limit laws of normed sums of independent but not necessarily identically distributed random variables. $\mathscr{H}$ is a "much larger" class than $\mathscr{L}$ but $\mathscr{L}$ is not contained in $\mathscr{H}$.

The proof of the theorem is in the next section. Incidentally, it turns out that we can prove Khintchine's result mentioned above with no extra work and then verify at the end that (1) implies that the F constructed is in $\mathscr{F}$. The final section consists of the comparison of $\mathscr{H}$ and $\mathscr{L}$, the proof that members of $\mathscr{H}$ have C^∞ densities, the observation that there are no universal laws in $\mathscr{F}$, i.e., there is no law in $\mathscr{F}$ which is in the domain of partial attraction of every law in $\mathscr{H}$, and the observation that $\mathscr{H} \subset \mathscr{F}$ if the degenerate laws are excluded. Note that this is true in spite of the fact that $\mathscr{F}$ is not closed under weak limits. In proving this, we also show that $Q(x)$ and $Q^\nu(x)$ are comparable for large x for any infinitely divisible X.

2. The proof. First we will show that if $H \in \mathscr{H}$ then (1) is satisfied. If F has finite second moment we will assume the mean is zero; this can be accomplished by incorporating the mean with δ_n. We define for $x > 0$

$$M(x) = x^{-1}\int_{|y|\le x} y\,dF(y).$$

Then, by the central convergence criterion [2, page 116] we have

(3) $$\lim_{k\to\infty} n_k G(x\gamma_{n_k}) = G^\nu(x),$$

for all x that are continuity points of $G^\nu(x)$ and

$$\lim_{\varepsilon\to 0}\limsup(\inf)_{k\to\infty} n_k\varepsilon^2(K(\varepsilon\gamma_{n_k}) - M^2(\varepsilon\gamma_{n_k})) = \sigma^2. \tag{4}$$

We will show that these imply that

$$\lim_{k\to\infty} n_k K(x\gamma_{n_k}) = K^\nu(x) \tag{5}$$

for all x that are continuity points of $G^\nu(x)$. First we have $M^2(x) = o(K(x))$ as $x \to \infty$ (see [5, page 11]; this is where the assumption that $EX = 0$ if $EX^2 < \infty$ is used) so that the M^2 term in (4) may be dropped. By Lemma 2.1 of [5] we have

$$x^2Q(x) = \int_0^x 2yG(y)\,dy \tag{6}$$

so that for $\varepsilon \in (0, x)$, ε a continuity point of $G^\nu(x)$,

$$\gamma_{n_k}^2(x^2Q(x\gamma_{n_k}) - \varepsilon^2 Q(\varepsilon\gamma_{n_k})) = \int_{\varepsilon\gamma_{n_k}}^{x\gamma_{n_k}} 2yG(y)\,dy = \gamma_{n_k}^2 \int_\varepsilon^x 2uG(u\gamma_{n_k})\,du.$$

Now we multiply by $n_k\gamma_{n_k}^{-2}$ and let $k \to \infty$. Since by (3)

$$n_k G(u\gamma_{n_k}) \le n_k G(\varepsilon\gamma_{n_k}) \to G^\nu(\varepsilon)$$

the integral will converge by bounded convergence. Thus

$$\limsup_{k\to\infty} n_k x^2 Q(x\gamma_{n_k}) = \limsup_{k\to\infty} n_k \varepsilon^2 Q(\varepsilon\gamma_{n_k}) + \int_\varepsilon^x 2uG^\nu(u)\,du$$

$$= \limsup_{k\to\infty} n_k\varepsilon^2 K(\varepsilon\gamma_{n_k}) + \varepsilon^2 G^\nu(\varepsilon) + \int_\varepsilon^x 2uG^\nu(u)\,du.$$

Now we let $\varepsilon \to 0$ and use (4);

$$\limsup_{k\to\infty} n_k x^2 Q(x\gamma_{n_k}) = \sigma^2 + \int_0^x 2uG^\nu(u)\,du \tag{7}$$

since the convergence of $\int_{|x|\le 1} x^2\,d\nu(x)$ implies the convergence of $\int_0^1 uG^\nu(u)\,du$ and that $\varepsilon^2 G^\nu(\varepsilon) \to 0$. The same argument works for $\liminf_{k\to\infty}$ so we have convergence in (7). Now by (6) applied to ν (the same proof applies even though ν may be an infinite measure since $\int_{|x|\le 1} x^2\,d\nu(x) < \infty$)

$$\sigma^2 + \int_0^x 2uG^\nu(u)\,du = \sigma^2 + \int_{|y|\le x} y^2\,d\nu(y) + x^2G^\nu(x) = x^2K^\nu(x) + x^2G^\nu(x) = x^2Q^\nu(x).$$

In conjunction with (7) this yields

$$\lim_{k\to\infty} n_k Q(x\gamma_{n_k}) = Q^\nu(x)$$

and using (3) once more gives (5). Finally (3) and (5) imply (1) since $F \in \mathscr{F}$ implies that we have an x_0 and a C such that

$$G(x) \le CK(x) \quad \text{for all} \quad x \ge x_0,$$

so that for k large enough that $x\gamma_{n_k} \ge x_0$ we have

$$n_k G(x\gamma_{n_k}) \le Cn_k K(x\gamma_{n_k}).$$

This gives (1) for all continuity points of $G^\nu(x)$ but this is enough since both G^ν and K^ν are right continuous.

Now suppose that we have an infinitely divisible law H. We will assume that it is nondegenerate since degenerate laws are clearly in $\mathscr{H}$. This implies that $Q^\nu(x) > 0$ for all $x > 0$. It is also no loss to assume that $G^\nu \not\equiv 0$ since normal laws are in $\mathscr{H}$. We will use the notation

$$G_+(x) = P\{X > x\}, \quad G_-(x) = P\{X < -x\}, \quad x > 0,$$

with similar conventions for $G_+^\nu(x)$ and $G_-^\nu(x)$. We need to construct a distribution F and sequences $\{\rho_k\}$, $\{n_k\}$ such that for all $x > 0$

$$\lim_{k\to\infty} n_k G_+(x\rho_k) = G_+^\nu(x), \tag{8}$$

$$\lim_{k\to\infty} n_k G_-(x\rho_k) = G_-^\nu(x), \tag{9}$$

and

$$\lim_{\varepsilon\to 0} \limsup\,(\inf)_{k\to\infty} n_k \varepsilon^2 (K(\varepsilon\rho_k) - M^2(\varepsilon\rho_k)) = \sigma^2. \tag{10}$$

Then the central convergence criterion guarantees that

$$\rho_k^{-1}(S_{n_k} - \delta_{n_k}) \Rightarrow H$$

for some centering sequence. This will prove Khintchine's Theorem. Then we will show that if (1) holds, then $F \in \mathscr{F}$. We start the construction by choosing sequences $\alpha_k \downarrow 0$ and $\beta_k \uparrow \infty$ which satisfy

$$G_+^\nu(\alpha_1) > 0, \quad G_-^\nu(\alpha_1) > 0 \tag{11}$$

(if either G_+^ν or G_-^ν is identically zero, then we will only assume (11) for the remaining one) and

$$\int_{|y| \le \beta_1} y^2 \, d\nu(y) > 0.$$

Next we let $\rho_1 = \alpha_1^{-1}$, $\lambda_1 = 1/Q^\nu(\alpha_1)$, and for $k > 1$

$$\rho_k = \frac{\beta_1 \cdots \beta_{k-1}}{\alpha_1 \cdots \alpha_k}, \quad \lambda_k = \frac{K^\nu(\beta_1) \cdots K^\nu(\beta_{k-1})}{Q^\nu(\alpha_1) \cdots Q^\nu(\alpha_k)}.$$

Now we may define F. Let $I_k = [-\beta_k\rho_k, -\alpha_k\rho_k) \cup (\alpha_k\rho_k, \beta_k\rho_k]$ and define

$$F(B) = c\lambda_k \nu(\rho_k^{-1}B), \quad B \subset I_k; \tag{12}$$

we will see that it will be possible to choose c so that the total mass is one. Since $\alpha_k\rho_k = \beta_{k-1}\rho_{k-1}$, F is defined on $\{x : |x| > 1\}$. There will be no mass in the unit interval. Next we list a few facts for later reference. If $G_+^\nu(x) \not\equiv 0$,

$$\frac{\lambda_{k+1} G_+^\nu(\alpha_{k+1})}{\lambda_k G_+^\nu(\alpha_k)} = \frac{K^\nu(\beta_k) G_+^\nu(\alpha_{k+1})}{Q^\nu(\alpha_{k+1}) G_+^\nu(\alpha_k)} \le \frac{K^\nu(\beta_k)}{G_+^\nu(\alpha_k)} \to 0 \tag{13}$$

since $K^\nu(x) \to 0$ as $x \to \infty$. An analogous statement holds with G_+^ν replaced by G_-^ν. We let

$$\xi = \int y^2 \, d\nu(y)$$

with the understanding that ξ may be infinite. Then

$$\frac{\int_{\alpha_k < |y| \le \beta_k} y^2 \, d\nu(y)}{K^\nu(\beta_k)\beta_k^2} = \frac{\int_{|y| \le \beta_k} y^2 \, d\nu(y) - \int_{|y| \le \alpha_k} y^2 \, d\nu(y)}{\sigma^2 + \int_{|y| \le \beta_k} y^2 \, d\nu(y)} \to \frac{\xi}{\sigma^2 + \xi} \tag{14}$$

with the convention that the limit is one when $\xi = \infty$. Next

$$\frac{\lambda_{k+1}\rho_{k+1}^2 \int_{\alpha_{k+1}<|y|\leq\beta_{k+1}} y^2\,d\nu(y)}{\lambda_k\rho_k^2 \int_{\alpha_k<|y|\leq\beta_k} y^2\,d\nu(y)} = \frac{K^\nu(\beta_k)\beta_k^2}{\int_{\alpha_k<|y|\leq\beta_k} y^2\,d\nu(y)} \cdot \frac{\int_{\alpha_{k+1}<|y|\leq\beta_{k+1}} y^2\,d\nu(y)}{Q^\nu(\alpha_{k+1})\alpha_{k+1}^2} \tag{15}$$
$$\to \frac{\sigma^2+\xi}{\xi}\cdot\frac{\xi}{\sigma^2} = 1 + \frac{\xi}{\sigma^2}$$

by using (14) since $x^2Q^\nu(x) \to \sigma^2$ as $x \to 0$; of course, ξ/σ^2 is to be interpreted as ∞ if either $\xi = \infty$ or $\sigma^2 = 0$. Now we are ready to estimate G and K and complete the proof. Recalling (12) we see that if $G_+^\nu(x) \equiv 0$, then $G_+(x) \equiv 0$ as well. Otherwise, for $\alpha_k \leq x < \beta_k$

$$G_+(x\rho_k) = c\lambda_k(G_+^\nu(x) - G_+^\nu(\beta_k)) + \textstyle\sum_{j=k+1}^\infty c\lambda_j(G_+^\nu(\alpha_j) - G_+^\nu(\beta_j))$$

so that by (13) there is a C_1 such that

$$c\lambda_k(G_+^\nu(x) - G_+^\nu(\beta_k)) \leq G_+(x\rho_k) \leq c\lambda_k G_+^\nu(x) + C_1\lambda_{k+1}G_+^\nu(\alpha_{k+1}). \tag{16}$$

Incidentally this and the analogous argument for G_- show that c may be chosen to make the total mass one. Now using (16) and (13) we see that for fixed x

$$\lim_{k\to\infty} \frac{1}{c\lambda_k} G_+(x\rho_k) = G_+^\nu(x).$$

This proves (8) for $n_k = [1/c\lambda_k]$. The same proof applies for (9). We will also need a uniform bound for $G(x)$ in order to prove that $F \in \mathscr{F}$. For this we use (16) and (13) once more to obtain

$$G(x\rho_k) \leq c\lambda_k G^\nu(x) + C_1\lambda_k K^\nu(\beta_k) \leq c\lambda_k Q^\nu(x) + C_1\lambda_k Q^\nu(\beta_k) \leq (c + C_1)\lambda_k Q^\nu(x). \tag{17}$$

Now we must estimate K. For $\alpha_k \leq x < \beta_k$, by (15)

$$x^2\rho_k^2K(x\rho_k) = \textstyle\sum_{j=1}^{k-1} c\lambda_j\rho_j^2 \int_{\alpha_j<|y|\leq\beta_j} y^2\,d\nu(y) + c\lambda_k\rho_k^2 \int_{\alpha_k<|y|\leq x} y^2\,d\nu(y)$$
$$\sim \left(1 + \frac{\sigma^2}{\xi}\right)c\lambda_{k-1}\rho_{k-1}^2 \int_{\alpha_{k-1}<|y|\leq\beta_{k-1}} y^2\,d\nu(y) + c\lambda_k\rho_k^2 \int_{\alpha_k<|y|\leq x} y^2\,d\nu(y) \tag{18}$$
$$\sim c\lambda_k\rho_k^2\left(\left(1 + \frac{\sigma^2}{\xi}\right)\frac{Q^\nu(\alpha_k)\alpha_k^2}{K^\nu(\beta_{k-1})\beta_{k-1}^2} \int_{|y|\leq\beta_{k-1}} y^2\,d\nu(y) + \int_{\alpha_k<|y|\leq x} y^2\,d\nu(y)\right)$$
$$\sim c\lambda_k\rho_k^2\left(\sigma^2 + \int_{\alpha_k<|y|\leq x} y^2\,d\nu(y)\right).$$

Thus, for fixed $\varepsilon > 0$,

$$\lim_{k\to\infty} \frac{1}{c\lambda_k} \varepsilon^2 K(\varepsilon\rho_k) = \sigma^2 + \int_{|y|\leq\varepsilon} y^2\,d\nu(y)$$

and this will approach σ^2 when $\varepsilon \to 0$. This is enough to prove (10) (and thus Khintchine's Theorem) with $n_k = [1/c\lambda_k]$ since $M^2(x) = o(K(x))$ as $x \to \infty$ as mentioned above as EX^2 cannot be finite for this would imply a normal limit. ($EX^2 = \infty$ can also be deduced from (18) since $\lambda_k\rho_k^2 \to \infty$.) Finally, we need a lower bound for K to complete the proof that $F \in \mathscr{F}$. We have by (14) that there is a $c_1 \in (0, 1)$ such that

$$\begin{aligned}
x^2\rho_k^2 K(x\rho_k) &\geq c\lambda_{k-1}\rho_{k-1}^2 \int_{\alpha_{k-1}<|y|\leq\beta_{k-1}} y^2\,d\nu(y) + c\lambda_k\rho_k^2 \int_{\alpha_k<|y|\leq x} y^2\,d\nu(y) \\
&= c\lambda_k\rho_k^2\left(\frac{Q^\nu(\alpha_k)\alpha_k^2}{K^\nu(\beta_{k-1})\beta_{k-1}^2}\int_{\alpha_{k-1}<|y|\leq\beta_{k-1}} y^2\,d\nu(y) + \int_{\alpha_k<|y|\leq x} y^2\,d\nu(y)\right) \\
&\geq c\lambda_k\rho_k^2\left(c_1 Q^\nu(\alpha_k)\alpha_k^2 + \int_{\alpha_k<|y|\leq x} y^2\,d\nu(y)\right) \\
&\geq cc_1\lambda_k\rho_k^2\left(\sigma^2 + \int_{|y|\leq\alpha_k} y^2\,d\nu(y) + \int_{\alpha_k<|y|\leq x} y^2\,d\nu(y)\right) \\
&= cc_1\lambda_k\rho_k^2 x^2 K^\nu(x)
\end{aligned}$$

or

$$K(x\rho_k) \geq cc_1\lambda_k K^\nu(x).$$

In conjunction with (17) this proves that $F \in \mathscr{F}$ since by (1)

$$Q^\nu(x) \leq (1 + C)K^\nu(x).$$

3. Final remarks. Lévy proved [2, page 149] that a distribution is in $\mathscr{L}$ if and only if both $G_+^\nu(x)$ and $G_-^\nu(x)$ have both right and left derivatives for every $x > 0$ and that $-xG_+^{\nu\prime}(x)$ and $-xG_-^{\nu\prime}(x)$ are nonincreasing. The class $\mathscr{L}$ is smaller than $\mathscr{H}$ in some sense but is not contained in $\mathscr{H}$. First, the condition for $\mathscr{L}$ puts separate conditions on the behavior of ν on the two half lines while (1) lumps this behavior together. Thus, for any ν satisfying (1) except the normal and degenerate laws it would be easy to redistribute the mass between the two half lines to violate Lévy's condition without changing G^ν and K^ν. Also the convolution of the normal and Poisson and other examples of this type are in $\mathscr{H}$ but not in $\mathscr{L}$. An example of a distribution in $\mathscr{L}$ but not in $\mathscr{H}$ is obtained by using ν with density

$$h(x) = \begin{cases} \dfrac{1}{x}, & 0 < x \leq 1, \\[2ex] \dfrac{1}{x(1 + \log^2 x)}, & x > 1. \end{cases}$$

Then Lévy's condition is satisfied but as $x \to \infty$

$$G^\nu(x) \sim \frac{1}{\log x}, \quad K^\nu(x) \sim \frac{1}{2\log^2 x}$$

so that (1) is violated. If, however, one asks a little more, namely that $-x^\alpha G^{\nu\prime}(x) \downarrow$ for some $\alpha > 1$, then the distribution must be in $\mathscr{H}$. To see this,

$$G^\nu(x) = \int_x^\infty - G^{\nu\prime}(y)\,dy \leq -x^\alpha G^{\nu\prime}(x)\int_x^\infty y^{-\alpha}\,dy = -\frac{1}{\alpha - 1}xG^{\nu\prime}(x),$$

while

$$\begin{aligned}
K^\nu(x) &\geq x^{-2}\int_{|y|\leq x} x^2\,d\nu(y) = -x^{-2}\int_0^x y^2 G^{\nu\prime}(y)\,dy \\
&\geq -x^{-2}xG^{\nu\prime}(x)\int_0^x y\,dy = -\frac{1}{2}xG^{\nu\prime}(x).
\end{aligned}$$

We conclude with a couple of observations. Doeblin showed that there are universal laws that are in the domain of partial attraction of every infinitely divisible law. There can be no such universal laws in $\mathscr{F}$ that are in the domain of partial attraction of every law in $\mathscr{H}$. To see this, we note that if $F \in \mathscr{F}$, then the proof of (1) shows that it holds with $C = \limsup_{x\to\infty} G(x)/K(x)$. This then restricts the class of laws in $\mathscr{H}$ to which F can be partially attracted. We can also show easily that the laws in $\mathscr{H}$ have C^∞ densities as claimed. Under (1), we have for $u > 0$

$$\operatorname{Re}\psi(u) = -\frac{1}{2}\sigma^2u^2 + \int (\cos ux - 1)\, d\nu(x) \le -cK^\nu\left(\frac{1}{u}\right) \le -c_1Q^\nu\left(\frac{1}{u}\right)$$

and (1) also implies (see Lemma 2.4 of [5]) that $x^\alpha Q^\nu(x) \downarrow$ for some $\alpha > 0$ so that for $u \ge 1$, $Q^\nu(u^{-1}) \ge Q^\nu(1)u^\alpha$ and thus

$$\operatorname{Re}\psi(u) \le -c_2|u|^\alpha, \quad |u| \ge 1.$$

This estimate allows the use of the inversion formula for the density and also shows that it may be repeatedly differentiated.

Finally we will show that $\mathscr{H} \subset \mathscr{F}$ if the degenerate laws are excluded. We need a lemma which may have independent interest. The notation $f(x) \approx g(x)$ will mean that there exist positive c_1, c_2, and x_0 such that $c_1 \le f(x)/g(x) \le c_2$ for $x \ge x_0$.

Lemma. *For any infinitely divisible X, $Q(x) \approx Q^\nu(x)$.*

Proof. Let Q_s, F_s denote the Q function and distribution function for the random variable $X_1 - X_2$, the symmetrization of X. Then

$$x\int_0^{x^{-1}} (1 - |\varphi(u)|^2)\, du = x\int_0^{x^{-1}} \int (1 - \cos uy)\, dF_s(y)\, du$$
$$= \int \left(1 - \frac{\sin x^{-1}y}{x^{-1}y}\right) dF_s(y) \approx Q_s(x).$$

On the other hand

$$1 - |\varphi(u)|^2 = 1 - e^{\psi(u)+\psi(-u)} = 1 - e^{-\sigma^2u^2 - 2\int(1-\cos uy)d\nu(y)}$$
$$\sim \sigma^2u^2 + 2\int (1 - \cos uy)\, d\nu(y) \quad \text{as } u \to 0.$$

Thus as $x \to \infty$ we have

$$x\int_0^{x^{-1}} (1 - |\varphi(u)|^2)\, du \sim x\int_0^{x^{-1}} \left(\sigma^2u^2 + 2\int (1 - \cos uy)\, d\nu(y)\right) du \approx Q^\nu(x)$$

as above. Thus we have $Q^\nu(x) \approx Q_s(x)$ and $Q_s(x) \approx Q(x)$ is proved in Lemma 2.7 of [4].

Now we can conclude the proof that $\mathscr{H} \subset \mathscr{F}$. Define an increasing sequence $\{a_n\}$ for large n by $Q(a_n) = n^{-1}$ (Q is continuous and strictly decreasing for large x). It is shown in Lemma 1 of [3] that if X is not in $\mathscr{F}$ then there exist sequences $m_j \le n_j$ tending to infinity such that

$$\frac{n_j}{m_j} \to 1, \quad \frac{a_{n_j}}{a_{m_j}} \to \infty.$$

But if the distribution of X is in $\mathscr{H}$ we know there is an $\alpha > 0$ such that $x^\alpha Q^\nu(x)\downarrow$. This leads to a contradiction: for large j,

$$\left(\frac{a_{n_j}}{a_{m_j}}\right)^\alpha \le \frac{Q^\nu(a_{m_j})}{Q^\nu(a_{n_j})} \le C\frac{Q(a_{m_j})}{Q(a_{n_j})} = C\frac{n_j}{m_j} \to C.$$

REFERENCES

[1] FELLER, W. (1967). On regular variation and local limit theorems. *Proc. Fifth Berkeley Symp. Math. Statist. Probab.* Vol. II, Part 1. University of California Press, Berkeley, 373–388.
[2] GNEDENKO, B. V., and KOLMOGOROV, A. N. (1954). *Limit Distributions for Sums of Independent Random Variables* (translated from the Russian). Addison-Wesley, Cambridge, Massachusetts.
[3] GRIFFIN, P., JAIN, N., and PRUITT, W. (1984). Approximate local limit theorems for laws outside domains of attraction. *Ann. Probab.,* to appear.
[4] GRIFFIN, P. (1983). Probability estimates for the small deviations of d-dimensional random walk. *Ann. Probab.* **11** 939–952.
[5] PRUITT, W. (1981). General one-sided laws of the iterated logarithm. *Ann. Probab.* **9** 1–48.

SCHOOL OF MATHEMATICS
UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MINNESOTA 55455