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THE MOTION OF A TAGGED PARTICLE IN THE SIMPLE SYMMETRIC EXCLUSION SYSTEM ON Z^1

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Consider a system of particles moving on the integers with a simple exclusion interaction: each particle independently attempts to execute a simple symmetric random walk, but any jump which would carry a particle to an already occupied site is suppressed. For the system running in equilibrium, we analyze the motion of a tagged particle. This solves a problem posed in Spitzer's 1970 paper "Interaction of Markov Processes." The analogous question for systems which are not one-dimensional, nearest-neighbor, and either symmetric or one-sided remains open. A key tool is Harris's theorem on positive correlations in attractive Markov processes. Results are also obtained for the rightmost particle in the exclusion system with initial configuration Z^-, and for comparison systems based on the order statistics of independent motions on the line.

1. Introduction. Infinite particle systems with a simple exclusion interaction were introduced in Spitzer's 1970 paper, "Interaction of Markov processes." These systems have been studied extensively; see Liggett (1977) for a survey, or Liggett (1980) for a recent study of a related model. Let $p(x, y) = p(0, y - x)$ be the transition probabilities for an irreducible random walk on the integers Z. The corresponding exclusion system η_t is a Markov process with state space $\{0, 1\}^Z \equiv \{\eta : \eta \subset Z\}$—it is an evolution of configurations of indistinguishable particles on Z with at most one particle per site. A particle at x waits an exponentially distributed time with mean 1, then chooses a site y with probability $p(x, y)$. If y is vacant at that time, the particle at x moves to y; otherwise it stays at x. All the holding times and choices according to p are independent. Since the holding times have a continuous distribution, only one particle moves at a time; one may tag an individual particle and follow its motion.

For each $\rho \in [0, 1]$, product measure ν_ρ on $\{0, 1\}^Z$, with marginals $\nu_\rho\{\eta : x \in \eta\} = \rho$, $\forall x \in Z$, is invariant for the simple exclusion process (Spitzer, 1970). Start an exclusion system η_t in its equilibrium ν_ρ, conditioned to have a particle at the origin initially, and let Y_t be the position at time t of that particle. This initial distribution, product measure ν_ρ conditioned to have a particle at 0, is invariant for the translated configuration $-Y_t + \eta_t$, i.e., for the configuration of particles relative to the frame of reference of the tagged particle. At the upper extreme of crowding, $\rho = 1$ and $\eta_0 = Z$, all attempted jumps are excluded and no particle ever moves; $Y_t = 0$, $\forall t, \omega$. At the opposite extreme, $\rho = 0$ and $\eta_0 = \{0\}$, the exclusion mechanism never comes into play and Y_t is the same as an ordinary continuous-time random walk X_t. Consider the case where the random walk X_t with jumps at rate one based on p has $\mathrm{Var}(X_t) = \sigma^2 t < \infty$. A naive guess is that $\mathrm{Var}(Y_t) \approx (1 - \rho)\sigma^2 t$ as $t \to \infty$; this gives the correct values at both $\rho = 1$ and $\rho = 0$, and for all ρ it is asymptotically correct as $t \to 0$. However, for the nearest-neighbor symmetric case, $p(x, x + 1) = p(x, x - 1) = ½$, the correct result is

$$\mathrm{Var}(Y_t) \sim \sqrt{2t/\pi}(1 - \rho)/\rho$$

as $t \to \infty$; this is Theorem 1 and the main result of this paper. This is in sharp contrast to

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the result for the one-sided nearest neighbor case, $p(x, x+1) = 1$, given in Spitzer (1970):

$$\mathrm{Var}(Y_t) = (1-\rho)t.$$

Let us consider these two cases of p in more detail. In any nearest-neighbor case ($p(x, y) = 0$ if $|x-y| > 1$), the paths of different tagged particles cannot cross over each other. A continuous-space system with this property was studied by Harris (1965) in "Diffusions with 'collisions' between particles." Particles in that system undergo independent Brownian motions apart from reflection upon collision with each other. The paths in Harris's system are the order statistics derived from a system of independent Brownian motions, so that the derived paths never cross each other, and the two systems have, for all t and ω, the same set of occupied sites. A Poisson point process on R is invariant in distribution for independent particle motions, and thus also invariant for the system with reflections. Harris considered the system with reflections starting from a Poisson point process on R with intensity one, conditioned to have a particle at the origin. By expressing the position $y_0(t)$ of that tagged particle in terms of the positions of particles in the system of independent motions, he proved that

$$t^{-1/4} y_0(t) \to_d n(0, \sqrt{2/\pi}).$$

This led Spitzer (1970) to conjecture, for the symmetric nearest-neighbor exclusion system starting in equilibrium ν_ρ for $\rho \in (0, 1)$, that the variance of the position of a tagged particle grows like some constant, depending on ρ, times $\sqrt{t}$. However, in the one-sided case $p(x, x+1) = 1$, a particle is never affected by other particles to its left; it can be checked that *even after conditioning* on the position Y_t of the tagged particle, the distribution of the configuration $(\eta_t(Y_t + i), i = 1, 2, \cdots)$ to the right is product measure with density ρ. Thus the process Y_t is Markovian with jumps of $+1$ at rate $1-\rho$, so Y_t is Poisson with mean and variance exactly $(1-\rho)t$; Spitzer (1970) attributes this result to Kesten.

These two cases of p, namely a one-dimensional nearest-neighbor random walk which is either one-sided or else symmetric, are the only cases in which asymptotics for $\mathrm{Var}(Y_t)$ have been established. The remaining nearest-neighbor cases, $p(x, x+1) = 1 - p(x, x-1) \in (\frac{1}{2}, 1)$ must somehow interpolate between $\mathrm{Var}(Y_t) \sim c\sqrt{t}$ when $p(x, x+1) = \frac{1}{2}$ and $\mathrm{Var}(Y_t) \sim ct$ when $p(x, x+1) = 1$. One might conjecture linear growth for $\mathrm{Var}(Y_t)$ in all the intermediate cases in the belief that the symmetric case is more "special" than the one-sided case. More boldly, one might conjecture that $\mathrm{Var}(Y_t) \sim |p(x, x+1) - p(x, x-1)| \cdot (1-\rho)t$ since the underlying random walk is the sum of a symmetric random walk at rate $\min(p(x, x+1), p(x, x-1))$ and a one-sided random walk at rate $|p(x, x+1) - p(x, x-1)|$. In non-nearest-neighbor exclusion systems on Z^1 and in exclusion systems on Z^d for $d \geq 2$, a particle is not "trapped" by its neighbors, which makes linear growth for $\mathrm{Var}(Y_t)$ seem plausible.

CONJECTURE. For the simple exclusion system on Z^d, $d \geq 1$ corresponding to an irreducible random walk with finite variance, starting in product measure ν_ρ with $\rho \in (0, 1)$, the position Y_t of a tagged particle has

$$\mathrm{Var}(Y_t) \sim ct \quad \text{as} \quad t \to \infty$$

for some constant depending on ρ and p, provided that p is *not* one-dimensional, symmetric and nearest-neighbor.

Theorem 2 concerns the non-equilibrium behavior of the exclusion system η_t starting from $\eta_0 = Z^-$. The position of the righmost particle

$$M_t \equiv \max\{y : y \in \eta_t^{Z^-}\}$$

is the same as the position of the tagged particle initially at zero in those cases where p is nearest-neighbor. In the one-sided case $p(x, x+1) = 1$, the lead particle is not affected by any other particle, so M_t is a rate one Poisson process, and the fluctuations of $M_t - t$ are described by the classical central limit theorem and law of the iterated logarithm. Rost

(1981) showed that the system spreads out linearly in time, with a density profile given by

$$P(\lfloor at \rfloor \in \eta_t^{Z^-}) \to (1-a)/2$$

for $a \in [-1, 1]$, where $\lfloor x \rfloor$ denotes the integer n with $n \le x < n+1$. We consider the symmetric, nearest neighbor case $p(x, x+1) = p(x, x-1) = ½$. The standard self-duality relation for symmetric exclusion systems:

$$P(\eta_t^A \cap B \neq \varnothing) = P(A \cap \eta_t^B \neq \varnothing)$$

with $A = Z^-$ and $B = \{\lfloor at^{1/2} \rfloor\}$ immediately yields a density profile result:

$$P(\lfloor at^{1/2} \rfloor \in \eta_t^{Z^-}) \to \int_a^\infty (2\pi)^{-1/2} e^{-x^2/2}\, dx.$$

The position M_t of the lead particle is described in Theorem 2 by

$$t^{-1/2} M_t - \sqrt{\log t} \to_{\text{a.s.}} 0;$$

it is the same result that would be obtained if M_t were the maximum of a countable collection of independent random walks, one starting from each $z \in Z^-$.

Theorems 1 and 2 are proved by considering order statistics of a system of *dependent* random walks, the stirring system. Both theorems use Lemma 1 to get an upper bound on the dependence of the stirring paths; Theorem 1 is sharp because it also exploits the strong dependence of the stirring paths, shown by formula (7). For the sake of comparison with Theorems 1 and 2, we conclude this paper with two theorems describing the order statistics of a system of *independent* paths in place of the stirring paths.

2. Symmetric exclusion motions in terms of stirring motions. For any symmetric random walk p on Z, the system of exclusion motions based on p can be expressed in terms of a system $\xi = \xi(\omega) = (\xi_t^x, x \in Z, t \ge 0)$ of random stirrings, as introduced by Harris (1972) and Lee (1974); see Griffeath (1979) for a recent exposition. The system ξ at all times has exactly one particle per site; the particles at sites $x < y$ are interchanged at rate $p(x, y) = p(y, x)$, with independent Poisson flows of event times for each pair of sites $\{x, y\}$ for which $p(x, y) > 0$. The position at time t of the particle initially at size x is denoted ξ_t^x. Equivalently, $\xi_t^\cdot$ is a random permutation of Z which starts as the identity permutation and to which the transposition (x, y) is appended after exponentially distributed times with mean $1/p(x, y)$, independently for each $\{x, y\}$ for which $p(x, y) > 0$. For each fixed x, the path $\xi_\cdot^x$ is a random walk based on p, starting at x. From now on we will take the realization of the symmetric exlcusion system given by:

$$\text{(1)} \qquad \text{for} \quad A \subset Z, \qquad \eta_t^A \equiv \{\xi_t^x : x \in A\}.$$

Notice that this gives an *additive coupling* of the family of exclusion processes corresponding to p: $\eta_t^{A \cup B} = \eta_t^A \cup \eta_t^B$ for all $A, B \subset Z$.

Following Harris (1965), we will express the set-valued evolution η_t^A in terms of a system of paths $(Y_t^x, x \in A)$ which never cross each other; we loosely describe these Y_t^x as the "order statistics" of the stirring paths ξ_t^x. Thus, we need to define paths Y_t^x for $x \in A$ so that for all $t \ge 0$

$$\text{(2)} \qquad \begin{aligned} &\{Y_t^x : x \in A\} = \eta_t^A \ (\equiv \{\xi_t^x : x \in A\}) \\ &\forall x < y \in A, \qquad Y_t^x < Y_t^y \\ &\forall x \in A, \qquad Y_0^x = x \ (= \xi_0^x) \\ &\forall x \in A,\ Y_\cdot^x \text{ is right-continuous with left limits, and} \\ &\quad |\{i \in A : Y_{t-}^i \neq Y_t^i\}| < \infty. \end{aligned}$$

To accomplish this, define for $x, z \in R$, $t \ge 0$

$$(3)\qquad \mu^+_{A,x}(z,t) = |\{y \in A: y \le x, \xi^y_t \ge z\}|$$

$$\mu^-_{A,x}(z,t) = |\{y \in A: y > x, \xi^y_t < z\}|.$$

Now define the paths $Y^x_t = Y^{x,A}_t$ for $x \in A$ by

$$(4)\qquad Y^x_t\ (= Y^{x,A}_t) = \sup\{z \in R: \mu^+_{A,x}(z,t) > \mu^-_{A,x}(z,t)\}.$$

Here is a proof that the paths $Y^x_t \equiv Y(x,t)$ defined above satisfy the four conditions in (2). Consider $A \subset Z$ and $t \ge 0$ as fixed. Let

$$D(x,z) = \mu^+_{A,x}(z,t) - \mu^-_{A,x}(z,t).$$

As a function of $z \in R$ for fixed x, $D(x,\cdot)$ is a left-continuous decreasing step function with jump -1 at each $z \in \{\xi^y_t: y \in A\}$, and $D(x,-\infty) = |A \cap (-\infty, x]|$, $D(x,\infty) = -|A \cap (x,\infty)|$. Similarly, for fixed z, $D(\cdot, z)$ is right-continuous increasing step function of $x \in R$ with jump $+1$ at each $x \in A$, and $D(-\infty, z) = -|\{y \in A: \xi^y_t < z\}|$, $D(\infty, z) = |\{y \in A: \xi^y_t \ge z\}|$. From the description of $D(x,\cdot)$ it follows that $\{Y(x,t): x \in A\} \subset \eta^A_t$. Fix integers x_0 and z_0 such that $x_0 \in A$, $Y(x_0,t) = z_0$. Label A and η^A_t so that $A = \{\cdots x_{-1} < x_0 < x_1 \cdots\}$ and $\eta^A_t = \{\cdots -z_1 < z_0 < z_1 \cdots\}$. From the jump description of D it follows that for $x \in [x_i, x_{i+1})$ and $z \in (z_{i+j-1}, z_{i+j}]$ we have $D(x,z) = 1 - j$ and hence $Y(x_i, t) = z_i$. This finishes the proof that the Y^x_t defined by equation (4) satisfy the relations in (2).

The definition of the path Y^x_t above is valid in the stirring system based on any symmetric random walk $p(x,y) = p(y,x)$. In the nearest-neighbor case, $p(x,y) = 0$ whenever $|x-y| > 1$, at most one of the paths $(Y^x_t, x \in A)$ moves at any time, so that Y^x_t *is* the path of the tagged particle initially at $x \in A$, in the exclusion system η^A_t defined by (1). When p is symmetric but *not* nearest-neighbor, the path of a tagged particle initially at x in the exclusion system η^A_t is *not* given by Y^x_t. To see this in detail, if $x \in \eta^A_{t-}$ and $y \notin \eta^A_{t-}$ and the particles in ξ_{t-} at sites x and y get stirred at time t, then the paths Y^z which jump at time t are exactly those for which Y^z_{t-} is in the closed interval between x and y.

3. Theorems for the nearest-neighbor, symmetric exclusion system.

LEMMA 1. *In a system $(\xi^x_t, x \in Z)$ of random stirrings, based on symmetric, nearest-neighbor random walk $(p(x, x+1) = p(x, x-1) = ½)$, the events*

$$\{\xi^v_t \ge a\} \quad and \quad \{\xi^w_t \ge b\}$$

are negatively correlated, for all $t \ge 0$ and $v, w, a, b \in Z$ with $v \ne w$.

PROOF. We apply Harris's theorem (1977), which states that for an attractive Markov process X_t on a finite partially ordered state space E, a necessary and sufficient condition for the evolution to preserve the class of measures having positive correlations is that the process can jump only between comparable states. Let $E = \{(x,y) \in Z^2: x \ne y\}$, with the partial order $(x_1, y_1) \le (x_2, y_2)$ iff $x_1 \le x_2$ and $y_1 \ge y_2$. (Note the reverse order for the second component.) The process X_t defined by $X^{(x,y)}_t = (\xi^x_t, \xi^y_t)$ for $(x,y) \in E$ is clearly a Markov process which only jumps up and down in E. We will present below a coupling to show that the process X_t is attractive. Although this is not a process with a finite state space, it is easy to approximate this process by finite-state processes for which Harris's theorem applies directly. For increasing functions on E into R use

$$f(x,y) = 1(x \ge a), \qquad g(x,y) = 1(y < b),$$

and start the process in the deterministic measure "unit mass at (v, w)." Now the statement that the distribution of X_t has positive correlations, applied to f and g, is precisely our goal:

$$P(\xi^v_t \ge a, \xi^w_t < b) = E(fg(X_t)) \ge (Ef(X_t))(Eg(X_t)) = P(\xi^v_t \ge a)P(\xi^w_t < b).$$

The process X_t which we must show to be attractive is the Markov process on E with flip rates

$$(x, y) \to (z, y) \quad \text{at rate} \quad p(x, z) \quad \text{if} \quad z \neq y;$$
$$(x, y) \to (x, z) \quad \text{at rate} \quad p(y, z) \quad \text{if} \quad z \neq x;$$
$$(x, y) \to (y, x) \quad \text{at rate} \quad p(x, y).$$

We will construct simultaneously, for each $e \in E$, a realization X_t^e of the process starting at e, so that for all $\omega \in \Omega$, $t \geq 0$,

$$e < f \in E \quad \text{implies} \quad X_t^e \leq X_t^f. \tag{5}$$

For each $(x, y) \in E$, let W_{xy} be a rate $p(x, y)$ flow of event times, with these flows mutually independent. At an event time in W_{xy}, any X^e in one of the states listed below will jump:

$$(x, y) \to (y, x),$$
$$(x, z) \to (y, z) \quad \text{for} \quad z \neq y, \quad \text{and}$$
$$(z, y) \to (z, x) \quad \text{for} \quad z \neq x.$$

It is trivial to check that at each event time τ, $X_{\tau-}^e < X_{\tau-}^f$ implies $X_\tau^e < X_\tau^f$, so that (5) holds. Note that this coupling differs from the coupling achieved by setting $X_t^{(x,y)} = (\xi_t^x, \xi_t^y)$; essentially, in this latter coupling, for each $x \neq y$ the two flows of event times W_{xy} and W_{yx} are taken to be identical rather than independent. □

Added in revision. The proof above does not work for the case of p symmetric but not nearest-neighbor, because the process X on E fails to be attractive: relation (5) fails in any coupling for $e = (0, x)$, $f = (1, x)$, and t small, if $x \geq 2$ and $p(0, x) > 0$. A referee showed how to prove Lemma 1 for an arbitrary symmetric random walk p on Z by calculating directly with the generator of our process X. Lemma 1 can also be generalized in a different direction. First note that for our application of Lemma 1, only the special case $a = b$ is used. Enrique Andjel pointed out that occupation of a *single* set by two different stirring paths involves negative correlation, and this has nothing to do with the order relation of the line, as shown below.

LEMMA 1′. *Let $p(x, y)$ be the Q matrix for a Markov process on a countable state space S, with $p(x, y) = p(y, x)$, and let $(\xi_t^x, x \in S, t \geq 0)$ be the corresponding stirring system. Thus $\forall x \in S$, $\xi_\cdot^x$ is a realization of the Markov process on S starting from x, and $\forall t \geq 0$, ξ_t is a permutation on S. Then for any $A \subset S$, $x \neq y \in S$, $t \geq 0$, the events $\{\xi_t^x \in A\}$ and $\{\xi_t^y \in A\}$ are negatively correlated.*

PROOF. The pair $\eta_t \equiv (\xi_t^x, \xi_t^y)$ is a realization of the exclusion process with state space $T_2 \equiv \{(i, j) \in S^2: i \neq j\}$. The function $g: S^2 \to R$ defined by $g((i, j)) \equiv 1(\{x, y\} \subset A)$ is bounded, symmetric, and positive definite in the sense of Liggett (1977, page 227). The conclusion of Lemma 2.3.4 in that reference, specialized to this g, is exactly the statement of Lemma 1′. □

THEOREM 1. *For the exclusion system of simple random walks on Z, starting from product measure with $P(x \in \eta_0) = \rho$ for $0 \neq x \in Z$, $P(0 \in \eta_0) = 1$, the position Y_t of that tagged particle initially at the origin satisfies*

$$t^{-1/4} Y_t \to_d n(0, \sqrt{2/\pi}\,(1 - \rho)/\rho).$$

Furthermore, $\operatorname{Var}(t^{-1/4} Y_t) \to \sqrt{2/\pi}\,(1 - \rho)/\rho$.

PROOF. The position $Y_t = Y_t^0$ of the tagged particle in the exclusion system is given by relations (3) and (4), where $(\xi_t^x, x \in Z)$ is a system of nearest neighbor stirrings, independent of the initial configuration $\eta_0 = A$, which is distributed according to the product measure ν_ρ conditioned to have $0 \in A$. These relations reduce to

(6) $$\{\omega\colon Y_t \geq z\} = \{\omega\colon (\textstyle\sum_{y\leq 0} 1(y \in A, \xi_y^t \geq z) - \sum_{y>0} 1(y \in A, \xi_y^t < z)) > 0\},$$

for any $z \in R$. Here is a quick argument to show that $\{t^{-1/4}Y_t, t > 1\}$ is tight; we don't use this result, but its proof illuminates the need to introduce relation (7) below. Fix $a \geq 0$ and let $z = at^{1/4}$. Since

$$\mu^+ \equiv \mu_{A,0}^+(z, t) = \textstyle\sum_{i\leq 0} 1(\xi_t^i \geq z, i \in A)$$

is the sum of indicators of events which, according to Lemma 1, are negatively correlated, it follows that $\mathrm{Var}(\mu^+) \leq E(\mu^+) \sim \rho\sqrt{t/(2\pi)}$. The same consideration for $\mu^- \equiv \mu_{A,0}^-(z, t)$ leads to the bound $\mathrm{Var}(\mu^+ - \mu^-) \leq ct^{1/2}$, and direct calculation yields $E(\mu^+ - \mu^-) \sim \rho z$. Now by relation (6),

$$P(Y_t > at^{1/4}) = P(\mu^+ - \mu^- > 0) \leq ct^{1/2}/(\rho z)^2 = c(\rho a)^{-2}.$$

This shows that $\{t^{-1/4}Y_t\}$ is tight, but this argument cannot establish the desired convergence in distribution or even identify the limiting variance.

A deterministic property of the stirring paths is the key to further progress; for all $0 < z \in R$, $t \geq 0$, and $\omega \in \Omega$,

(7) $$\mu_{Z,0}^+(z, t) - \mu_{Z,0}^-(z, t) = -|Z \cap (0, z)|,$$

where the μ are defined by (3). To prove this claim, check that it is true at $t = 0$ and that any stirring changes the left side of (7) by zero. In detail, the left side is $|B| - |C|$ with $B = \{y \in Z\colon y \leq 0, \xi_t^y \geq z\}$ and $C = \{y \in Z\colon y > 0, \xi_t^y < z\}$. The only stirring which affects either B or C is the transposition $(i, i + 1)$ where $i < z \leq i + 1$. Suppose that x and y are the integers such that $\xi_{t-}^x = i$, $\xi_t^x = i + 1$, $\xi_{t-}^y = i + 1$, and $\xi_t^y = i$. The four possible cases and the resulting changes at time t are

$x \leq 0, y \leq 0$	B gains x, B loses y;
$x \leq 0, y > 0$	B gains x, C gains y;
$x > 0, y \leq 0$	C loses x, B loses y;
$x > 0, y > 0$	C loses x, C gains y;

so that the net change in $|B| - |C|$ is always zero.

Write $N = N(\omega, z, t)$ for the first term in relation (7), so that

$$N = \mu_{Z,0}^+(z, t) = \textstyle\sum_{y\leq 0} 1(\xi_t^Y \geq z)$$

is a sum of indicators of dependent events, which by Lemma 1 are negatively correlated. If $z(t) = o(t^{1/2})$, then $\mathrm{Var}(N) \leq EN = E((\xi_t^0 - \lceil z \rceil + 1)^+) \approx (t/(2\pi))^{1/2}$. By Chebyshev's inequality it follows that

$$P(|N - (t/(2\pi))^{1/2}| > t^{3/8}) \to 0 \quad \text{as} \quad t \to \infty.$$

Now consider the role of the initial state A of the exclusion system. Write $A = \eta \cup \{0\}$ where the distribution of η is product measure ν_ρ. Take z to be a positive integer. Label the two sets of sites which would contribute to expression (6) if A were Z:

$$\{y \leq 0\colon \xi_t^y \geq z\} = \{y_{-N} < y_{-N+1} < \cdots < y_{-1}\},$$

$$\{y > 0\colon \xi_t^y < z\} = \{y_1 < y_2 < \cdots < y_{N+z-1}\}.$$

Let

$$S = \textstyle\sum_{i=-N \text{ to } -1} 1(y_i \in \eta) - \sum_{i=1 \text{ to } N+z-1} 1(y_i \in \eta),$$

so that $Y_t \geq z$ iff $S + 1(\xi_t^0 \geq z, 0 \notin \eta) > 0$. Fix $a \geq 0$ and let $z = z(t) = \lfloor a((1-\rho)/\rho)^{1/2}\cdot$

$(2t/\pi)^{1/4}\rfloor$. After conditioning on the value of N, which depends only on the stirring paths ξ, the distribution of S is of the form described in Lemma 2 below. Let $E = E_t$ be the event $\{|N - (t/(2\pi))^{1/2}| < t^{3/8}\}$, so that $P(E) \to 1$ as $t \to \infty$, and let X denote a $n(0, 1)$ random variable. We have

$$P(Y_t \ge z) = P(S + 1(\xi_t^0 \ge z, 0 \notin \eta) > 0)$$
$$\le P(S > -1) \le P(E^c) + P(S > -1|E)P(E) \to 0 + P(X > a).$$

Similarly $P(Y_t \ge z) \ge P(S > 1) \ge P(S > 1|E)P(E) \to P(X > a)$. We have shown that $P(Y_t \ge z) \to P(X > a)$, where X is $n(0, 1)$ and $z = a((1 - \rho)/\rho)^{1/2}(2t/\pi)^{1/4}$. Thus $t^{-1/4}Y_t \to_d n(0, (2/\pi)^{1/2}(1 - \rho)/\rho)$.

To establish that $\mathrm{Var}(Y_t) \sim (2t/\pi)^{1/2}(1 - \rho)/\rho$, it remains to show that $\{t^{-1/2}Y_t^2, t \ge 1\}$ is uniformly integrable. Here are some of the estimates, to show that $\limsup E((t^{-1/4}|Y_t|)^3) < \infty$. Take $a > 1$ and $z = \lfloor at^{1/4}\rfloor$. Now

$$P(Y_t > at^{1/4}) \le P(N > t^{1/2} + t^{1/4}a^{7/4})$$
$$+ 1 \textstyle\sum_{n \le t^{1/2} + t^{1/4}a^{7/4}} P(N = n)P(X_1 + \cdots + X_n + 1 > Y_1 + \cdots + Y_{n+z-1}).$$

For the first term, since $\mathrm{Var}(N) \le EN \approx (t/(2\pi))^{1/2}$, $P(N > t^{1/2} + t^{1/4}a^{7/4}) \le P(N - EN > t^{1/4}a^{7/4}) \le \mathrm{Var}(N)t^{-1/2}a^{-7/2} < a^{-7/2}$, for all $a > 1$ provided t is large enough that $EN < t^{1/2}$. Routine bounds on the second term lead to $P(Y_t > at^{1/4}) \le ca^{-7/2}$ for all $a > 1$ and $t > t_0$, so that $\sup_{t > t_0} E(t^{-1/4}|Y_t|)^3 < \infty$. □

LEMMA 2. *Let* $S = S_{t,n} = X_1 + \cdots + X_n - (Y_1 + \cdots + Y_{n+z})$, *where the* $X_1, X_2, \cdots, Y_1, Y_2, \cdots$ *are* i.i.d., $P(X_1 = 1) = \rho$, $P(X_1 = 0) = 1 - \rho$, *and* $z = z(t) = \lfloor a((1 - \rho)/\rho)^{1/2} (2t/\pi)^{1/4}\rfloor$. *Then for any* $a \in R$, $\lim_{t \ge 0} \sup_{n:|n - t/\sqrt{2\pi}| < t^{3/8}} |P(S > a) - \int_a^\infty (2\pi)^{-1/2}e^{-x^2/2}\, dx| = 0$.

PROOF. For any $n(t)$ such that $z(t) = o(n(t))$, $\mathrm{Var}(X_1 + \cdots + X_n)/\mathrm{Var}(S) = n/(2n + z) \to ½$, so that $(X_1 + \cdots + X_n - n\rho)/(\mathrm{Var}(S))^{1/2} \to n(0, ½)$ and $(Y_1 + \cdots + Y_{n+z} - (n + z)\rho)/(\mathrm{Var}(S))^{1/2} \to_d n(0, ½)$ as $t \to \infty$. For $n(t) \approx (t/2\pi)^{1/2}$, $ES/(\mathrm{Var}(S))^{1/2} = -z\rho/(\rho(1 - \rho)(2n + z))^{1/2} \approx -a$, so $S/(\mathrm{Var}(S))^{1/2} \to_d n(-a, 1)$ as $t \to \infty$. □

THEOREM 2. *For the exclusion system* $\eta_t = \eta_t^{Z^-}$ *of simple symmetric random walks on* Z *starting with particles on the negative integers, the position of the rightmost particle,* $M_t = \max(\eta_t)$, *satisfies*

$$t^{-1/2}M_t - \sqrt{\log t} \to_{\text{a.s.}} 0.$$

PROOF. We will first give the argument to show that $t^{-1/2}M_t - \sqrt{\log t}$ converges to zero in probability; a routine extension of this using a skeleton of times and the Borel-Cantelli lemma then shows the almost sure convergence. As in relation (1), the set η_t of sites occupied by the exclusion system with $\eta_0 = Z^- = \{\cdots, -2, -1, 0\}$ is realized in terms of a system $(\xi_t^x, x \in Z)$ of nearest neighbor stirrings by

$$\eta_t = \eta_t^{Z^-} = \{\xi_t^i: i \in Z, i \le 0\}.$$

Given some deterministic $z = z(t)$, define $N = N(t, \omega)$ by

$$N = \textstyle\sum_{i \le 0} 1(\xi_t^i \ge z),$$

so that for each ω, $M_t \ge z$ iff $N \ge 1$. (This is the special case of relations (3) and (4) with $A = Z^-$, $x = 0$, $N = \mu_{A,x}^+(z, t)$, $\mu^- = 0$, $M_t = Y_t^0$.) Fix $a \in R$ and let $z = z(a, t)$ be given by

$$z = t^{1/2}((\log t)^{1/2} + a).$$

Since ξ_t^i is a simple random walk starting from i,

$$EN = E((\xi_t^0 - \lceil z\rceil + 1)^+) \to \infty \quad \text{if} \quad a < 0,$$
$$\to 0 \quad \text{if} \quad a > 0,$$

as $t \to \infty$; the calculation is given in Lemmas 3 and 4 below. Thus for $a > 0$, Chebyshev's inequality yields

$$P(t^{-1/2}M_t - (\log t)^{1/2} > a) = P(N \geq 1) \leq EN \to 0$$

as $t \to \infty$. For $a < 0$, we need to use the negative correlations of stirring paths given by Lemma 1 to conclude that $\mathrm{Var}(N) \leq EN$ so that

$$P(t^{-1/2}M_t - (\log t)^{1/2} > a) = P(N = 0) \leq \mathrm{Var}(N)/(EN)^2 \leq 1/(EN) \to 0$$

as $t \to \infty$. This shows that

$$t^{-1/2}M_t - (\log t)^{1/2} \to 0 \quad \text{in probability.}$$

The argument to get almost sure convergence proceeds differently on the two sides of zero. On the right, to get

$$\limsup(t^{-1/2}M_t - (\log t)^{1/2}) \leq 0 \quad \text{a.s.},$$

it suffices to find a skeleton of times $t_n \uparrow \infty$ for which, for every $a > 0$, with $z(a, n) = (t_n)^{1/2}((\log t_n)^{1/2} + a)$, $P((\max_{s \leq t_{n+1}} M_s) \geq z(a, n) \text{ i.o.}) = 0$. Now

$$\sum_{n\geq 0} P(\max_{s\leq t_{n+1}} \max_{i\leq 0} \xi_s^i \geq z(a, n)) \leq \sum_{n\geq 0, i\leq 0} P(\max_{s\leq t_{n+1}} \xi_s^i \geq z(a, n))$$

$$\leq \sum_{n\geq 0, i\leq 0} 2P(\xi_{t_{n+1}}^i \geq z(a, n)) = \sum_{n\geq 0} 2E((\xi_{t_{n+1}}^0 - \lceil z(a, n) \rceil + 1)^+) < \infty.$$

Estimates like those in Lemma 3 show that this last sum is finite when $t_n = \exp(n^\alpha)$, for $\alpha \in (0, 2/3)$.

On the other side, to get

$$\liminf(t^{-1/2}M_t - (\log t)^{1/2}) \geq 0 \quad \text{a.s.,} \tag{8}$$

we take $t_n = e^{n^\alpha}$ for any choice $\alpha \in (0, 1)$ and show, using estimates like those in Lemma 4, that for every $a > 0$,

$$\sum_{n\geq 0} P(M_{t_n} < z(-a, n)) < \infty, \quad \text{and}$$

$$\sum_{n\geq 0} \sum_{i\leq 0} P((\min_{s\in[t_n, t_{n+1}]} \xi_s^i) < z(-2a, n+1) \mid M_{t_n} = \xi_{t_n}^i \geq z(-a, n)) < \infty.$$

From these it follows that for all $a > 0$,

$$\sum_n P(t^{-1/2}M_t - (\log t)^{1/2} < -2a \text{ for some } t \in [t_n, t_{n+1}]) < \infty,$$

which implies (8). This completes the proof that $t^{-1/2}M_t - (\log t)^{1/2} \to 0$ a.s. □

The next two lemmas give routine estimates for rate one continuous time random walk X_t, with $P(X_t = i) = e^{-t} \sum_{n\geq 0} t^n/(n!)P(S_n = i)$, where S_n is discrete time simple symmetric random walk.

LEMMA 3. *For rate one simple random walk X_t, for $a > 0$, with $z(a, t) = \sqrt{t}(\sqrt{\log t} + a)$,*

$$E(X_t; X_t > z(a, t)) \to 0 \quad \text{as} \quad t \to \infty.$$

PROOF. Take $t \to \infty$ along the integers, so that X_t is the sum of t independent copies of X_1. Since $E(\exp(\lambda X_1)) < \infty$ for all λ, a standard large deviation result applies (Feller, 1971, Section XVI.6), namely that $P(X_t > \sqrt{t}\, x(t))/P(X > x(t)) \to 1$, whenever $x(t) \to \infty$ and $x(t) = o(t^{1/6})$, where X is $n(0, 1)$. We have for large t

$$E(X_t; X_t > z(a, t)) = E(X_t; t^{-1/2}X_t \in (\sqrt{\log t} + a, \log t)) + E(X_t; t^{-1/2}X_t \geq \log t).$$

The first term above is dominated by $\sqrt{t}\, \log t\, P(t^{-1/2}X_t > \sqrt{\log t} + a) \sim \sqrt{t}\, \log t\, P(X > \sqrt{\log t} + a) \sim c\sqrt{t} \log t(\sqrt{\log t} + a)^{-1} \exp(-(\sqrt{\log t} + a)^2/2) \sim c'\sqrt{\log t}\, \exp(-a\sqrt{\log t}) =$

$o(1)$. The second term, using Cauchy-Schwartz, is dominated by $E(X_t^2)P(t^{-1/2}X_t > \log t) = o(1)$. □

LEMMA 4. *For rate one simple random walk X_t, with $z(a, t) = \sqrt{t}\ (\sqrt{\log t} + a)$, for $a > 0$,*

$$E((X_t - z(-a, t))^+) \to \infty \quad as \quad t \to \infty.$$

PROOF. As in Lemma 3, consider $t \to \infty$ along the integers to use a large deviation result for i.i.d. sums; let X denote a $n(0, 1)$ random variable. Now for any $a > 0$, for sufficiently large t,

$$\begin{aligned} E((X_t - z(-3a, t))^+) &\ge E(X_t - z(-3a, t);\, z(-2a, t) \le X_t \le z(-a, t)) \\ &\ge a\sqrt{t}\, P(t^{-1/2}X_t \in (-2a + \sqrt{\log t}, -a + \sqrt{\log t})) \sim a\sqrt{t}\, P(X > -2a + \sqrt{\log t}) \\ &\sim c\sqrt{t}\,(-2a + \sqrt{\log t})^{-1} \exp(-(-2a + \sqrt{\log t})^2/2) \\ &\sim c'(\log t)^{-1/2} \exp(2a\sqrt{\log t}) \to \infty. \end{aligned}$$ □

4. Comparison results: order statistics of independent motions. For comparison with Theorem 2 on the rightmost particle in the symmetric exclusion process on Z, we consider in Theorem 3 an analogous system in which the underlying stirring paths are replaced by independent random walks. The estimates in Theorem 3, combined with the step $\mathrm{Var}(N) \le EN$ from Theorem 2, would only show for the exclusion system that $\{b(t)^{-1}M_t - a(t), t > t_0\}$ is tight.

THEOREM 3. *For a system of independent Brownian motions on R, starting with particles occupying an intensity λ Poisson point process on $(-\infty, 0]$, the position M_t of the rightmost particle at time t satisfies*

$$P\left(\frac{M_t}{b(t)} - a(t) < x\right) \to H_{3,0}(x) = \exp(-e^{-x})$$

with scaling and centering constants

$$b(t) = \sqrt{\frac{t}{\log t}}, \qquad a(t) = \log\left(\frac{\lambda t}{\sqrt{2\pi}\log t}\right).$$

This same result, with $\lambda = 1$ above, is obtained for $M_t = \max_{i\le 0} x_i(t)$, where the x_i are independent simple random walks, starting with $x_i(0) = i$ for each $i \in Z$.

PROOF. (The limiting distribution $H_{3,0}$ is well known; see Galambos (1978).) Here is the argument for the system of Brownian motions. Let $z = z(c, t) = \sqrt{\log t} + (\log t)^{-1/2}(c - \log\log t)$. Let $N = N(z\sqrt{t}, t)$ be the number of particles which are to the right of $z\sqrt{t}$ at time t. Let X_t be a standard Brownian motion, so that $X \equiv X_1$ is $n(0, 1)$. By decomposing N as the integral over $x \in [0, \infty)$ of how many particles start between $-x - dx$ and $-x$ and then travel to the right of $z\sqrt{t}$ by time t, we see that N is Poisson with parameter

$$EN = \int_{x\ge 0} \lambda P(X_t - x \ge z\sqrt{t})dx = \lambda E((X_t - z\sqrt{t})^+) = \lambda\sqrt{t}\,E((X - z)^+).$$

Writing $f(z) = (2\pi)^{-1/2}\exp(-z^2/2)$ for the standard normal density, Feller, volume I, gives $P(X > z) \sim f(z)(z^{-1} - z^{-3} + 1\cdot 3z^{-5} - \cdots)$, in the sense of being caught between the oscillations. Thus

$$\begin{aligned} E(X - z)^+ &= E(X; X > z) - zP(X > z) \\ &\sim f(z) - zf(z)(z^{-1} - z^{-3} + 3z^{-5} - \cdots) \sim f(z)(z^{-2} - 3z^{-4} + \cdots). \end{aligned}$$

Plugging in $z = z(c, t)$ which has $z^2 = \log t + 2(c - \log\log t) + o(1)$, we get

$$E(X - z)^+ = (2\pi)^{-1/2}\exp(-\tfrac{1}{2}\log t - (c - \log\log t) - o(1))$$
$$\cdot((\log t + \cdots)^{-1} - 3(\log t + \cdots)^{-2} + \cdots)$$
$$\sim (2\pi)^{-1/2}t^{-1/2}e^{-c}.$$

Thus N is Poisson with parameter approaching $\lambda(2\pi)^{-1/2}e^{-c}$, so

$$P(M_t < z\sqrt{t}) = P(N = 0) \sim \exp(-\lambda(2\pi)^{-1/2}e^{-c}).$$

With $x = c - \log\lambda + \tfrac{1}{2}\log(2\pi)$ this is exactly our goal.

For the system of simple random walks on Z in place of the system of Brownian motions, proceed as above. Now the distribution of $N = N(z\sqrt{t}, t)$ is no longer exactly Poisson, but it still converges to Poisson with parameter $(2\pi)^{-1/2}e^{-c}$. To check this, for large t

$$EN = \textstyle\sum_{i\le 0} P(x_i(t) \ge z\sqrt{t}) = E((x_0(t) - \lfloor z\sqrt{t}\rfloor + 1)^+) \sim \sqrt{t}E((t^{-1/2}x_0(t) - z)^+)$$

$$(10) \qquad = \sqrt{t}\int_{u=z}^{\infty} P(t^{-1/2}x_0(t) > u)du \sim \sqrt{t}\int_{u=z}^{\log t} P(t^{-1/2}x_0(t) > u)\, du$$

$$\sim \sqrt{t}\int_{u=z}^{\log t} P(X > u)\, du \sim \sqrt{t}E((X - z)^+) \to (2\pi)^{-1/2}e^{-c}.$$

To justify the replacement of ∞ by $\log t$ as the upper limit in (10), use Cauchy-Schwartz: $E(x_0(t), x_0(t) > \sqrt{t}\log t) \le E(x_0^2(t))P(x_0(t) > \sqrt{t}\log t) = o(1)$. To justify the replacement of $t^{-1/2}x_0(t)$ in (10), where x_0 is rate one simple random walk, by X in the next line, where X is $n(0, 1)$, use the large deviation result: for integers t, since $x_0(1)$ has a finite Laplace transform in a neighborhood of zero, $|1 - P(t^{-1/2}x_0(t) > u)/P(X > u)| = O(u^3/\sqrt{t})$. The rest of the proof for the random walk case is identical to the proof given for the Brownian motion case, taking $\lambda = 1$. □

The next theorem is intended to highlight the effect of crowding in the exclusion system. Theorem 1 says that for Y_t = a tagged particle in the exclusion system = an order statistic of stirring paths, starting with particles distributed according to product measure ν_ρ, $\mathrm{Var}(t^{-1/4}Y_t) \to \sqrt{2/\pi}(1 - \rho)/\rho$. Clearly the same function of ρ cannot arise in analyzing the order statistics of independent random walks in place of stirring motions, since the factor $1 - \rho$ is zero when $\rho = 1$. Harris (1965) considered the analogous order statistic $y_0(t)$ for a system of independent Brownian motions, starting from a Poisson point process on R with intensity 1, and showed that $t^{-1/4}y_0(t) \to_d n(0, \sqrt{2/\pi})$. By rescaling space by λ and time by λ^2, one transforms this result so that if the initial intensity is $\lambda > 0$, then $t^{-1/4}y_0(t) \to_d n(0, \sqrt{2/\pi}/\lambda)$. Theorem 4 gives exactly the same constant, $\sqrt{2/\pi}/\lambda$, in a system of independent random walks starting with a Poisson point process on $\mathbb{Z}$ with intensity λ. A system starting in product measure ν_ρ with $\rho \in (0, 1]$ corresponds to $\lambda = \rho$, $\sigma^2 = \rho(1 - \rho)$ in Theorem 4, in which case (12) simplifies to

$$t^{-1/4}y(t) \to_d n(0, \sqrt{2/\pi}/\rho - (\sqrt{2} - 1)/\sqrt{\pi});$$

the variance of this limit is $1/\sqrt{\pi}$ at the extreme $\rho = 1$, in contrast to $\sqrt{2/\pi}$ for the intensity-one Poisson initial configuration.

By a system of simple random walks with collisions we mean the following analogue of the system in Harris (1965). Start with a countably infinite collection of particles on the integers, at positions $x_i \le x_{i+1}$, $i \in Z$, with $x_0 = 0 < x_1$, $-\infty = \lim_{i\to-\infty}x_i$, and $+\infty = \lim_{i\to\infty}x_i$. Let $(x_i(t), i \in Z)$ be independent rate one simple random walks with $x_i(0) = x_i$ for all $i \in Z$, with sample paths that are right-continuous with left limits. Except for a set of zero probability, there will be a unique system $(y_i(t), i \in Z, t \ge 0)$ for which for all $i \in Z$, $t > 0$

$$
(11)\quad
\begin{aligned}
&y_i(0) = x_i(0), \quad \text{and} \quad y_i(t) \le y_{i+1}(t)\\
&y_i(\cdot) \text{ is right-continuous with left limits, and}\\
&\quad y_i(t) \ne y_i(t-) \quad \text{implies} \quad y_j(t) = y_j(t-), \qquad \forall j \ne i\\
&N_k(t) \equiv |\{i \in Z: y_i(t) = k\}| = |\{i \in Z: x_i(t) = k\}|.
\end{aligned}
$$

We refer to the $y_i(t)$ as "the order statistics of independent simple random walks," or as "simple random walks with collisions."

THEOREM 4. *In a system* $(y_i(t), i \in Z, t \ge 0)$ *of simple random walks with collisions, starting with* $N_i + 1_{i=0}$ *particles at site i, if the* N_i *are independent of each other and the future evolution, with* $EN_i = \lambda > 0$ *and* $\mathrm{Var}(N_i) = \sigma^2 \in [0, \infty)$, *then for* $y(t) = y_0(t)$, *the path of the "rightmost" particle starting at the origin,*

$$
t^{-1/4} y(t) \to n(0, \lambda^{-1}(2/\pi)^{1/2} + (\sigma^2 - \lambda)\lambda^{-2}(\sqrt{2} - 1)/\sqrt{\pi}). \tag{12}
$$

PROOF. The paths $y_i(t)$ are obtained from a system $(x_i(t), i \in Z)$ of independent motions by the analogue of relations (3) and (4): for $i, z \in Z, t \ge 0$ let

$$
\mu^+ \equiv \mu^+(i, z, t) = \textstyle\sum_{j \le i} 1(x_j(t) \ge z); \qquad \mu^- \equiv \mu^-(i, z, t) = \textstyle\sum_{j > i} 1(x_j(t) < z),
$$

$$
y_i(t) = \max\{z \in Z: \mu^+(i, z, t) > \mu^-(i, z, t)\}.
$$

Fix $i \in Z$. Since λ is the expected number of particles N_k starting at any site k,

$$
E(\mu^+ - \mu^-) \sim -\lambda z \quad \text{as} \quad |z| \to \infty,
$$

uniformly in t. To get $\mathrm{Var}(\mu^+)$, consider the contribution $\sum_j 1(x_j(0) = k, x_j(t) \ge z)$ to μ^+ from particles starting at site k. For i.i.d. Bernoulli variables $I, I_1, I_2, \cdots$ with $P(I = 1) = 1 - P(I = 0) = p$, independent of N, we have

$$
\mathrm{Var}(I_1 + \cdots + I_N) = (EN)\mathrm{Var}(I) + (EI)^2\mathrm{Var}(N) = (EN)p + (\mathrm{Var}\, N - EN)p^2,
$$

so, with $p(t, x) = P(x_0(t) > x)$ for simple random walk x_0,

$$
\mathrm{Var}(\mu^+) = p(t, z)(1 - p(t, z) + \textstyle\sum_{k \le o} (\lambda p(t, k + z) + (\sigma^2 - \lambda)p^2(t, k + z)).
$$

Taking $z = \lceil a t^{1/4} \rceil$, and with $F(x) = P(X > x)$ for $X \in n(0, 1)$,

$$
\begin{aligned}
\mathrm{Var}(\mu^+) &\sim \sqrt{t}\left(\lambda \int_{x \ge 0} F(x)\, dx + (\sigma^2 - \lambda) \int_{x \ge 0} F^2(x)\, dx\right)\\
&= \sqrt{t}(\lambda(2\pi)^{-1/2} + (\sigma^2 - \lambda)(\sqrt{2} - 1)\pi^{-1/2}/2).
\end{aligned}
$$

The same holds for $\mathrm{Var}(\mu^-)$, so

$$
\begin{aligned}
E(\mu^+ - \mu^-)(\mathrm{Var}(\mu^+ - \mu^-))^{-1/2} &\sim \lambda a t^{1/4}[2t^{1/2}(\lambda(2\pi)^{-1/2} + (\sigma^2 - \mu)(\sqrt{2} - 1)\pi^{-1/2}/2)]^{-1/2}\\
&= \lambda a[\lambda(2/\pi)^{1/2} + (\sigma^2 - \lambda)(\sqrt{2} - 1)/(\sqrt{\pi})]^{-1/2}.
\end{aligned}
$$

Now $(\mu^+ - \mu^- - E(\mu^+ - \mu^-))(\mathrm{Var}(\mu^+ - \mu^-))^{-1/2} \to_d X$, so

$$
P(y(t) \ge a t^{1/4}) = P(\mu^+ - \mu^- \ge 0) \sim P(X > -E(\mu^+ - \mu^-)(\mathrm{Var}(\mu^+ - \mu^-))^{-1/2}),
$$

which shows (12). □

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