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SOME RESULTS ON INCREMENTS OF THE WIENER PROCESS WITH APPLICATIONS TO LAG SUMS OF I.I.D. RANDOM VARIABLES[1]

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Let $W(t)$ be a standardized Wiener process. In this paper we prove that

$$\limsup_{T\to\infty} \max_{a_T \le t \le T} \frac{|W(T) - W(T-t)|}{\{2t[\log(T/t) + \log\log t]\}^{1/2}} = 1 \quad \text{a.s.}$$

under suitable conditions on a_T. In addition we prove various other related results all of which are related to earlier work by Csörgő and Révész.

Let $\{X_k\}$ be an i.i.d. sequence of random variables and let $S_N = X_1 + \cdots + X_N$. Our original objective was to obtain results similar to the ones obtained for the Wiener process but with N replacing T and S_N replacing $W(T)$. Using the work of Komlós, Major, and Tusnády on the invariance principle, we obtain the desired results for i.i.d. sequences as immediate corollaries to our work for the Wiener process.

1. Introduction. The motivation for the work done here came from a specific statistical problem. Suppose $\{X_k\}$ is a sequence of random variables whose distribution functions approach (in some appropriate sense) a fixed distribution with mean μ. If the sequence is independent (or is appropriately dependent), then under mild regularity conditions $S_n/n \to \mu$ almost surely. There will usually be bias (or non-random error) associated with the earlier X_k's. One might hope to reduce this bias by "throwing away" some of these earlier X_k's, by considering averages of the form $(S_n - S_{n-k_n})/k_n$ so that only the k_n most recent observations are used in computing the nth average.

In [9] we showed that $\max_{k_n \le k \le n} |S_n - S_{n-k}|/k \to 0$ almost surely for appropriate sequences $\{k_n\}$ under assumptions on $E|X_k|^r$ or on $E\{\exp(\theta X_k)\}$. We assumed that $\{X_k\}$ was an i.i.d. sequence of random variables but our method of proof can be extended to cover certain non-i.i.d. cases as well.

We were originally interested only in obtaining $(S_n - S_{n-k_n})/k_n \to 0$ almost surely. The maximum came "free" (i.e., with no additional assumptions).

The purpose of this paper is to continue the investigation of the asymptotic behavior of weighted sums of the form $(S_n - S_{n-k_n})/d(n, k_n)$. The sequence X_k will clearly not be i.i.d. in most statistical cases of interest; however, we restrict our attention to the i.i.d. case in this paper. Our results in the non-i.i.d. case involve entirely different methods of proof from those used in this paper and will be put into manuscript form later.

Our proofs here involve three things: a sort of duality between certain types of asymptotic results; results on the Wiener process and arguments involving the Wiener process, both of which are similar to those of Csörgő and Révész in [5]; and the invariance principles of Komlós, Major, and Tusnády from [12] and [15].

In the next section we present our simple duality argument and use it to show how closely related our work is to the Erdős-Rényi new law of large numbers [8] and the body of work which has arisen in that area (see, e.g., [1], [2], [7], and [20]).

In Section 3 we state and prove our results on the Wiener process. They are similar to those given by Csörgő and Révész in [5] and are extensions of those results. Though our

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original purpose in obtaining the results of Section 3 was to use them in our investigation of weighted sums, the results in Section 3 are of independent interest. Section 4 contains some discussion of these results.

Our results on weighted sums are presented in Section 5 and discussed in Section 6.

2. On the relationship between our work and the Erdös-Rényi new law of large numbers. Since completing our work in [9], we have become interested in the complete limiting behavior of sequences of the form $(S_n - S_{n-k_n})/k_n$. We have become aware that our work is very closely related to a body of work involving generalizations of the Erdős-Rényi new law of large numbers and that results related to ours have already been obtained for Wiener processes.

The following extremely simple lemma is highly useful in our work on the Wiener process, as well as being the main tool we will use in showing the relationship between Theorem 2.1 of [9] and the Erdős-Rényi new law of large numbers.

LEMMA 2.1. *Suppose $S = (0, \infty)$ or $S = \{1, 2, \cdots\}$ and that for each x in S we have sets A_x and B_x of real numbers. Suppose $m: S \to S$, $\lim_{x\to\infty} m(x) = \infty$, and*

(2.1) $$A_x \subset \cup_{y \geq m(x)} B_y \text{ for every } x \text{ in } S.$$

Then

(2.2) $$\limsup_{x\to\infty}(\sup A_x) \leq \limsup_{x\to\infty}(\sup B_x).$$

The proof is easy and omitted. Note that if the relationship which exists between the A_x's and the B_x's also holds when the roles of the A_x's and B_x's are interchanged, then we have equality in (2.2). Note also that Lemma 2.1 says nothing about the relationship between $\liminf_{x\to\infty}(\sup A_x)$ and $\liminf_{x\to\infty}(\sup B_x)$ unless $\lim_{x\to\infty}(\sup B_x)$ exists. Similar results hold for $\inf A_x$ and $\inf B_x$ but here we are interested only in (2.2).

The following theorem is an immediate consequence of Lemma 2.1.

THEOREM 2.1. *Suppose k_n is a non-decreasing sequence of integers such that $k_n \to \infty$ and for each n either $k_{n+1} = k_n$ or $k_{n+1} = k_n + 1$. Then the following are all equal:*

(2.3) $$\limsup_{n\to\infty} \max_{k_n \leq k \leq n}\left(\frac{S_n - S_{n-k}}{d(n, k)}\right),$$

(2.4) $$\limsup_{n\to\infty} \max_{0 \leq k < m \leq n, k_n \leq m-k}\left(\frac{S_m - S_k}{d(m, m-k)}\right), \quad \text{and}$$

(2.5) $$\limsup_{n\to\infty} \max_{0 \leq k \leq n-k_n}\left(\frac{S_{k+k_n} - S_k}{d(k+k_n, k_n)}\right).$$

The notation is meant to imply that $d(n, k)$ is a real valued function defined (at least) when its arguments are non-negative integers such that $n_0 \leq k \leq n$, and also to imply that S_n is either a sequence of real numbers or a sequence of random variables. If the S_n's are random variables we obtain equality of (2.3), (2.4), and (2.5), not just equality almost everywhere.

PROOF. Let

$$A_n = \{(S_n - S_{n-k})/d(n, k): k_n \leq k \leq n\},$$

$$B_n = \{(S_m - S_k)/d(m, m-k): 0 \leq k < m \leq n \text{ and } k_n \leq m - k\}, \quad \text{and}$$

$$C_n = \{(S_{k+k_n} - S_k)/d(k+k_n, k_n): 0 \leq k \leq n - k_n\}.$$

Applying Lemma 2.1 all six possible ways to the sequences A_n, B_n, and C_n gives the result.

An easy way to do the "bookkeeping" necessary to verify that (2.1) holds in the various cases is to define $\phi(n, k) = (S_n - S_{n-k})/d(n, k)$ and note that applying ϕ to various sets in the plane gives A_n, B_n, and C_n. For example, if $A_n^* = \{(n, k): n = N, k_N \le k \le N\}$, then $\phi(A_N^*) = A_N$, and if $B_N^* = \{(m, m-k): 0 \le k < m \le N \text{ and } k_N \le m-k\}$, then $\phi(B_N^*) = B_N$; the observation that $B_N^* \subset \cup_{j=k_N}^N A_j^*$ implies that $B_N \subset \cup_{j=k_N}^N A_j$, and an application of Lemma 2.1 shows that (2.4) ≤ (2.3).

Erdős and Rényi [8] proved (their new law of large numbers) that if X_n is an i.i.d. sequence of random variables having zero mean and moment generating function $m(\cdot)$ existing in some open interval about zero, then if $\exp(-1/c) = \inf\{m(\theta)\exp(-\theta\alpha)\}$ we have

$$\lim_{n\to\infty} \max_{0\le k\le n-[c(\log n)]} \left(\frac{S_{k+[c(\log n)]} - S_k}{[c(\log n)]}\right) = \alpha \text{ almost surely.} \tag{2.6}$$

Using Theorem 2.1 with $d(n, k) = k$ and k and $k_n = [c(\log n)]$ we immediately obtain

$$\limsup_{n\to\infty} \max_{[c(\log n)]\le k\le n} \left(\frac{S_n - S_{n-k}}{k}\right) = \alpha \quad \text{almost surely} \tag{2.7}$$

so that (in the i.i.d. case)

$$\lim_{n\to\infty} \max_{[c(\log n)]\le k\le n} \left(\frac{S_n - S_{n-k}}{\phi(k)}\right) = 0 \quad \text{a.s.} \quad \text{if} \quad \frac{\phi(k)}{k} \to \infty. \tag{2.8}$$

Since $\alpha \to 0$ as $c \to \infty$, we see that (2.7) plus a little argument will also give

$$\lim_{n\to\infty} \max_{k_n\le k\le n} \left(\frac{S_n - S_{n-k}}{k}\right) = 0 \quad \text{a.s.} \quad \text{if } k_n/(\log n) \to \infty. \tag{2.9}$$

Thus, when X_k is an i.i.d. sequence, our Theorem 2.1 in [9] is a consequence of the Erdős-Rényi new law of large numbers.

Using the methods of this section; Theorem 1 of Csörgő and Révész [5]; and the invariance principle results of Komlós, Major, and Tusnády (Theorem 2 of [12] and the Corollary to Theorem 1 in [15]) we can obtain (fairly quickly) (2.3) of our Theorem 3.1 from [9] when $r > 2$.

3. Results similar to those of Csörgő and Révész on the increments of a Wiener process. Throughout this paper C will denote various positive constants whose exact numerical values do not matter so that, for example, $1 + C = C$ might appear in this notation. We will use "log log x" to mean

$$\log\log x = \log\log(\max\{x, e\}). \tag{3.1}$$

$W(t)$ for $0 \le t < \infty$ will be a standardized Wiener process (with continuous sample paths). Sometimes $[x]$ will denote the greatest integer less than or equal to x; we hope it is clear when this is the case.

In [5] Csörgő and Révész presented the following two theorems.

THEOREM A. *Let a_T for $T > 0$ satisfy*

$$a_T \text{ is nondecreasing,} \tag{3.2}$$

$$0 < a_T \le T, \text{ and} \tag{3.3}$$

$$a_T/T \text{ is nonincreasing.} \tag{3.4}$$

Define $\beta_T = \{2a_T[\log(T/a_T) + \log\log T]\}^{-1/2}$. Then

$$\limsup_{T\to\infty} \sup_{0\le t\le T-a_T} \beta_T |W(t + a_T) - W(t)| = 1 \quad \text{a.s.} \tag{3.5a}$$

and

$$\limsup_{T\to\infty} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} \beta_T |W(t + s) - W(t)| = 1 \quad \text{a.s.} \tag{3.5b}$$

If, in addition,

$$\lim_{T\to\infty} \frac{\log(T/a_T)}{\log\log T} = \infty, \tag{3.6}$$

then "lim sup" *may be replaced by* "lim" *in both* (3.5a) *and* (3.5b).

THEOREM B. *If a_T satisfies* (3.2), (3.3), *and* (3.4) *then*

$$\limsup_{T\to\infty} \beta_T | W(T) - W(T - a_T) | = 1 \quad \text{a.s.} \tag{3.7}$$

This is not Theorem 2 from Csörgő and Révész [5], but instead is the result of Step 1 on page 735 in the proof of their Theorem 1. This form is more suitable for our purposes than is their Theorem 2 which is a form of interest to Lai (see [13] and [14]).

In this section we will prove the following theorems. They are similar to, or extensions of, the Csörgő-Révész results. Some will be used in Section 5. All are of interest independent of their value in Section 5.

THEOREM 3.1. *Suppose $0 < a_T \le T$ for $T > 0$ and that*

$$a_T T^{\alpha} \to \infty \text{ as } T \to \infty \text{ for every } \alpha > 0. \tag{3.8}$$

Then

$$\limsup_{T\to\infty} \max_{a_T\le t\le T} \frac{| W(T) - W(T-t) |}{\{2t[\log(T/t) + \log\log t]\}^{1/2}} = 1 \quad \text{a.s.} \tag{3.9a}$$

and

$$\limsup_{T\to\infty} \max_{a_T\le t\le T} \max_{0\le s\le t} \frac{| W(T) - W(T-s) |}{\{2t[\log(T/t) + \log\log t]\}^{1/2}} = 1 \quad \text{a.s.} \tag{3.9b}$$

THEOREM 3.2A.

$$\lim_{a\to\infty} \sup_{0\le t} \frac{| W(t+a) - W(t) |}{\{2a[\log((t+a)/a) + \log\log a]\}^{1/2}} = 1 \quad \text{a.s.} \tag{3.10a}$$

$$\lim_{a\to\infty} \sup_{0\le t} \sup_{0\le s\le a} \frac{| W(t+s) - W(t) |}{\{2a[\log((t+a)/a) + \log\log a]\}^{1/2}} = 1 \quad \text{a.s.} \tag{3.10b}$$

THEOREM 3.2B. *Suppose $0 < a_T \le T$ for $T > 0$ and that*

$$a_T \to \infty \text{ as } T \to \infty. \tag{3.11}$$

Then

$$\limsup_{T\to\infty} \max_{0\le t\le T-a_T} \frac{| W(t+a_T) - W(t) |}{\{2a_T[\log((t+a_T)/a_T) + \log\log a_T]\}^{1/2}} \le 1 \quad \text{a.s.} \tag{3.12a}$$

and

$$\limsup_{T\to\infty} \max_{0\le t\le T-a_T} \max_{0\le s\le a_T} \frac{| W(t+s) - W(t) |}{\{2a_T[\log((t+a_T)/a_T) + \log\log a_T]\}^{1/2}} \le 1 \quad \text{a.s.} \tag{3.12b}$$

If, in addition, a_T is onto, then we have equality in (3.12a) *and* (3.12b).

THEOREM 3.3A.

$$\lim_{a\to\infty} \sup_{0\le u\le v-a} \frac{| W(v) - W(u) |}{\{2(v-u)[\log(v/(v-u)) + \log\log(v-u)]\}^{1/2}} = 1 \quad \text{a.s.} \tag{3.13a}$$

and

$$\text{(3.13b)}\qquad \lim_{a\to\infty}\sup_{0\le u\le s\le t\le v, a\le v-u} \frac{|W(t)-W(s)|}{\{2(v-u)[\log(v/(v-u)) + \log\log(v-u)]\}^{1/2}} = 1 \quad \text{a.s.}$$

THEOREM 3.3B. *Suppose* $0 < a_T \le T$ *for* $T > 0$ *and that* $a_T \to \infty$ *as* $T \to \infty$. *Then*

$$\text{(3.14a)}\qquad \limsup_{T\to\infty}\max_{0\le u<v\le T, a_T\le v-u} \frac{|W(v)-W(u)|}{\{2(v-u)[\log(v/(v-u)) + \log\log(v-u)]\}^{1/2}} = 1 \quad \text{a.s.}$$

and

$$\text{(3.14b}\qquad \limsup_{T\to\infty}\max_{0\le u\le s\le t\le v\le T, a_T\le v-u} \frac{|W(t)-W(s)|}{\{2(v-u)[\log(v/(v-u)) + \log\log(v-u)]\}^{1/2}} = 1 \quad \text{a.s.}$$

Some of the relationships between Theorems 3.1, 3.2(A and B), and 3.3(A and B) via the methods of Lemma 2.1 will become apparent in our proofs.

We will need the following lemma in our proofs. Define

$$\text{(3.15)}\qquad d(T,t) = \{2t[\log(T/t) + \log\log t]\}^{1/2}.$$

LEMMA 3.1. *For each fixed* $t > 0$, $d(T, t)$ *is an increasing function of* T *for* $T > t$. *For each fixed* $T \ge e^e$, $d(T, t)$ *is an increasing function of* t *for* $0 < t \le T$. *For each fixed* $s > 0$, $d(s + t, t)$ *is an increasing function of* t *for* > 0.

PROOF. The first assertion is obvious. The second requires an analysis of the partial derivative of $(d(T, t))^2$ with respect to t; the cases $0 < t < e$, $e < t < e^e$, and $e^e < t < T$ must be considered individually. The third assertion again requires calculus arguments and uses the fact that $f(x) = \log(1 + x) + (1 + x)^{-1} - 1$ is positive for $x > 0$.

PROOF OF THEOREM 3.2A. The proof is similar to the Csörgő-Révész proof, but the change in denominators requires some alterations in the proof.

Suppose $\theta > 1$. Let $a_k = \theta^k$,

$$\text{(3.16)}\qquad A(a) = \sup_{0\le t}\max_{0\le s\le a}|W(t+s) - W(t)|/d(t+a, a),$$

and $A_k = A(a_k)$. Then if $\varepsilon > 0$

$$\text{(3.17)}\qquad \begin{aligned}&\textstyle\sum_{k=1}^{\infty} P\{A_k \ge 1+\varepsilon\}\\ &\le \textstyle\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} P\left(\max_{(n-1)a_k\le t\le na_k, 0\le s\le a_k} \frac{|W(t+s)-W(t)|}{d(t+a_k, a_k)} \ge 1+\varepsilon\right)\end{aligned}$$

$$\text{(3.18)}\qquad \le \textstyle\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} P(\max_{0\le t\le a_k, 0\le s\le a_k}|W(t+s)-W(t)|/d(na_k, a_k) \ge 1+\varepsilon).$$

Using Lemma 1* from [5] we bound (3.18) by

$$\textstyle\sum_{k=1}\sum_{n=1} C\exp\{-(2(1+\varepsilon)^2/(2+\varepsilon))(\log n + \log\log a_k)\}.$$

Let $\gamma = 2(1+\varepsilon)^2/(2+\varepsilon)$. We note that $\gamma > 1$ and obtain

$$(3.16) \le \textstyle\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} C(n\log(a_k))^{-\gamma} \le \sum_{k=1}^{\infty} C\, k^{-\gamma} < \infty.$$

Thus, via the Borel-Cantelli lemma, $\limsup_{k\to\infty} A(a_k) \le 1+\varepsilon$ for every $\varepsilon > 0$ so

$$\text{(3.19)}\qquad \limsup_{k\to\infty} A(\theta^k) \le 1 \quad \text{a.s.}\quad \text{for every } \theta > 1.$$

It follows that (still using $a_k = \theta^k$) for every $\theta > 1$ we have

$$\limsup_{a\to\infty}\sup_{0\le t}\sup_{0\le s\le a}|W(t+s)-W(t)|/d(t+a,a)$$

$$= \limsup_{k\to\infty}\sup_{a_k\le a\le a_{k+1}}\sup_{0\le t}\sup_{0\le s\le a}|W(t+s)-W(t)|/d(t+a,a)$$

(3.20a)
$$\le \left(\limsup_{k\to\infty}\sup_{0\le s\le a_{k+1},0\le t}|W(t+s)-W(t)|/d(t+a_{k+1},a_{k+1})\right) \times \left(\sup_{0\le t}\frac{d(t+a_{k+1},a_{k+1})}{d(t+a_k,a_k)}\right)$$

$$\le 1 \times \left(\sup_{0\le t}\frac{\theta[\log((t+a_{k+1})/a_{k+1})+\log\log a_{k+1}]}{\log((t+a_k)/a_k)+\log\log a_k}\right)^{1/2} = \theta^{1/2} \quad \text{a.s.}$$

We deal with "lim inf" as follows:

(3.20b)
$$\liminf_{a\to\infty}\sup_{0\le t}|W(t+a)-W(t)|/d(t+a,a)$$

$$\ge \liminf_{a\to\infty}\sup_{0\le t\le a^2-a}|W(t+a)-W(t)|/d(t+a,a)$$

$$= \liminf_{T\to\infty}\sup_{0\le t\le T-\sqrt{T}}|W(t+\sqrt{T})-W(t)|/d(t+\sqrt{T},\sqrt{T})$$

(3.20c)
$$\ge \liminf_{T\to\infty}\sup_{0\le t\le T-\sqrt{T}}\beta_T|W(t+\sqrt{T})-W(t)|$$

where $\beta_T = \{2\sqrt{T}[\log(T/\sqrt{T})+\log\log T]\}^{-1/2}$ so that β_T^{-1} is the Csörgő-Révész denominator with $a_T = \sqrt{T}$. By (3) of their Theorem 1 [5] we see that (3.20c) = 1 a.s. and hence (3.20b) ≥ 1 a.s. It follows that (3.10a) is true. The "lim inf" in (3.10b) is greater than or equal to the "lim inf" in (3.10a), so is greater than or equal to one. Hence (3.10b) is true also.

PROOF OF THEOREM 3.2B. (3.12a) and (3.12b) follow immediately from (3.10a) and (3.10b) respectively.

If a_T is onto then, since $a_T \to \infty$,

$$\limsup_{T\to\infty}\max_{0\le t\le T-a_T}|W(t+a_T)-W(t)|/d(t+a_T,a_T)$$

$$\ge \limsup_{T\to\infty}|W(a_T)-W(0)|/d(a_T,a_T)$$

$$= \limsup_{t\to\infty}|W(t)|/\{2t(\log\log t)\}^{1/2} = 1 \quad \text{a.s.}$$

by the law of the iterated logarithm. Thus we have equality in (3.12a), and hence in (3.12b).

LEMMA 3.2. *If* $0 < \alpha \le 1$ *then*

(3.21)
$$\limsup_{T\to\infty}\max_{T^\alpha\le t\le T}|W(T)-W(T-t)|/d(T,t) = 1 \quad \text{a.s.}$$

PROOF. This is an immediate corollary to (3.12a) of Theorem 3.2B. Set $a_T = T^\alpha$. Then an application of Lemma 2.1 gives

$$\limsup_{T\to\infty}\max_{T^\alpha\le t\le T}\frac{|W(T)-W(T-t)|}{d(T,t)}$$

$$\le \limsup_{T\to\infty}\max_{0\le t\le T-T^\alpha}\frac{|W(t+a_T)-W(t)|}{d(t+a_T,a_T)},$$

and (3.12a) bounds this above by one. Equality is obtained by noting that a lower bound is $\limsup_{T\to\infty}|W(T)|/d(T,T)$ which is at least one by the law of the iterated logarithm.

LEMMA 3.3. *Suppose* $0 < a_T \le T$ *for* $T > 0$, *that* $a_T \downarrow 0$ *as* $T \to \infty$, *that* (3.8) *holds, and that* $\varepsilon > 0$. *Then there is a* β *in* (0, 1) *such that*

(3.22)
$$\limsup_{T\to\infty}\max_{a_T\le t\le T^\beta,0\le s\le t}|W(T)-W(T-s)|/d(T,t) \le 1+\varepsilon \quad \text{a.s.}$$

PROOF. Let δ in $(0, \varepsilon)$ be arbitrary and choose $0 < \beta < \min\{1, \varepsilon/(1+\varepsilon)\}$. Let $b_N = [1/\delta a_N]$, let $u_N = [(1 + (N-1)^\beta)/\delta a_N] + 1$, let $x_j^{(N)} = N - j\delta a_N$, and for $0 \le j \le b_N$ and $1 \le k \le u_N \simeq N^\beta/\delta a_N$ let

$$P_{jk}^{(N)} = P\{\max_{(k-1)\delta a_N \le t \le k\delta a_N} | W(x_j^{(N)}) - W(x_j^{(N)} - t)| \ge (1+\varepsilon)\, d(N, k\delta a_N)\}.$$

Since $P\{\max_{a\le t\le b} | W(b) - W(t)|/(b-a)^{1/2} \ge s\} \le 4e^{-s^2/2}$ for all $s \ge 1$, if N is large enough then $\log(N/u_N\delta a_N) > 1$ and (for $j = 0, \cdots, b_N$ and $k = 1, \cdots, u_N$) we have $P_{jk}^{(N)} \le 4\exp\{-(1+\varepsilon)^2\log(N/k\delta a_N)\} \le 4(k\delta a_N/N)^{1+2\varepsilon}$; thus

$$(P_N =)\ \sum_{j=0}^{b_N} \sum_{k=1}^{u_N} P_{jk}^{(N)} \le C(b_N + 1)(a_N/N)^{1+2\varepsilon} \int_1^{u_N+1} x^{1+2\varepsilon}\, dx$$

$$\simeq C(a_N)^{-1}(a_N/N)^{1+2\varepsilon}(N^\beta/a_N)^{2+2\varepsilon} = C(a_N)^{-2}N^{-1-2(\varepsilon-\beta-\varepsilon\beta)}$$

and, from (3.8) and our choice of β, we get $\sum P_N < \infty$. Thus

$$\limsup_{N\to\infty} \max_{0\le j\le b_N, 1\le k\le u_N} \max_{(k-1)\delta a_N \le t\le k\delta a_N} | W(x_j^{(N)}) - W(x_j^{(N)} - t)|/d(N, k\delta a_N) \le 1 + \varepsilon \quad \text{a.s.} \tag{3.23}$$

Consider any sufficiently large T and any t in $[a_T, T^\beta]$. Then for some large N we have $N - 1 < T \le N$; for some $j = 0, \cdots, b_N$ we have $x_j^{(N)} - \delta a_N \le T \le x_j^{(N)}$; and for some $k \ge 1/\delta$ we have $x_j^{(N)} - k\delta a_N < T - t \le x_j^{(N)} - (k-1)\delta a_N$ so that

$$\begin{aligned}
\max_{a_T\le s\le t\le T^\beta} | W(T) - W(T-s)|/d(T, t) &= \max_{a_T\le s\le T^\beta} | W(T) - W(T-s)|/d(T, s) \\
&\le \max_{0\le j\le b_N}\{\max_{a_T\le t\le T^\beta}[| W(x_j^{(N)}) - W(T)|/d(T, t) \\
&\qquad + | W(x_j^{(N)}) - W(T-t)|/d(T, t)]\} \\
&\le \{\max_{0\le j\le b_N}\max_{0\le t\le \delta a_N} | W(x_j^{(N)}) - W(x_j^{(N)} - t)|/d(N, \delta a_N)\} \\
&\qquad \cdot \{d(N, \delta a_N)/d(T, a_T)\} \\
&\quad + \left(\max_{0\le j\le b_N}\max_{1/\delta\le k\le u_n}\max_{(k-1)\delta a_N\le t\le k\delta a_N} \frac{| W(x_j^{(N)}) - W(x_j^{(N)} - t)|}{d(N, k\delta a_N)}\right) \\
&\qquad \times (\max_{1/\delta\le k\le u_N} d(N, k\delta a_N)/d(T, (k-1)\delta a_N)) \\
&= (A)(B) + (C)(D).
\end{aligned} \tag{3.24}$$

Now from (3.23) we see that

$$\limsup_{N\to\infty}(A) \le 1 + \varepsilon \quad \text{a.s.} \quad \text{and} \quad \limsup_{N\to\infty}(C) \le 1 + \varepsilon \quad \text{a.s.} \tag{3.25}$$

$$\begin{aligned}\limsup_{N\to\infty}\max_{N-1\le T\le N}(B) &\le \limsup_{N\to\infty} d(N, \delta a_N)/d(N-1, a_N) \\ &= \limsup_{N\to\infty}\{\delta\log(N/\delta a_N)/\log((N-1)/a_N)\}^{1/2} = \delta^{1/2}.\end{aligned} \tag{3.26}$$

$$\begin{aligned}&\limsup_{N\to\infty}\max_{N-1\le T\le N}(D) \\ &\quad\le \limsup_{N\to\infty}\max_{1/\delta\le k\le u_N} d(N, k\delta a_N)/d(N-1\ (k-1)\delta a_N) \\ &\quad\le \limsup_{N\to\infty}\max_{1/\delta\le k\le u_N}(k/(k-1))^{1/2} \le (1-\delta)^{-1/2}.\end{aligned} \tag{3.27}$$

Thus

$$\limsup_{T\to\infty}\max_{a_T\le s\le t\le T^\beta} \frac{| W(T) - W(T-s)|}{d(T, t)} \le (\delta^{1/2} + (1-\delta)^{-1/2})(1+\varepsilon) \quad \text{a.s.} \tag{3.28}$$

Now

$$\max_{0\le s\le a_T\le t\le T^\beta}|W(T)-W(T-s)|/d(T,t)=\max_{0\le s\le a_T}|W(T)-W(T-s)|/d(T,a_T)$$

$$\le\{\max_{0\le j\le b_N}\max_{0\le t\le\delta a_N}|W(x_j^{(N)})-W(x_j^{(N)}-t)|/d(N,\delta a_N)\}$$

$$\cdot\{d(N,\delta a_N)/d(T,a_T)\}$$

$$+\left(\max_{0\le j\le b_N}\max_{1\le k\le(1/\delta)+1}\max_{(k-1)\delta a_N\le t\le k\delta a_N}\frac{|W(x_j^{(N)})-W(x_j^{(N)}-t)|}{d(N,k\delta a_N)}\right)\cdot \tag{3.29}$$

$$\times(\max_{1\le k\le(1/\delta)+1}d(N,k\delta a_N)/d(T,a_T))$$

$$=(A)(B)+(C)(D).$$

(A) and (B) are (A) and (B) from our treatment of (3.24). (C) is treated (via (3.23)) like (C) from our treatment of (3.24).

$$\limsup_{T\to\infty}\max_{N-1\le T\le N}(D)$$
$$\le\limsup_{N\to\infty}d(N,(1+\delta)a_N)/d(N-1,a_N)\le(1+\delta)^{1/2}. \tag{3.30}$$

Thus

$$\limsup_{T\to\infty}\max_{0\le s\le a_T\le t\le T^\beta}|W(T)-W(T-s)|/d(T,t)$$
$$\le((1+\delta)^{1/2}+\delta^{1/2})(1+\varepsilon)\quad\text{a.s.} \tag{3.31}$$

We put (3.28) and (3.31) together, observe that $\delta>0$ was arbitrary (and independent of β), and let $\delta\downarrow 0$ to finish the proof of the lemma.

LEMMA 3.4. *Suppose a_T satisfies the hypotheses of Lemma 3.3. Then*

$$\limsup_{T\to\infty}\max_{a_T\le t\le T}|W(T)-W(T-t)|/d(T,t)\le 1\quad\text{a.s.} \tag{3.32}$$

PROOF. Fix $\varepsilon>0$ and choose $\beta=\beta(\varepsilon)$ from Lemma 3.3. Then, from Lemma 3.2 and Lemma 3.3

$$\limsup_{T\to\infty}\max_{a_T\le t\le T}|W(T)-W(T-t)|/d(T,t)$$
$$\le\max\left(\limsup_{T\to\infty}\max_{T^\beta\le t\le T}\frac{|W(T)-W(T-t)|}{d(T,t)},\ \limsup_{T\to\infty}\max_{a_T\le t\le T^\beta}\frac{|W(T)-W(T-t)|}{d(T,t)}\right)$$
$$\le\max\{1+\varepsilon,1\}.$$

Since $\varepsilon>0$ was arbitrary we are done.

PROOF OF THEOREM 3.1. Since $d(T,t)$ is an increasing function of t for $T\ge e^e$ (via Lemma 3.1), if $T\ge e^e$ then

$$\max_{a_T\le t\le T,a_T\le s\le t}\frac{|W(T)-W(T-s)|}{d(T,t)}=\max_{a_T\le t\le T}\frac{|W(T)-W(T-t)|}{d(T,t)}.$$

Thus, if $a_T\downarrow 0$ we see that Lemma 3.4 gives

$$\limsup_{T\to\infty}\max_{a_T\le s\le t\le T}|W(T)-W(T-s)|/d(T,t)\le 1\quad\text{a.s.} \tag{3.33}$$

If $a_T\downarrow 0$ then Lemma 3.3 implies that

$$\limsup_{T\to\infty}\max_{0\le s\le a_T\le t\le T}|W(T)-W(T-s)|/d(T,t)$$
$$=\limsup_{T\to\infty}\max_{0\le s\le a_T=t}|W(T)-W(T-s)|/d(T,t)\le 1+\varepsilon\quad\text{a.s.} \tag{3.34}$$

for every $\varepsilon > 0$. Putting (3.33) and (3.34) together and letting $\varepsilon \downarrow 0$ gives an upper bound of one in (3.9b) and hence in (3.9a). Looking at what happens when $t = T$ in (3.9a), and when $s = t = T$ in (3.9b), we get a lower bound of one via the law of the iterated logarithm. Finally, we note that we have proved the theorem only under the assumption that $a_T \downarrow 0$, but that once it is known to be true for such a_T's then the theorem is clearly true also when the assumption $a_T \downarrow 0$ is removed.

PROOF OF THEOREM 3.3A. We will prove (3.13b). Note that

$$\begin{aligned}(3.35)\quad &\limsup_{a\to\infty} \sup_{0\le u\le s\le t\le v, a\le v-u} |W(t) - W(s)|/d(v, v-u)\\ &= \lim_{a_0\to\infty}\sup_{0\le u\le s\le t\le v, a_0\le a\le v-u} |W(t) - W(s)|/d(v, v-u)\\ &= \lim_{a\to\infty}\sup_{0\le u\le s\le t\le v, a\le v-u} |W(t) - W(s)|/d(v, v-u)\end{aligned}$$

so the limit (possibly infinite) in (3.13b) exists.

Let $a_T = \log T$ and suppose ω is such that the "lim sup" in (3.9b) is equal to one. Fix ω. Suppose $\varepsilon > 0$. Choose T_0 so that $T_0 \ge e^e$ and so that

$$(3.36)\quad \sup_{T_0\le T}\max_{a_T\le t\le T}\max_{0\le s\le t}| W(T) - W(T-s)|/d(T, t) \le 1 + \varepsilon.$$

If $a_0 = \log T_0$ then, using Lemma 3.1 and letting $\tau = \min\{v-u, t\}$, we see that

$$\begin{aligned}&\sup_{0\le u\le s\le t\le v, a_0\le a\le v-u, T_0\le t}| W(t) - W(s)|/d(v, v-u)\\ (3.37)\quad &\le \sup_{T_0\le t}\sup_{a_t\le v-u}\max_{0\le t-s\le v-u, 0\le s}| W(t) - W(s)|/d(v, v-u)\\ &\le \sup_{T_0\le t}\max_{a_t\le \tau\le t}\max_{0\le t-s\le \tau}| W(t) - W(t-(t-s))|/d(t, \tau) \le 1 + \varepsilon.\end{aligned}$$

In addition, $\max_{0\le s\le t\le T_0}| W(t) - W(s)|$ is finite, and $d(v, v-u) \to \infty$ uniformly in v (note that $v \ge v-u$) as $v - u \to \infty$, so that

$$(3.38)\quad \limsup_{a\to\infty}\sup_{0\le u\le s\le t\le v, a\le v-u, t\le T_0}| W(t) - W(s)|/d(v, v-u) = 0.$$

Combining (3.37) and (3.38) and noting that $\varepsilon > 0$ was arbitrary shows that (3.35) ≤ 1 a.s. Letting $u = s = 0$ and $t = v = a$ and using the law of the iterated logarithm gives (3.13b). (3.13a) follows from (3.13b) and the law of the iterated logarithm.

PROOF OF THEOREM 3.3B. The upper bounds in (3.14a) and (3.14b) follow easily from (3.13a) and (3.13b) respectively.

We note that if a_t is replaced by some $b_T \ge a_T$ then the "lim sup's" in (3.14a) and (3.14b) will not be increased. We choose b_T so as to be continuous and non-decreasing. We then obtain equality in (3.14a) and (3.14b) via the law of the iterated logarithm.

4. Remarks. The second form of each of our theorems (as given in (3.9b), (3.10b), (3.12b), (3.13b), and (3.14b)) is given for comparison with the second form of the original Csörgő-Révész results. The additional generality provided is not that great and the relationships between the various results are obscured by the technicalities involved in the proofs of the second forms. On the other hand, if a_T satisfies (in addition to (3.11))

$$(4.1)\quad a_T \text{ is continuous and non-decreasing,}$$

then the "lim sup's" in (3.9a), (3.12a), and (3.14a) are all equal via six applications of Lemma 2.1. As an example:

$$\{(t + a_T, a_T): 0 \le t \le T - a_T\} \subset \cup_{a_T\le\tau\le T}\{(\tau, t): a_T \le t \le \tau\} \subset \cup_{a_T\le\tau}\{(\tau, t): a_\tau \le t \le \tau\}$$

so that

$$\begin{aligned}&\{| W(t + a_T) - W(t)|/d(t + a_T, a_T): 0 \le t \le T - a_T\}\\ &\qquad\subset \cup_{a_T\le\tau}\{| W(\tau) - W(\tau - t)|/d(\tau, t): a_\tau \le t \le \tau\};\end{aligned}$$

it follows (via Lemma 2.1) that

$$\limsup_{T\to\infty}\max_{0\le t\le T-a_T}|W(t+a_T)-W(t)|/d(t+a_T,a_T)$$
$$\le \limsup_{T\to\infty}\max_{a_T\le t\le T}|W(T)-W(T-t)|/d(T,t).$$

The denominators which we use did not arise from an attempt to generalize the Csörgő-Révész results, but rather from an attempt (via Lemma 2.1) to understand the relationship between weighted differences of the form $(W(T)-W(T-t))/(\text{denominator})$—which appear in Theorem 3.1 and are of the right form for our application to weighted sums of i.i.d. random variables—and weighted differences of the form $(W(t+a_T)-W(t))/(\text{denominator})$—which appear in the Csörgő-Révész work and in our generalizations of it, Theorems 3.2A and 3.2B.

Think of T as time. Then in the "A" version of each of our theorems, at a given time we are looking at various differences $W(t)-W(s)$ divided by the denominator $d(t,t-s)$ which may depend on T indirectly, but should be thought of as depending *only* on the leading index t and either the other index s or the difference $t-s$. In the original Csörgő-Révész results a particular difference $W(t+a_T)-W(t)$ may have different denominators at different "times" if $a_T=a_S$ for $T\ne S$ or if different functions a_T are used (i.e., when $a_S'=a_T$).

As long as $a_T\to\infty$ our Theorem 3.2B gives stronger upper bound results than do the Csörgő-Révész theorems; our denominators for certain differences may be quite a bit smaller than theirs and, in addition, we require neither (3.2) nor (3.4). Furthermore, because of our denominator, we can get the more elegant results contained in Theorem 3.2A. Note that in both (3.12a) and (3.12b) of Theorem 3.2B we could replace $\max_{0\le t\le T-a_T}$ by $\max_{0\le t}$, but then the relationships between (3.12a) and both (3.9a) and (3.14a) become less clear.

The assumption $a_T\to\infty$ is required in Theorem 3.2B. If $0<\liminf_{T\to\infty}a_T<\infty$ then there exist a Δ in $(0,\infty)$ and an increasing sequence T_k such that $T_k\to\infty$ and $a_{T_k}\to\Delta$. Then taking $t=0$ gives

$$\limsup_{T\to\infty}\max_{0\le t\le T-a_T}|W(t+a_T)-W(t)|/d(t+a_T,a_T)$$
$$\ge \lim_{k\to\infty}|W(a_{T_k})-W(0)|/d(a_{T_k},a_{T_k})$$

which is $|W(\Delta)|/d(\Delta,\Delta)$ if a_T is continuous. Since $W(\Delta)$ is $N(0,\Delta)$, this last quantity is larger than one with positive probability. In Theorem 3.2B we could replace the assumption that $a_T\to\infty$ by the assumption $\liminf_{T\to\infty}a_T>0$ if we replace our denominator $d(t+a_T,a_T)$ by some denominator $D(T,t+a_T,a_T)$ such as $\{2a_T[\log((t+a_T)/a_T)+\max\{\log\log a_T,\log\log\log T\}]\}^{1/2}$ which is, in a sense, intermediate asymptotically to $d(t+a_T,a_T)$ and $(\beta_T)^{-1}$. The sole function of the addition of the term $\log\log\log T$ is to insure that the denominator goes to infinity as $T\to\infty$ no matter what t and a_T are.

In most of our results, once we obtained an upper bound of one on our lim sup's, showing that one is also a lower bound was easy via the law of the iterated logarithm. The same was true in Theorem 3.2B once we assumed that a_T is continuous. (Note, as pointed out by Csörgő and Révész, that continuity of a_T is implied by Conditions (3.2), (3.3), and (3.4) of their theorem.) However, the fact that our various lim sup's are bounded below by one is of only minor interest compared to questions related to behavior like that in (3.7) of Theorem B of Csörgő and Révész which provides much more precise lower bound information. Note that, because of the particular term they pick to look at, their denominator is almost ours for that term. I.e.,

$$\beta_T|W(T)-W(T-a_T)|\simeq|W(T)-W(T-a_T)|/d(T,a_T).$$

The difference in the denominators is that the $\log\log T$ term in β_T was replaced by $\log\log a_T$ in our denominator. Most of the time this will make no asymptotic difference, but in any case $\beta_T\le 1/d(T,a_T)$ so that the lower bound results obtained by Csörgő and Révész apply to our case as well.

QUESTION 4.1. Under what conditions on a_T do we get

$$\limsup_{T\to\infty} | W(T) - W(T-a_T)|/d(T, a_T) \geq 1 \quad \text{a.s.?}$$

I.e., can the Csörgő-Révész conditions be relaxed?

In Theorems 3.2B and 3.3B the requirement that $a_T \to \infty$ as $T \to \infty$ is necessary. Otherwise a fixed term $| W(t; \omega) - W(0; \omega)|/d(t, t)$ for a fixed ω will appear (in the sets over which maxima or suprema are taken) for arbitrarily large values of T. If this term is bigger than one, then the lim sup must be bigger than one. The following discussion is pertinent only to Theorem 3.1 unless our denominator is changed.

We originally proved Theorem 3.1 under the assumption that $a_T \to \infty$ (as in Theorems 3.2B and 3.3B), then assuming that $a_T \equiv \Delta > 0$, and finally under our condition (3.8) which allows $a_T \to 0$ if it does so slowly enough. The following questions about Theorem 3.1 arise:

QUESTION 4.2. Can assumption (3.8) be weakened or eliminated?

QUESTION 4.3. In (3.9a) can $\max_{a_T \leq t \leq T}$ be replaced by $\sup_{0<t\leq T}$?

QUESTION 4.4. If the answer to the Question 4.3 is "no", can the denominator in (3.9a) be changed so as to allow this replacement, and if so, how?

Let $S(h, a, b) = \max_{a\leq t\leq b}| W(t + h) - W(t)|$. An extension of the arguments of P. Lévy (see the discussion in [17]) shows that

$$P\{\lim_{h\downarrow 0} S(h, a, b)/[2h(\log h^{-1})]^{1/2} = 1\} = 1. \tag{4.2}$$

This shows that $\sup_{0<t\leq T}| W(T) - W(T-t)|/d(T, t)$ is finite a.s. Thus the obvious way to get "no" as an answer to Question 4.3 doesn't work.

Note that under condition (3.6) (which amounts to $a_T = T(\log T)^{-\phi(T)}$ where $\phi(T) \to \infty$) Csörgő and Révész were able to replace "$\limsup_{T\to\infty}$" by "$\lim_{T\to\infty}$". Because the arguments required were simple, we used "lim" instead of "lim sup" in Theorem 3.2A and Theorem 3.3A, but our results contain no detailed study of the "lim inf's".

QUESTION 4.5. Under what conditions can we replace "lim sup" by "lim" in the theorems of Section 3? When we can't replace "lim sup" by "lim", what is "lim inf"?

We note in passing that for our applications in the next section we can assume that $\liminf_{T\to\infty} a_T > 0$ so that Questions 4.2, 4.3, and 4.4 are of interest in the study of Brownian motion but would seem to be of no interest in connection with its application to sums of i.i.d. random variables.

5. Results for i.i.d. sequences. As mentioned in the introduction, the original motivation for all the work in this paper was the investigation of the limiting behavior of (properly normed) sums of the form $S_n - S_{n-k_n}$. In the i.i.d. case (to which this paper is restricted) our results are all fairly easy corollaries to our results in Section 3 on the increments of a Wiener process and the results of Komlós, Major, and Tusnády on the invariance principle. When we began this investigation we were hunting for results like those in Theorems 5.1 and 5.1*. The other theorems came about when we discovered the relationships between our work and that of other investigators.

The numbering of our theorems is intended to draw attention to the relationship between the theorems of this section and the theorems (on which they are based) from Section 3.

Throughout this section we assume that:

(5.1a) $X, X_1, X_2, \cdots$ is an i.i.d. sequence of random variables,

(5.1b) $EX = 0$ and $\operatorname{Var} X = EX^2 = 1$,

(5.1c) a_N is a sequence of non-negative integers such that $1 \le a_N \le N$, and
(5.2) $m(\theta) = E(e^{\theta X})$.

THEOREM 5.1. *Suppose* (5.1) *and* (5.2) *hold,*
(5.3a) *there is a* $\delta > 0$ *such that* $m(\theta) < \infty$ *for all* θ *in* $(-\delta, \delta)$, *and*
(5.3b) $a_N/\log N \to \infty$.
Then

$$\text{(5.4a)} \qquad \limsup_{N\to\infty} \max_{a_N \le k \le N} \frac{|S_N - S_{N-k}|}{\{2k[\log(N/k) + \log\log k]\}^{1/2}} = 1 \quad \text{a.s.}$$

and

$$\text{(5.4b)} \qquad \limsup_{N\to\infty} \max_{a_N \le k \le N} \max_{0 \le j \le k} \frac{|S_N - S_{N-j}|}{\{2k[\log(N/k) + \log\log k]\}^{1/2}} = 1 \quad \text{a.s.}$$

PROOF OF THEOREM 5.1. We go to a new probability space on which we have an image of our i.i.d. sequence and a Wiener process, related via the Komlós, Major, and Tusnády construction. From (3.9b) of Theorem 3.1 we get

$$\text{(5.5a)} \qquad \limsup_{N\to\infty} \max_{a_N \le k \le N} \max_{0 \le j \le k} |W(N) - W(N-j)|/d(N,k) \le 1 \quad \text{a.s.}$$

From Theorem 1 of Komlós, Major, and Tusnády [12], $W(N) - S_N = O(\log N)$ a.s. so that, using Lemma 3.1 and considering separately those N's for which $a_N < (\log N)^3$ and those N's for which $a_N \ge (\log N)^3$, we get

$$\text{(5.5b)} \qquad \limsup_{N\to\infty} \max_{a_N \le k \le N} |W(N) - S_N|/d(N,k) = 0 \quad \text{a.s.}$$

and

$$\text{(5.5c)} \qquad \limsup_{N\to\infty} \max_{a_N \le k \le N} \max_{0 \le j \le k} |W(N-j) - S_{N-j}|/d(N,k) = 0 \quad \text{a.s.}$$

The fact that one is an upper bound in (5.4) follows immediately from (5.5a, b, c). Equality follows by using the law of the iterated logarithm and $j = k = N$.

THEOREM 5.1*. *Suppose* (5.1) *holds,*

$$\text{(5.6a)} \qquad R > 2 \text{ and } E|X|^R < \infty, \text{ and}$$

$$\text{(5.6b)} \qquad \liminf_{N\to\infty} a_N (\log N)/N^{2/R} > 0.$$

Then (5.4a) *and* (5.4b) *hold.*

PROOF OF THEOREM 5.1*. This proof is like that of Theorem 5.1 and is omitted. Instead of using Theorem 1 from [12] we use Theorem 2 of [12] and the Corollary to Theorem 1 in Major's paper [15]. Combined they show that $W(N) - S_N = o(N^{1/R})$.

THEOREM 5.2*A*. *Suppose* (5.1), (5.2), *and* (5.3a) *hold, and that*

$$\text{(5.7)} \qquad (\log a_k)/k \to 0.$$

Then

$$\text{(5.8a)} \qquad \limsup_{k\to\infty} \max_{0 \le n \le a_k} \frac{|S_{n+k} - S_n|}{\{2k[\log((n+k)/k) + \log\log k]\}^{1/2}} = 1 \quad \text{a.s.}$$

and

$$\text{(5.8b)} \qquad \limsup_{k\to\infty} \max_{0 \le n \le a_k} \max_{0 \le j \le k} \frac{|S_{n+j} - S_n|}{\{2k[\log((n+k)/k) + \log\log k]\}^{1/2}} = 1 \quad \text{a.s.}$$

THEOREM 5.2A*. *Suppose* (5.1) *and* (5.6a) *hold, and that*

$$\limsup_{k\to\infty} a_k/(k \log k)^{R/2} < \infty. \tag{5.9}$$

Then (5.8a) *and* (5.8b) *hold.*

PROOF OF THEOREM 5.2A. The proof is similar to the proof of Theorem 5.1. Those n's for which $n \le \exp\{k^{1/3}\}$ should be considered separately from those n's for which $\exp\{k^{1/3}\} < n \le a_k$.

PROOF OF THEOREM 5.2A*. The proof is similar to the proof of Theorem 5.1*. Those n's for which $n \le k^{R/2}$ should be considered separately from those n's for which $k^{R/2} < n \le a_k$.

THEOREM 5.2B. *Suppose* (5.1), (5.2), *and* (5.3) *hold. Then*

$$\limsup_{N\to\infty} \max_{0\le k\le N-a_N} \frac{|S_{k+a_N} - S_k|}{\{2a_N[\log((k + a_N)/a_N) + \log\log a_N]\}^{1/2}} \le 1 \quad \text{a.s.} \tag{5.10a}$$

and

$$\limsup_{N\to\infty} \max_{0\le k\le N-a_N} \max_{0\le j\le a_N} \frac{|S_{k+j} - S_k|}{\{2a_N[\log((k + a_N)/a_N) + \log\log a_N]\}^{1/2}} \le 1 \quad \text{a.s.} \tag{5.10b}$$

THEOREM 5.2B*. *Suppose* (5.1) *and* (5.6) *hold. Then* (5.10a) *and* (5.10b) *also hold.*

Theorems 3.3A and 3.3B have the same analog in the context of this section. The analog of Theorem 3.3A requires some sort of bound (in terms of a) on v in order to use our method of proof. In Theorem 3.3B, as long as $a_T \to \infty$ as $T \to \infty$ it does not matter whether we take $\limsup_{T\to\infty}$ or $\limsup_{a_T\to\infty}$.

THEOREM 5.3. *Suppose* (5.1), (5.2), *and* (5.3) *hold. Then*

$$\limsup_{N\to\infty} \max_{0\le m\le n\le N, a_N\le n-m} \frac{|S_n - S_m|}{\{2(n - m)[\log(n/(n - m)) + \log\log (n - m)]\}^{1/2}} = 1 \quad \text{a.s.} \tag{5.11a}$$

and

$$\limsup_{N\to\infty} \max_{0\le m\le j\le k\le n\le N, a_N\le n-m} \frac{|S_k - S_j|}{\{2(n - m)[\log(n/(n - m)) + \log\log(n - m)]\}^{1/2}} = 1 \quad \text{a.s.} \tag{5.11b}$$

THEOREM 5.3*. *Suppose* (5.1) *and* (5.6) *hold. Then* (5.11a) *and* (5.11b) *hold.*

As indicated, some proofs were omitted since they are like another proof that has been given.

6. More remarks. In comparing the results of Section 5 with those of Section 3, one immediately notices that rates of convergence of a_N to infinity were required for the theorems in Section 5 while no corresponding rates were required for those in Section 3. These rates were required to eliminate the error obtained when S_N was approximated by $W(N)$. It is natural to ask whether these rates are necessary for our theorems or only for our method of proof (i.e., whether our results are "sharp").

Suppose $R > 2$ and that $\{X_n\}$ is an i.i.d. sequence such that $P\{X_n = 0\} = a$ and such that, except for the jump at zero, the distribution of the X_n's is absolutely continuous with density function $f(x) = C|x|^{-R-1}(\log|x|)^{-1-\delta}$ for $|x| \geq 2$. We adjust the constants a and C so that we have a probability distribution and so that $E(X_n^2) = 1$. $EX_n = 0$ automatically from symmetry. We note that $E[|X_n|^R(\log^+|X_n|)^\alpha] < \infty$ if $\alpha < \delta$, but that this expected value is infinite if $\alpha \geq \delta$. It follows that if $A_N = \{\omega: |X_N|^R\log(|X_N|)^\delta \geq N\}$ and $B_N = \{\omega: |X_N|^R\log(|X_N|)^{\delta/2} \leq N\}$, then $P\{\omega \mid \omega \in A_N \cap B_N \text{ infinitely often}\} = 1$. Now $R(\log x) + (\delta/2)(\log\log x) \sim R(\log x)$ so that if N is large enough and $\omega \in A_N B_N$ then we get $R(\log|X_N|) \leq 2(\log N)$ and $|X_N| \geq N^{1/R}(\log|X_N|)^{-\delta/R} \geq CN^{1/R}(\log N)^{-\delta/R} \geq N^{1/R}(\log N)^{-\delta/2}$ so that $P\{|X_N| \geq N^{1/R}(\log N)^{-\delta/2} \text{ i.o.}\} = 1$. Now suppose $a_N \sim N^{2/R}(\log N)^{-1-\varepsilon}$. Then $d(N, a_N) \sim CN^{1/R}(\log N)^{-\varepsilon/2}$. If $\delta < \varepsilon$ then for almost all ω's we have

$$\limsup_{N\to\infty}|X_N|/d(N, a_N) = +\infty. \tag{6.1}$$

Then if $k_N = \min\{a_N, a_{N+1}\} - 1$, for the ω's for which (6.1) holds we get

$$\limsup_{N\to\infty}\max\{|S_N - S_{N-k_N}|/d(N, k_N), |S_{N+1} - S_{N-k_N}|/d(N+1, k_N+1)\} = \infty. \tag{6.2}$$

Thus the "lim sup's" in (5.4a) and (5.4b) are infinite almost surely. The point of this example is that (5.6b) can't be relaxed significantly, that if the conditions under which Theorem 5.1* holds can be improved, the improvement will not be a major one.

We can make a similar argument about Theorem 5.1. If $a_N \sim C(\log N)$ then $d(N, a_N) \sim (2C)^{1/2}(\log N)$. For every $\varepsilon > 0$ we can choose a distribution so that $EX_N = 0$, $E(X_N^2) = 1$, $E[\exp(\varepsilon|X_n|/2)] < \infty$, but $E[\exp(\varepsilon|X_n|)] = \infty$ so that $P\{|X_N| \geq \varepsilon^{-1}(\log N) \text{ i.o.}\} = 1$. By adjusting ε we can make the "lim sup's" in (5.4a) and (5.4b) arbitrarily (though not infinitely) large with probability one.

We note that our assumption in (5.3a) is only that $m(\theta) < \infty$ for θ in a finite interval.

Question 6.1. Can (5.3b) be relaxed if more is assumed about the tails of the distribution of the X_n's? Suppose, for example, that the X_n's are assumed to be bounded or that $P\{|X_n| \geq t\} \leq 1 - \Phi(ct)$ for some $c > 0$ where Φ is the standardized normal distribution function.

Book and Shore [3] have done some work in this direction; see their Theorem 3.

As mentioned in the introduction, we have results like those in Section 5 but for the non-i.i.d. case. (We are putting them in a separate manuscript.) Our results in the non-i.i.d. case are not nearly as good as those obtained in Section 5. Due to the lack of really good rates results for the invariance principle in the non-i.i.d. case, we have been forced to use entirely different methods of proof in the non-i.i.d. case from the method used in Section 5. Can the invariance principle results of Komlós, Major, and Tusnády be extended to cover the non-i.i.d. case?

Question 6.2. If one assumes appropriate bounds on the tails of the distributions involved (e.g., those given by (1.1) or by (1.4) and (1.5)), can the results of Section 5 be extended to cover the non-i.i.d. case?

Book has worked on results like Theorems 5.2B and 5.2B* for some time. His denominators were a little different from ours. Book and Shore [5] investigate the "lim inf" in their setting (as well as obtaining other results). Under Assumption (3.6) where the Csörgő-Révész theorem gives a limit, not just a "lim sup", Csörgő and Steinebach [7] have shown the existence of a limit for the corresponding weighted sums. Their situation is again like that in our Theorems 5.2B and 5.2B* and our method of proof in Section 5 copies theirs.

Question 6.3. What happens in the results of Section 5 when "lim sup" is replaced by "lim inf"?

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