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SPATIAL GROWTH OF A BRANCHING PROCESS OF PARTICLES LIVING IN $\mathbb{R}^d$

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Consider a branching process in which particles are located in $\mathbb{R}^d$, do not move during their life times, die according to the exponential holding law, and at their deaths give birth to random number of particles which are located at distances from their parents. The total number process is supposed supercritical. We are interested in the number of particles living in a shifted region $D + tc$, denoted by $Z_t(D + tc)$, where $c \in \mathbb{R}^d$ and D is a bounded set of $\mathbb{R}^d$, and observe a.s. convergences of $Z_t(D + tc)/E[Z_t(D + tc)]$ as $t \to \infty$. The result is applied to an associated non-linear evolution equation, which reduces, in a special case, to the equation of a deterministic model of simple epidemics.

1. Introduction. Consider a branching process in which particles are located in $S \equiv \mathbb{R}^d$ (d-dimensional Euclidean space), do not move during their life times, and die and produce offsprings in the following way: if a particle is alive at time t and located at position $x \in S$, then the particle dies in the time interval $(t, t + dt)$ with probability $\kappa dt + o(dt)$ and at the moment of its death it gives birth, with probability $\pi_n(d\mathbf{y})$, to n particles which are located in $x + \mathbf{y} + d\mathbf{y}$, $\mathbf{y} \in S^n$. Here $\pi_n(\cdot)$ is a finite measure on $\mathscr{B}^n$ the totality of Borel sets of S^n (S^n is the n-fold direct sum of the vector space S) and we write

$$x + \mathbf{y} = (x + y_1, \cdots, x + y_n)$$

for $\mathbf{y} = (y_1, \cdots, y_n) \in S^n$. We assume of course $\sum_{n=1}^{\infty} \pi_n(S^n) \leq 1$. The development stated is not affected by the existence of any other particles and the past history of the process. Both the constant κ and the system of measures π_n are independent of x and t. It should be noted that if $\pi_1(S) > 0$, the true rate of holding time for splitting (or vanishing) of a particle is not κ but $\kappa(1 - \pi_1(S))$. For convenience we let $S^0 = \{\partial\}$, $p_0 = 1 - \sum_{n=1}^{\infty} \pi_n(S^n)$ and $\pi_0(\cdot) = p_0\delta_\partial(\cdot)$, where ∂ is an extra point and δ_∂ is the delta measure at ∂.

Let $Z_t(D)$ be the total number of particles living in $D \in \mathscr{B}^1$ at time t and let $Z_t = Z_t(S)$. We will consider $Z_t(\cdot)$ as a (stochastic) measure on S. Our main objective in this paper is to determine the limiting behavior of shifted measures $Z_t(\cdot + tc)$ for all $c \in S$, as t approaches the infinity ($D + x$ denotes $\{y + x : y \in D\}$). Now let the process start with one particle at 0 at time 0 and set $A(\xi) = \log E[\int_S e^{\xi \cdot x} Z_1(dx)]$ for $\xi \in S$ and

$$\nu(c) = \inf_{\xi \in S}(A(\xi) - c \cdot \xi) \quad \text{for} \quad c \in S. \tag{1.1}$$

(Here the dot denotes the usual inner product of d-vectors.) Then under certain regularity conditions on $\{\pi_n\}$ it will be proved that if $\nu(c) > 0$, there exists a set Ω_0 of full probability such that for all bounded $D \in \mathscr{B}^1$ with $\int_{\partial D} dx = 0$ (∂D is the boundary of D)

$$\lim_{t\uparrow\infty} \sqrt{t}^d e^{-\nu(c)t} Z_t(D + tc) = C_\xi \left(\int_D e^{-\xi \cdot x}\, dx \right) W^\xi \quad \text{on} \quad \Omega_0, \tag{1.2}$$

where ξ is a unique solution of $\nu(c) = A(\xi) - c \cdot \xi$, C_ξ is a positive constant and W^ξ is a random variable with $W^\xi > 0$ a.s. on $\{Z_t \to \infty\}$ (Theorem 1; see also Remark 1 after Theorem 1), and that if $\nu(c) \leq 0$, $\lim_{t\to\infty} Z_t(D + tc) = 0$ for all bounded D a.s. (Theorem 2).

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For the proof of the first result we will follow S. Watanabe [28]. The basic idea in [28] is to expand the stochastic measure in (1.2) by a system of martingales W_t^λ (see (5.3) of this paper). In the present case W_t^λ is expressed as

$$W_t^\lambda = e^{-A(\lambda)t} \int e^{\lambda \cdot x} Z_t(dx), \quad \lambda = \xi + i\eta$$

where $\xi, \eta \in S$, $i = \sqrt{-1}$ and $A(\lambda)$ is defined for λ with $A(\xi)$ finite; W^ξ in (1.2) is the limit of W_t^ξ as $t \to \infty$. The main and most hard task in this paper is to show an L_α-boundedness and an L_α-continuity (w.r.t. η at 0) for the martingales W_t^λ with some $\alpha > 1$. In the case treated by Watanabe, the martingales were L_2-bounded which followed readily from one of basic assumptions in view of a simple relation for the second moment of $Z_t(D)$ (cf. the equations (2.23) and (3.21) in [28]), while in the present case the L_2-boundedness is violated for some ξ appearing in (1.2) and there is no simple relation for estimating $| W_t^\lambda |^\alpha$. Once the estimates of W_t^λ are obtained, the path to (1.2) is a saddle point method in which the stochastic nature is managed by martingale inequalities. The same estimates of W_t^λ and computations as leading to (1.2) will prove, without any additional cost, the convergence of the following stochastic measures, provided $\nu(c) > 0$:

$$\text{(1.3)} \qquad \sqrt{t}^d e^{-\nu(c)t + (\xi \cdot a)\sqrt{t}} e^{(\xi \cdot x)} Z_t(dx + \sqrt{t}a + tc) \quad (a \in S),$$

$$\text{(1.4)} \qquad e^{-\nu(c)t} e^{(\xi \cdot x)\sqrt{t}} Z_t(\sqrt{t}\, dx + tc).$$

The limiting measures are $\Phi^\xi(a) W^\xi\, dx$ for (1.3) and $W^\xi \Phi^\xi(x)\, dx$ for (1.4), where $\Phi^\xi(x)$ is the density of a Gaussian probability measure on S and dx is the Lebesgue measure on S.

J. D. Biggins [7] has obtained results similar to (1.2) for discrete time and one dimensional processes by a different method. His results would yield the corresponding ones for continuous time processes by applying a technique which was used by Asmussen and Kaplan [2]. When c the velocity of shift takes a particular value which corresponds to $\xi = 0$ in the present situation, convergences of measures as defined in (1.3) and (1.4) have been established in a number of papers for the various kinds of processes [2], [13], [15], [21], [23], [24]. A recent work of Y. Ogura [22] is also relevant: he treates a binary splitting branching process of Brownian particles with drift on a positive real axis and investigates the limiting behavior of the process according to the intensity of the drift.

In Section 7 we shall study asymptotic behavior of a stochastic region of propagation of the process without assuming main conditions imposed in Theorems 1 or 2.

In the last section (Section 9) we shall make use of the results as mentioned above for investigating the behavior of solutions of a non-linear evolution equation, which in a special case reduces to

$$\text{(1.5)} \qquad \frac{\partial v}{\partial t} = \left(\int v(t, x + y) H(dy) \right) (1 - v)$$

where $H(dy)$ is a probability measure on $\mathscr{B}^1$. This equation appears as the deterministic model for simple epidemics (cf. [4], [12], [19], [20]).

2. Preliminaries and main results. Let us denote by $x_{t,k}$ the position of the kth particle alive at time t and set

$$\mathbf{x}_t = (x_{t,1}, \cdots, x_{t,n})$$

if $Z_t = n \geq 1$ and $\mathbf{x}_t = \partial$ if $Z_t = 0$, where Z_t is the total number of particles alive at t as defined in Section 1. Then $\mathbf{x}_t$ is a strong Markov process on the state space $\mathbf{S} = \cup_{n=0}^\infty S^n$ with trap ∂ (see Ikeda *et al.* [14] for a detailed description and construction of the process). We will denote by $P_\mathbf{x}$ the probability measure of the process $\mathbf{x}_t$ whose initial state is $\mathbf{x}$ and by $E_\mathbf{x}$ the integration by $P_\mathbf{x}$. If $\mathbf{x} = x \in S$, we will write simply P_x or E_x. For a Borel measurable complex valued function f on S we define a function on $\mathbf{S}$:

(2.1) $$\check{f}(\mathbf{x}) = \begin{cases} f(x_1) + \cdots + f(x_n) & \text{if } \mathbf{x} = (x_1, \cdots, x_n) \\ 0 & \text{if } \mathbf{x} = \partial \end{cases}$$

and then set

(2.2) $$\mathcal{M}f(x) = \sum_{n=1}^{\infty} \int_{S^n} \check{f}(x + \mathbf{y})\pi_n(d\mathbf{y}), \quad x \in S$$

if the right hand side is absolutely convergent (i.e. $\mathcal{M}|f|(x)$ is finite) for all $x \in S$. From the condition (C.2) which will be introduced soon it evidently follows that $\mathcal{M}1(x) = \mathcal{M}1(0) < \infty$ and we can define a measure $G(dx)$ on $\mathscr{B}^1$ through the relation that $\int f(x)G(dx) = \mathcal{M}f(0)$ for all bounded f: then (2.2) is rewritten as

(2.2)′ $$\mathcal{M}f(x) = \int_S f(x + y)G(dy).$$

Let for $\lambda = (\lambda^1, \cdots, \lambda^d) \in \mathbb{C}^d$ (a complex d-vector)

$$\phi_\lambda(x) = e^{\lambda \cdot x}, \quad x \in S$$

where $\lambda \cdot x$ is the usual inner product, and let

$$A(\lambda) = \kappa\left(\int_S \phi_\lambda(x)G(dx) - 1\right),$$

$$W_t^\lambda = e^{-A(\lambda)t}\,\check{\phi}_\lambda(\mathbf{x}_t),$$

which will be regarded as making sense if and only if $A(\mathscr{R}\lambda)$ is finite. (Here $\mathscr{R}\lambda = (\mathscr{R}\lambda^1, \cdots, \mathscr{R}\lambda^d)$; $\mathscr{R}z$ denotes the real part of $z \in \mathbb{C}$.) It will be seen in the next section that if $A(\mathscr{R}\lambda) < \infty$, W_t^λ is a martingale with respect to P_x and to the monotone family of σ-fields generated by $\{\mathbf{x}_s : 0 \le s \le t\}$, $t \ge 0$; in particular $E_0[\check{\phi}_\lambda(\mathbf{x}_t)] = e^{A(\lambda)t}$ and the definition of $A(\lambda)$ is consistent with what we have made in Section 1.

The following conditions are assumed throughout the paper:

(C.1) $A(0) > 0$;
(C.2) $\sum_{n=1}^{\infty} n^3\pi_n(S^n) < \infty$;
(C.3) $G(\cdot)$ *is not supported by any* $(d-1)$*-dimensional hyperplane of* S;
(C.4) *for each* $\xi \in S$ *there is an* $\varepsilon > 0$ *such that* $A(\varepsilon\xi) < \infty$.

The conditions (C.1) and (C.2) are equivalent, respectively, to $P_x[Z_t \to \infty] > 0$ (i.e. the total number process Z_t, which is clearly a Galton-Watson process, is super-critical) and to $E_x[(Z_t)^3] < \infty$ $(t > 0)$. Let Θ be the unit sphere in S and for each $\theta \in \Theta$ define the positive number $r(\theta) \le \infty$ through the relation

(2.3) $$\inf_{0<r<\infty} \frac{A(r\theta)}{r} = \lim_{r\uparrow r(\theta)} \frac{A(r\theta)}{r}.$$

Since $A(r\theta)$ is a strictly convex function of $r > 0$ as far as it is finite, $r(\theta)$ is uniquely determined and $A(r\theta)/r$ is strictly decreasing in $0 < r < r(\theta)$. If $r(\theta) < \infty$ and $A(r\theta)$ is finite beyond $r(\theta)$, $r(\theta)$ is a unique point at which $A(r\theta)/r$ attains its minimum and the minimum agrees with the value of the derivative $(d/dr)A(r\theta)$ at $r = r(\theta)$. Set

$$T = \{r\theta : 0 \le r < r(\theta) \text{ and } \theta \in \Theta\},$$

and let us have as given an $\alpha > 1$ and $\alpha\xi \in T$. Then $A(\alpha\xi) < \alpha A(\xi)$ and therefore there is a $\delta > 0$ such that

(2.4) $$A(\alpha\xi) < \alpha\mathscr{R}A(\lambda) \quad \text{for} \quad \lambda = \xi + i\eta, \qquad \eta \in S \quad \text{with} \quad |\eta| < \delta$$

($i = \sqrt{-1}$, $|\eta| = \sqrt{\eta \cdot \eta}$). The next proposition is essential to prove the main theorem (Theorem 1) by the present method. We need the following condition

$$\text{(A.1)} \qquad B(\xi) \equiv \textstyle\sum_{n=1}^{\infty} n \left[\int_{S^n} \check{\phi}_\xi(\mathbf{y}) \pi_n(d\mathbf{y}) \right] < \infty \quad \text{for all} \quad \xi \in T.$$

PROPOSITION 1. *Suppose* (A.1) *holds. Let* $1 < \alpha \le 2$ *and* $\alpha\xi \in T$, *and choose* $\delta > 0$ *so that* (2.4) *is valid. Then for* λ *appearing in* (2.4), W_t^λ *is* L_α*-bounded, namely* $\sup_{t>0} E[|W_t^\lambda|^\alpha] < \infty$.

For $\xi \in S$ with $A(\xi) < \infty$ let

$$W^\xi \equiv \lim_{t \uparrow \infty} W_t^\xi \quad \text{(a.s.)},$$

which exists (finite valued) since W_t^ξ is a positive martingale. In the last section it will be deduced from Proposition 1 that under (A.1)

$$\text{(2.5)} \qquad W^\xi > 0 \quad \text{a.s. on} \quad \{Z_t \to \infty\} \quad \text{for} \quad \xi \in T.$$

To state the main theorem we need further a few notations. Let X be the set of all inner points of the domain of convergence of $A(\xi)$ ($\xi \in S$). $A(\xi)$ is regular in X. Using the convexity of the exponential function e^t and applying (C.4) to any d independent unit vectors $\theta_1, \cdots, \theta_d$, we see $\sup_{\theta \in \Theta} A(\varepsilon\theta) < \infty$ for some positive ε; in particular $0 \in X$. By an application of Hölder's inequality, we see that ξ belongs to X if and only if $A(\alpha\xi) < \infty$ for some $\alpha > 1$. Therefore $T \subset X$. For $\xi \in X$ we denote the first and the second derivatives of $A(\xi)$ by $\mathbf{D}A(\xi)$ and by $\mathbf{D}^2 A(\xi)$, respectively; in other words $\mathbf{D}A(\xi)$ is the vector $\kappa \int x\phi_\xi(x) G(dx)$ and $\mathbf{D}^2A(\xi)$ the matrix whose (j, k)-component is $\kappa \int x_j x_k \phi_\xi(x) G(dx)$. Because of (C.3) $\mathbf{D}^2A(\xi)$ is positive definite. Therefore by the inverse mapping theorem $X^* \equiv \{\mathbf{D}A(\xi) : \xi \in X\}$ is open and the mapping

$$\text{(2.6)} \qquad \xi \rightsquigarrow c = \mathbf{D}A(\xi)$$

is a homeomorphism of X onto X^*. Now we set

$$T^* = \{\mathbf{D}A(\xi) : \xi \in T\}$$

and formulate the main theorem as the convergence of a measure valued process μ_t^c defined for $c \in T^*$:

$$\mu_t^c(D) = \sqrt{t}^d e^{-(A(\xi) - c\cdot\xi)t} Z_t(D + tc), \quad D \in \mathscr{B}^1,$$

where ξ corresponds to c as the inverse image by the mapping (2.6).

THEOREM 1. *Assume the Condition* (A.1) *and*

$$\text{(A.2)} \qquad G(\cdot) \text{ is not supported by a set } \{ne : n \in \mathbb{Z}\} \text{ for any } e \in S.$$

($\mathbb{Z}$ *is the set of all integers.*) *If* $c \in T^*$, *then* μ_t^c *converges vaguely to* $C_\xi W^\xi \mu^\xi$ *with probability one* (P_x), *where* ξ *corresponds to* c *by* (2.6), $C_\xi = (\det(\mathbf{D}^2A(\xi))/(2\pi)^d)^{1/2}$ *and* $\mu^\xi(dx) = e^{-\xi\cdot x}\, dx$.

REMARK 1. (i) Let $\nu(c)$ be defined by (1.1) and set $M \equiv \{c \in S : \nu(c) > 0\}$. Then $T^* \subset M$, because $T = \{\xi \in X : A(\xi) - \mathbf{D}A(\xi)\cdot\xi > 0\}$ as is easily seen. Under the assumption of Theorem 2 below we will show $M = T^*$ (cf. Lemma 11 in Section 8), but in general the identity is violated. (ii) Let $\xi(c)$ be the inverse image of c by (2.6). Then the infimum $\nu(c)$ is attained at $\xi(c)$:

$$\nu(c) = A(\xi(c)) - c\cdot\xi(c) \quad \text{for} \quad c \in X^*.$$

Differentiating both sides w.r.t. c, we have a dual formula $\xi(c) = -\text{grad}\, \nu(c)$. We also note that T^* agrees with the set $\{c \in X^* : \nu(c) > 0\}$ and hence with the set of admissible velocity parameters in Biggins (1979) (Theorem B). (See Biggins (1978) for additional information on these matters.) (iii) The condition (A.2) is equivalent to the condition

$$\mathscr{R}A(i\eta) < A(0) \quad \text{for all} \quad \eta \neq 0, \quad \eta \in S$$

or to an apparently stronger one that if $A(\xi) < \infty$,

$$\mathscr{R}A(\xi + i\eta) < A(\xi) \quad \text{for all} \quad \eta \neq 0, \quad \eta \in S. \tag{2.7}$$

In this paper (A.2) will be always applied in this form. When $G(dx)$ is supported by a centered lattice (i.e., a lattice which contains the origin as a lattice point), a complete analogue will be obtained (Section 6). (iv) In Section 9 for the purpose of an application to a non-linear equation we shall prove L_α-convergences of μ_t^c which occur uniformly for $c \in F$, provided F is a closed set of S and contained in T^*.

The next theorem is essentially due to Biggins (1977).

THEOREM 2. *Assume the following two conditions*

$$G(\{x : \theta \cdot x > 0\}) > 0 \quad \textit{or} \quad G(\{x : \theta \cdot x = 0\}) < 1 \quad \textit{for each} \quad \theta \in \Theta \tag{A.3}$$

$$A(\xi) \textit{ is finite in a neighborhood of } T. \tag{A.4}$$

Then for each $c \notin T^*$ *there exists a* $\xi \in S$ *such that if* $\inf_{x \in D} \xi \cdot x > -\infty$, $D \subset S$, *then* $Z_t(D + tc) \to 0 (t \to \infty)$ a.s.

REMARK 2. (i) Let the process start with one particle at the origin and let the first inequality in (A.3) fail to hold for a $\theta \in \Theta$. Then there lives no particle in $\{x : \theta \cdot x > 0\}$ a.s. and the restriction of $\mathbf{x}_t$ to the hyperplane $S_\theta \equiv \{x : \theta \cdot x = 0\}$ is still a branching process of the same kind as presently considered (but $(d - 1)$-dimensional), whose total number process ($= Z_t(S_\theta)$) is supercritical or not according as $G(S_\theta) > 1$ or ≤ 1. In the supercritical case the problem is reduced to that of the $(d - 1)$-dimensional process. In the critical case the conclusion of Theorem 2 can be false according to the behavior of $G(\{x : -b < \theta \cdot x < 0\})$ as $b \downarrow 0$ (cf. Bramson, 1978). (ii) Assume $p_0 = 0$ for simplicity. By Theorems 1 and 2, for all nonempty bounded open D

$$\lim_{t \uparrow \infty} Z_t(D + tc) = 0 \quad \text{or} \quad \infty \quad \text{(a.s.) according as} \quad c \notin T^* \quad \text{or} \quad c \in T^*, \tag{2.8}$$

provided the assumptions of theorems hold. In this respect the Condition (A.4) is crucial. In fact we will see in Section 7 that $\lim Z_t(D + tc) = \infty$ for all nonempty open D if c is an inner point of $M = \{c : \nu(c) > 0\}$ and (A.2) holds. As mentioned in Remark 1, T^* and M are identical under (A.3) and (A.4). If (A.4) fails, not only the identity is violated, but the subtraction of T^* from M contains an inner point and hence we have a consequence considerably different from (2.8).

What are asserted for the measure valued processes in (1.3) and (1.4) in Section 1 follow from the following variants of Theorem 1 as special cases. Given a positive function ρ_t and a S-valued function a_t of $t \geq 0$ such that $a = \lim_{t \uparrow \infty} a_t$ exists in S, we set for $c \in T^*$

$$\tilde{\mu}_t^c = (\sqrt{t}/\rho_t)^d \exp\{-(A(\xi) - c \cdot \xi)t + (\xi \cdot a_t)\sqrt{t}\} e^{(\xi \cdot x)\rho_t} Z_t(\rho_t\, dx + \sqrt{t} a_t + tc) \tag{2.9}$$

where ξ corresponds to c via (2.6) and $\int f(x)\mu(\rho\, dx)$ is understood to be $\int f(x/\rho)\mu(dx)$.

THEOREM 3. *Assume* (A.1). *Let* $c \in T^*$ *and* $a = \lim a_t$ *and assume either* (a) $\lim \rho_t = \infty$ *and* $\lim \rho_t/\sqrt{t} = 0$ *or* (b) ρ_t *is bounded and satisfies*

$$\limsup_{t \to \infty} \frac{1}{t} \log(1/\rho_t) < \left[A(\xi) - \max\left\{ \inf_{1 < \alpha < 2} \frac{A(\alpha\xi)}{\alpha}, \limsup_{|\eta| \to \infty} \mathscr{R}A(\xi + i\eta) \right\} \right] \Big/ d. \tag{2.10}$$

Then $\tilde{\mu}_t^c$ *converges vaguely to* $\Phi^\xi(a) W^\xi\, dx$ *with probability one* (P_x), *where* $\Phi^\xi(x)$ *is the density function of a Guassian probability distribution on* S *with mean* 0 *and covariance matrix* $\mathbf{D}^2 A(\xi)$.

THEOREM 4. *Assume* (A.1) *and that* ρ_t *is increasing and continuous and there exists* $\lim \rho_t/\sqrt{t} = m \le \infty$. *If* $c \in T^*$ *and* $a = \lim a_t$, *then with probability one* (P_x)

(i) $$(\rho_t/\sqrt{t})^d \tilde{\mu}_t^c \ \textit{weakly converges to}\ \ W^\xi \delta_0 \ \ \textit{if}\ \ m = \infty,$$

(ii) $$\tilde{\mu}_t^c \ \textit{weakly converges to}\ \ m^{-d} W^\xi \Phi^\xi(x - a/m)\, dx \ \ \textit{if}\ \ 0 < m < \infty.$$

We shall prove Proposition 1 in Section 4, Theorems 1, 3 and 4 in Section 5 and Theorem 2 in Section 8.

3. Martingales. Let $\mathbb{B}$ be the set of all bounded Borel measurable functions on S which take values in $\mathbb{C}$. For $f \in \mathbb{B}$ the function $u(t, x) = E_x[\check{f}(\mathbf{x}_t)]$, $x \in S$, is a unique global solution of the evolution equation

$$\frac{\partial u}{\partial t} = \kappa(\mathbb{M}u - u) \tag{3.1}$$

with $u(0, x) = f(x)$, in which the operator $\mathbb{M}$ acts on x only (cf. [14]). This is the fundamental equation in the later arguments. Instead of (3.1) we will often deal with the following integral equation

$$u(t, x) = e^{-\kappa t} f(x) + \int_0^t \kappa e^{-\kappa s} \mathbb{M} u(t - s, \cdot)(x)\, ds, \tag{3.1$'$}$$

which is understood to be equivalent to (3.1) in a sense.

LEMMA 1. *Let* $\xi \in S$ *and* $A(\xi) < \infty$. *If* $|f(x)| \le \phi_\xi(x)$ *for all* $x \in S$ *where* f *is a Borel measurable* (*not necessarily bounded*) *function, then there exists a unique solution* u *of* (3.1)$'$ *such that* $|u(t, x)| \le \phi_\xi(x) e^{A(\xi)t}$.

PROOF. The solution is constructed by means of iteration. Let $u_0 = e^{-\kappa t} f(x)$ and recursively define

$$u_n(t, x) = e^{-\kappa t} f(x) + \int_0^t \kappa e^{-\kappa s} \mathbb{M} u_{n-1}(t - s, \cdot)(x)\, ds.$$

From $|f| \le \phi_\xi(x)$ and $\kappa \mathbb{M} \phi_\xi = (A(\xi) + \kappa)\phi_\xi$ it follows by induction that $|u_n| \le e^{A(\xi)t}\phi_\xi$. If $f \ge 0$, u_n is increasing in n and $u = \lim u_n$ solves (3.1)$'$. Because of the linearity of the equation, this proves the existence part of the lemma. To show the uniqueness let u be the difference of two solutions. So $|u| \le \psi \equiv 2\phi_\xi e^{A(\xi)t}$ and $u = \kappa \int_0^t e^{-\kappa s} \mathbb{M} u(t - s, x)\, ds$, and we have

$$|u| \le (A(\xi) + \kappa) t \psi, \qquad |u| \le \tfrac{1}{2}[(A(\xi) + \kappa)t]^2 \psi, \qquad \text{etc.}$$

which yields $u \equiv 0$.

LEMMA 2. *If* $A(\mathscr{R}\lambda) < \infty$, *then* $E_x[\check{\phi}_\lambda(\mathbf{x}_t)] = \phi_\lambda(x) e^{A(\lambda)t}$.

PROOF. First we assume λ real and let $\xi = \lambda$.

Then by taking a sequence $0 \le f_n \in \mathbb{B}$ which is dominated by and converges to $\phi_\xi(x)$, we see $u(t, x) = E_x[\check{\phi}_\xi(\mathbf{x}_t)]$ is a solution of (3.1)$'$, which is not greater than $\phi_\xi(x) e^{A(\xi)t}$; but $\phi_\xi(x) e^{A(\xi)t}$ is also a solution and therefore by the uniqueness they coincide. When λ is not real, the similar argument can be applied by noting $|\phi_\lambda(x)| = \phi_{\mathscr{R}\lambda}(x)$.

COROLLARY. *If $A(\mathscr{R}\lambda) < \infty$, W_t^λ is a $\mathscr{F}_t$-martingale, where $\mathscr{F}_t$ is the σ-field generated by $\mathbf{x}_s$, $0 \le s \le t$.*

PROOF. Using the additive nature of $\check{\phi}_\lambda$ and the Markov property of $\mathbf{x}_t$ and then applying Lemma 2 we have

$$E_x[\check{\phi}_\lambda(\mathbf{x}_{t+s}) \mid \mathscr{F}_s] = (\mathrm{E}.[\check{\phi}_\lambda(\mathbf{x}_t)])\check{}(\mathbf{x}_s) = e^{A(\lambda)t}\check{\phi}_\lambda(\mathbf{x}_s)$$

a.s., which is the same as what is to be shown.

4. Proof of Proposition 1. For a complex valued measurable function f on S, let us define a function on $\mathbf{S}$:

$$F_f(\mathbf{x}) = |\check{f}(\mathbf{x})|^\alpha,$$

where α is a fixed number between 1 and 2 and will be taken as in the statement of Proposition 1, and set formally

$$\mathscr{G}\,\mathrm{F}_f(\mathbf{x}) = \kappa\left[\textstyle\sum_{n=0}^\infty \sum_{k=1}^m \displaystyle\int_{S^n} (|\check{f}(\mathbf{x}) + \check{f}(x_k + \mathbf{y}) - f(x_k)|^\alpha - |\check{f}(\mathbf{x})|^\alpha)\pi_n(d\mathbf{y})\right] \tag{4.1}$$

if $\mathbf{x} = (x_1, \cdots, x_m)(m \ge 1)$ and $\mathscr{G}F_f(\partial) = 0$. If f is bounded, this definition makes sense by virtue of (C.2) and it is standard to see that

$$\lim_{h\downarrow 0} \frac{1}{h}(E_{\mathbf{x}}[F_f(\mathbf{x}_h)] - F_f(\mathbf{x})) = \mathscr{G}\mathrm{F}_f(\mathbf{x}).$$

(This would be a usual formula for the generator of a Markov process if F_f were bounded.) To find bounds for $\mathscr{G}F_f$ we prepare the following elementary lemma.

LEMMA 3. *There exists a constant N such that for any $1 \le \alpha \le 2$ and any $z \in \mathbb{C}$*

$$1 + \alpha\mathscr{R}z \le |1 + z|^\alpha \le 1 + N|z|^\alpha + \alpha\mathscr{R}z.$$

PROOF. As $z \to 0$, $|1 + z|^\alpha = (1 + z + \bar{z} + |z|^2)^{\alpha/2} = 1 + (\alpha/2)(z + \bar{z} + |z|^2) + O(|z|^2)$ ($\bar{z}$ is the complex conjugate of z). Therefore $|1 + z|^\alpha \le 1 + N_1|z|^\alpha + \alpha\mathscr{R}z$ for $|z| < 1/2$ for some constant N_1. Since $|1 + z|^\alpha/|z|^\alpha$ and $(\mathscr{R}z)/|z|^\alpha$ are bounded as $z \to \infty$, we have $|1 + z|^\alpha \le N_2|z|^\alpha + \alpha\mathscr{R}z$, $|z| \ge 1/2$. Thus we can take $N = \max\{N_1, N_2\}$ for the second inequality of the lemma. Letting $x = \mathscr{R}z$ we have $|1 + z|^\alpha \ge |1 + x|^\alpha \ge 1 + \alpha x$, proving the first inequality.

Substitute z/w for z in the inequalities of Lemma 3. Then after elementary manipulations we have

$$\alpha|w|^{\alpha-2}\mathscr{R}(z\bar{w}) \le |w + z|^\alpha - |w|^\alpha \le N|z|^\alpha + \alpha|w|^{\alpha-2}\mathscr{R}(z\bar{w}) \tag{4.2}$$

for $w, z \in \mathbb{C}$, where $|w|^{\alpha-2}\mathscr{R}(z\bar{w})$ is understood as zero if $w = 0$. Apply the second inequality in (4.2) to the integrands on the right side of (4.1) with $w = \check{f}(\mathbf{x})$ and $z = \check{f}(x_k + \mathbf{y}) - f(x_k)$. Then by noting $|z_0 + \cdots + z_n|^\alpha \le (n + 1)(|z_0|^\alpha + \cdots + |z_n|^\alpha)$, $\mathscr{G}F_f(\mathbf{x})$ is bounded above by

$$\kappa\alpha|\check{f}(\mathbf{x})|^{\alpha-2}\mathscr{R}\{\bar{\check{f}}(\mathbf{x}) \textstyle\sum_{k=1}^m (\mathscr{M}f(x_k) - f(x_k))\}$$

$$+ \kappa N \textstyle\sum_{k=1}^m \left[\sum_{n=1}^\infty (n + 1)\displaystyle\int_{S^n} (|f|^\alpha)\check{}(x_k + \mathbf{y})\pi_n(d\mathbf{y}) + (2 + A(0)/\kappa)|f(x_k)|^\alpha\right]$$

and if we write $K = N(2\kappa + A(0))$ and

$$\mathscr{M}_2 f(x) = \textstyle\sum_{n=1}^\infty (n + 1)\displaystyle\int_{S^n} \check{f}(x + \mathbf{y})\pi_n(d\mathbf{y}),$$

$$\mathscr{G}\mathrm{F}_f(\mathbf{x}) \le \kappa\alpha|\check{f}(\mathbf{x})|^{\alpha-2}\mathscr{R}\{\bar{\check{f}}(\mathbf{x})(\mathscr{M}f - f)\check{}(\mathbf{x})\} + K(\mathscr{M}_2\{|f|^\alpha\} + |f|^\alpha)\check{}(\mathbf{x}). \tag{4.3}$$

This inequality will furnish the key to our proof of Proposition 1.

By virtue of the Condition (C.2) we obtain

$$\frac{d}{dt}E_x[F_f(\mathbf{x}_t)] = E_x[\mathcal{G}F_f(\mathbf{x}_t)] \tag{4.4}$$

for $f \in \mathcal{B}$, as will be proved below.

First we let $\Psi_h F(\mathbf{x}) = (1/h)(E_x[F(\mathbf{x}_h)] - F(\mathbf{x}))(h > 0, \mathbf{x} \in \mathbf{S})$ for a non-negative function F on $\mathbf{S}$ and prove.

LEMMA 4. *If* $F(\mathbf{x}) \le C(\#\mathbf{x})^2$, *then* $\Psi_h F(\mathbf{x}) \le C_1(\#\mathbf{x})^3$. ($\#\mathbf{x} = n$ if $\mathbf{x} \in S^n$; *C and C_1 are constants independent of* $\mathbf{x}$.)

PROOF. Let τ be the first jump time of $\mathbf{x}_t$. By the fact that Z_t is a super-critical Galton-Watson process we see

$$E_{\mathbf{x}}[Z_h^2 \mid \tau < h] \le (\#\mathbf{x})^2 E_0[Z_h^2 \mid \tau < h]$$

and then

$$|\psi_h F(\mathbf{x})| \le \frac{1}{h} P_{\mathbf{x}}(\tau < h) E_{\mathbf{x}}[C(Z_h^2 + (\#\mathbf{x})^2) \mid \tau < h] \le \kappa C(\#\mathbf{x})^3 (E_0[Z_h^2 \mid \tau < h] + 1),$$

but

$$E_0[Z_h^2 \mid \tau < h] \le E_0[E_{\mathbf{x}_\tau}[Z_h^2]] \le E_0[Z_\tau^2] E_0[Z_h^2] < \infty,$$

proving the lemma.

Coming back to the proof of (4.4), let $u(t, x) = E_x[F_f(\mathbf{x}_t)]$ where $f \in \mathcal{B}$. Since $\lim_{h\downarrow 0} \Psi_h F_f(\mathbf{x}) = \mathcal{G}F_f(\mathbf{x})$ and $(1/h)(u(t+h, x) - u(t, x)) = E_x[\Psi_h F(\mathbf{x}_t)]$, an application of the dominated convergence theorem with the help of Lemma 4 and the moment condition (C.2) implies that the right derivative $(\partial^+/\partial t)u(t, x)$ exists and is equal to $E_x[\mathcal{G}F_f(\mathbf{x}_t)]$. By (4.3), $|\mathcal{G}F_f(\mathbf{x})| \le \text{const}(\#\mathbf{x})^2$ and by using Lemma 4 and (C.2) again we see that $(\partial^+/\partial t)u(t, x)$ is continuous in t. Since a continuous function having a continuous right-derivative is differentiable, $u(t, x)$ is differentiable in t. Accordingly we obtain (4.4).

LEMMA 5. *Assume* (A.1). *If* $1 \le \alpha \le 2$, $\alpha\xi \in T$ *and there exists a constant C such that* $|f(x)| \le C\phi_\xi(x)$ *for all* $x \in S$, *then* (4.4) *holds.*

PROOF. It follows from the first inequality of (4.2) that $\mathcal{G}F_f$ *is bounded below by the first term on the right-hand side of* (4.3). *Using this bound together with* (4.3) *we see*

$$|\mathcal{G}F_f| \le \text{const}\{(\check{g})^{\alpha-1}(\mathcal{M}g + g)^{\vee} + (\mathcal{M}_2 g^\alpha)^{\vee}\}, \quad g = |f| \tag{4.5}$$

where we also used the inequality $(g^\alpha)^{\vee} \le (\check{g})^\alpha$. Let us prove that if $\alpha\xi \in T$, there is a constant b such that

$$E_x[|\check{\phi}_\xi(\mathbf{x}_t)|^\alpha] \le \phi_{\alpha\xi}(x)e^{bt}. \tag{4.6}$$

For this purpose let $g_n(x) = \min\{\phi_\xi(x), n\}$, $n \ge 1$. Then $\mathcal{M}g_n(x) \le \min\{\mathcal{M}\phi_\xi(x), n\,\mathcal{M}1(x)\} \le C_1 g_n(x)$ where C_1 is independent of n. Similarly, by using the Assumption (A.1), $\mathcal{M}_2\{g_n^\alpha\}(x) \le C_2 g_n(x)^\alpha$. Hence by (4.5) we have $|\mathcal{G}F_{g_n}(\mathbf{x})| \le bF_{g_n}(\mathbf{x})$ for some b independent of n. This combined with (4.4) which is true for F_{g_n} yields $E_x[|\check{g}_n(\mathbf{x}_t)|^\alpha] \le g_n(x)^\alpha e^{bt}$ and by letting $n \to \infty$ we have (4.6).

Now the lemma is proved by approximating $f(x)$ by $f_n \in \mathcal{B}$ with $|f_n| \le C\phi_\xi$: the Relation (4.4) is preserved in the limit because of (4.6) and the inequality

$$|\mathcal{G}F_{f_n}(\mathbf{x})| \le \text{const}(\check{\phi}_\xi(\mathbf{x}))^\alpha,$$

which is verified from (4.5) and (A.1) as above.

PROOF OF PROPOSITION 1. Assume (A.1) and let $\lambda = \xi + i\eta$, $1 < \alpha \le 2$ and $\alpha\xi \in T$. We see by (4.3) and (A.1)

$$\mathcal{G}F_{\phi_\lambda}(\mathbf{x}) \le \alpha \mid \check{\phi}_\lambda(\mathbf{x})|^{\alpha-2}\mathcal{R}\{\bar{\check{\phi}}_\lambda(\mathbf{x})(A(\lambda)\check{\phi}_\lambda(\mathbf{x}))\} + K(\mathcal{M}_2\phi_{\alpha\xi} + \phi_{\alpha\xi})\check{}(\mathbf{x})$$
$$\le \alpha\mathcal{R}A(\lambda)F_{\phi_\lambda}(x) + K_1\check{\phi}_{\alpha\xi}(\mathbf{x})$$

where $K_1 = K(2B(\alpha\xi) + 1)$. Hence from Lemma 5 it follows that if $u(t, x) = E_x[| \check{\phi}_\lambda(\mathbf{x}_t)|^\alpha]$,

$$\frac{\partial u(t, x)}{\partial t} \le \alpha\mathcal{R}A(\lambda)u(t, x) + K_1\phi_{\alpha\xi}(x)e^{A(\alpha\xi)t}. \tag{4.7}$$

Rewritting this differential inequality in the following form

$$\frac{\partial}{\partial t}(e^{-qt}u(t, x)) \le ae^{(p-q)t},$$

where $p = A(\alpha\xi)$ and $q = \alpha\mathcal{R}A(\lambda)$, and choosing $\delta > 0$ so small that $p - q = A(\alpha\xi) - \alpha\mathcal{R}A(\lambda) < 0$ for $|\eta| < \delta$ (see (2.4)), we conclude that $E_x[| W_t^\lambda|^\alpha] = e^{-qt}u(t, x)$ is bounded for large t. The proof of Proposition 1 is complete.

The differential inequality (4.7) incidentally proves the following lemma, which will be used in the next section.

LEMMA 6. *If $1 \le \alpha \le 2$ and $\alpha\xi \in T$, then for any $\lambda = \xi + i\eta$*

$$E_x[| W_t^\lambda |^\alpha] \le \phi_{\alpha\xi}(x)(K_1te^{-(\alpha\mathcal{R}A(\lambda)-A(\alpha\xi))t} + K_1t + 1).$$

5. Proofs of Theorems 1, 3 and 4.

For $h > 0$ define functions on S:

$$K_h(x) = \prod_{k=1}^d \left(\frac{\sin(hx^k)}{hx^k}\right)^2, \quad x = (x^1, \cdots, x^d) \in S.$$

Then for $\eta = (\eta^1, \cdots, \eta^d) \in S$

$$K_h^*(\eta) = (2h)^{-2d}\prod_{k=1}^d (2h - |\eta^k|) \quad \text{if} \quad \max_k |\eta^k| < 2h$$
$$= 0 \quad \text{otherwise}$$

where f^* denotes the Fourier transform of f:

$$f^*(\eta) = 2(\pi)^{-d}\int_S f(x)e^{-i\eta\cdot x}\,dx$$

(cf. Breiman's book, 1968, page 218). Let $h_1, \cdots, h_{d+1}$ be positive numbers such that if $k \ne j$, h_k and h_j are not rational multiples of each other. Then the function

$$K(x) = \sum_{k=1}^{d+1} K_{h_k}(x) \tag{5.1}$$

is continuous, everywhere positive and integrable: $\int K(x)\,dx < \infty$. In the rest of this section except in the proof of Theorem 4 we assume the Conditions (A.1) and (A.2); also it is assumed that $\xi \in T$ and

$$c = \mathbf{D}A(\xi).$$

PROOF OF THEOREM 1. What we prove below is the a.s. weak convergence of $K(x)e^{\xi\cdot x}\mu_t^c(dx)$ to $C^\xi W^\xi K(x)\,dx$, or equivalently

$$\int_S e^{ip\cdot x}K(x)e^{\xi\cdot x}\mu_t^c(dx) \to C_\xi W^\xi \int_S e^{ip\cdot x}K(x)\,dx \quad \text{as} \quad t \to \infty \tag{5.2}$$

for all points p in S with probability one (P_x), which is somewhat stronger than the required vague convergence. Let

$$f_p(x) = K(x)e^{ip\cdot x}, \quad p \in S.$$

Then by Fourier's inversion formula

$$\int f_p(x)e^{\xi\cdot x}\mu_t^c(dx) = \left(\int f_p^*(\eta)e^{i\eta\cdot x}\, d\eta e^{\xi\cdot x}\right)^{\vee}(\mathbf{x}_t - tc)\cdot\Xi(t)$$

(5.3)
$$= \int f_p^*(\eta)\check{\phi}_\lambda(\mathbf{x}_t - tc)\, d\eta\cdot\Xi(t), \quad \lambda = \xi + i\eta$$

$$= \int f_p^*(\eta)\, W_t^\lambda e^{(A(\lambda)-c\cdot\lambda)t}\, d\eta\cdot\Xi(t)$$

where

$$\Xi(t) = t^{d/2}e^{-(A(\xi)-c\cdot\xi)t}.$$

We decompose the last integral: $\int f_p(x)e^{\xi\cdot x}\mu_t^c(dx) = I(t) + II(t) + III(t)$ where

$$I(t) = \int_{|\eta|<t^{-1/3}} e^{(A(\lambda)-c\cdot\lambda)t}W_t^\xi f_p^*(\eta)\, d\eta\cdot\Xi(t)$$

$$II(t) = \int_{|\eta|<t^{-1/3}} e^{(A(\lambda)-c\cdot\lambda)t}(W_t^\lambda - W_t^\xi)f_p^*(\eta)\, d\eta\cdot\Xi(t)$$

$$III(t) = \int_{|\eta|>t^{-1/3}} e^{(A(\lambda)-c\cdot\lambda)t}W_t^\lambda f_p^*(\eta)\, d\eta\cdot\Xi(t).$$

First we observe that ξ is chosen as the Saddle point of the last integral in (5.3), because of the positive definiteness of $\mathbf{D}^2A(\xi)$:

(5.4)
$$A(\lambda) - c\cdot\lambda = A(\xi) - c\cdot\xi - \tfrac{1}{2}\mathbf{D}^2A(\xi)\eta\cdot\eta + iB\,|\,\eta\,|^3 \quad \text{as} \quad \eta \to 0,$$

where $B = B(\lambda)$ is real and bounded as $\eta \to 0$. By the change of variable according to $\eta \to \eta/\sqrt{t}$,

$$I(t) = \int_{|\eta|<t^{-1/6}} e^{-J(\eta)}\exp(iB\,|\,\eta\,|^3/\sqrt{t})f_p^*(\eta/\sqrt{t})\, d\eta\, W_t^\xi,$$

where $J(\eta) = (\tfrac{1}{2})(\mathbf{D}^2A(\xi)\eta)\cdot\eta$. Since

$$\int e^{-J(\eta)}\, d\eta = (2\pi)^{d/2}[\det(\mathbf{D}^2A(\xi))]^{-1/2},$$

by writing $\Omega_1 = \{W_t^\xi \to W^\xi\}$ we have

(5.5)
$$I(t) \to C_\xi \int f_p(x)\, dx\cdot W^\xi \;(t\to\infty) \quad \text{on} \quad \Omega_1.$$

Clearly Ω_1 is independent of p and $P_x(\Omega_1) = 1$.

Next we prove

(5.6)
$$III(t) \to 0 \;(t\to\infty) \quad \text{on} \quad \Omega_2$$

for some Ω_2 independent of p and $P_x(\Omega_2) = 1$. Let $q(\eta) = A(\xi) - \mathscr{R}A(\lambda)$. Then

$$|\,III(t)| \le t^{d/2}\int_{|\eta|>t^{-1/3}} e^{-q(\eta)t}\,|\,W_t^\lambda\,|\cdot|\,f_p^*(\eta)|\, d\eta.$$

Since $A(\xi) - \mathscr{R}A(\lambda) = (\tfrac{1}{2})(\mathbf{D}^2A(\xi)\eta)\cdot\eta(1 + o(1))$ as $\eta \to 0$ and $f_p^*(\eta) = K^*(\eta - p)$ is zero outside a compact set, for each $m = 1, 2, \cdots$ we can choose an $\varepsilon > 0$ so small that

(5.7)
$$\text{if} \quad |\,\eta\,| > t^{1/3} \quad \text{and} \quad f_p^*(\eta) \ne 0 \quad \text{for some} \quad |\,p\,| < m, \quad \text{then} \quad q(\eta)t \ge \varepsilon t^{1/3}$$

by virtue of (2.7). By a martingale inequality (cf. Meyer [8], $II - 8$) and Lemma 6 we see for $1 < \alpha \le 2$ with $\alpha\xi \in T$

$$E_x[\sup_{n\le t\le n+1}|W_t^\lambda|^\alpha] \le \left(\frac{\alpha}{\alpha-1}\right)^\alpha E_x[|W_{n+1}^\lambda|^\alpha] \tag{5.8}$$
$$\le \text{const}\ \phi_{\alpha\xi}(x)n(e^{-(\alpha\Re A(\lambda)-A(\alpha\xi))n}+1), \quad n\ge 1.$$

On the other hand, by noting the factorization $|f^*(\eta)| = |f^*(\eta)|^{1/\alpha}|f^*(\eta)|^{1/\beta}$ where $1/\beta = 1 - 1/\alpha$, an application of the Hölder's inequality yields

$$|III(t)|^\alpha \le \int_{|\eta|>t^{-1/3}} t^{\alpha d/2}e^{-\alpha q(\eta)t}|W_t^\lambda|^\alpha\,|f_p^*(\eta)|\,d\eta\left[\int |f_p^*(\eta)|\,d\eta\right]^{\alpha/\beta}.$$

Therefore by (5.7) and (5.8)

$$E_x[\sup_{n\le t\le n+1}\sup_{|p|<m}|III(t)|^\alpha] \le \text{const}\ \phi_{\alpha\xi}(x)n^{1+\alpha d/2}(e^{-(\alpha A(\xi)-A(\alpha\xi))n}+e^{-\alpha\varepsilon n^{1/3}})$$

which implies

$$\textstyle\sum_{n=0}^\infty E_x[\sup_{n\le t\le n+1}\sup_{|p|<m}|III(t)|^\alpha] < \infty, \quad m = 1, 2, \cdots.$$

Accordingly we have (5.6).

To prove $II(t) \to 0$, we prepare Lemmas 7, 8 and 9 below.

LEMMA 7. *For any* $\alpha \ge 1$, $E_0[\sup_{1\le k\le Z_t}|x_{t,k}|^\alpha] = O(t^\alpha)\,(t\to\infty)$.

PROOF. It suffices to show that for each $\theta \in \Theta$

$$E_0[(\sup_k(x_{t,k}\cdot\theta)^+)^\alpha] = O(t^\alpha) \quad \text{as } t\to\infty.$$

Here a^+ denotes the positive part of $a \in \mathbb{R}$. Let us fix θ and set

$$I_b(x) = 1 \text{ or } 0 \text{ according as } \quad x\cdot\theta > b \text{ or } \quad x\cdot\theta \le b$$

where $b \in \mathbb{R}$ and $x \in S$. Take an $\varepsilon > 0$, possibly by (C.4), so that $A(\varepsilon\theta) < \infty$. Since $I_b(x) \le e^{\varepsilon(x\cdot\theta - b)} = e^{-\varepsilon b}\phi_{\varepsilon\theta}(x)$, we have

$$P_0[\sup_k x_{t,k}\cdot\theta > b] \le E_0[\check{I}_b(\mathbf{x}_t)] \le E_0[\check{\phi}_{\varepsilon\theta}(\mathbf{x}_t)]e^{-\varepsilon b} = e^{A(\varepsilon\theta)t}e^{-\varepsilon b}.$$

Then, letting $r = \max\{0, A(\varepsilon\theta)/\varepsilon\}$

$$E_0[(\sup_k(x_{t,k}\cdot\theta)^+)^\alpha] = \int_0^\infty \alpha b^{\alpha-1}P_0[\sup_k x_{t,k}\cdot\theta > b]\,db$$
$$\le \int_0^{rt}\alpha b^{\alpha-1}\,db + \int_{rt}^\infty \alpha b^{\alpha-1}e^{A(\varepsilon\theta)t}e^{-\varepsilon b}\,db$$
$$\le O(t^\alpha).$$

Let

$$\Phi_t(x) = e^{-A(\lambda)t}\phi_\lambda(x) - e^{-A(\xi)t}\phi_\xi(x), \quad x\in S.$$

LEMMA 8. *Let* $\xi \in T$, $1 < \alpha \le 2$, $\alpha\xi \in T$ *and* $\lambda = \xi + i\eta$. *Then*

$$E_0[|W_t^\lambda - W_t^\xi|^\alpha] \le K\int_0^t E_0[(\mathcal{M}_2|\Phi_s|^\alpha + |\Phi_s|^\alpha)\check{\ }(\mathbf{x}_s)]\,ds$$

where K and $\mathcal{M}_2$ *appear in* (4.3).

PROOF. Noting that $W_t^\lambda - W_t^\xi = \check{\Phi}_t(\mathbf{x}_t)$, let

$$u(t) = E_0[|\check{\Phi}_t(\mathbf{x}_t)|^\alpha].$$

Then, writing $F_t(\mathbf{x}) = |\check{\Phi}_t(\mathbf{x})|^\alpha$, it follows from Lemma 5

$$\frac{du(t)}{dt} = E_0\left[\left(\frac{\partial}{\partial t}F_t\right)(\mathbf{x}_t) + \mathcal{G}F_t(\mathbf{x}_t)\right].$$

Since for $z, w \in \mathbb{C}$, $w \neq 0$

$$|z|^\alpha - |w|^\alpha = \alpha\mathcal{R}\{\bar{w}(z-w)\}|w|^{\alpha-2} + o(|z-w|) \quad \text{as } z \to w$$

(see (4.2)), on the one hand we have

$$\frac{\partial}{\partial t}F_t = \alpha|\Phi_t|^{\alpha-2}\mathcal{R}\left\{\check{\bar{\Phi}}_t\left(\frac{\partial}{\partial t}\Phi_t\right)^{\check{}}\right\}.$$

On the other hand by (4.3)

$$\mathcal{G}F_t \leq \kappa\alpha|\Phi_t|^{\alpha-2}\mathcal{R}\{\check{\bar{\Phi}}_t(\mathcal{M}\Phi_t - \Phi_t)^{\check{}}\} + K(\mathcal{M}_2|\Phi_t|^\alpha + |\Phi_t|^\alpha).$$

But $\frac{\partial}{\partial t}\Phi_t = -\kappa(\mathcal{M}\Phi_t - \Phi_t)$ and hence

$$\frac{du(t)}{dt} \leq KE_0[(\mathcal{M}_2|\Phi_t|^\alpha + |\Phi_t|^\alpha)^{\check{}}(\mathbf{x}_t)]$$

which implies the desired inequality.

LEMMA 9. *Let ξ, α and δ be taken as in Proposition* 1. *Then for a constant C*

$$E_0[\sup_{t>0}|W_t^\lambda - W_t^\xi|^\alpha] \leq C|\eta|^\alpha \quad \textit{for} \quad |\eta| < \delta.$$

PROOF. First we observe that

$$\begin{aligned}|\Phi_t(x)|^\alpha &\leq 2[e^{-\alpha A(\xi)t}|\phi_\xi - \phi_\lambda|^\alpha(x) + |e^{-A(\xi)t} - e^{-A(\lambda)t}|^\alpha|\phi_\lambda|^\alpha(x)] \\ &\leq 2[|1 - e^{-\iota\eta\cdot x}|^\alpha e^{-\alpha A(\xi)t} + (t|A(\xi) - A(\lambda)|)^\alpha e^{-\alpha\mathcal{R}A(\lambda)t}]\phi_{\alpha\xi}(x) \\ &\leq \text{const}(1 + t^\alpha)e^{-\alpha\mathcal{R}A(\lambda)t}\phi_{\alpha\xi}(x)(|x|^\alpha + 1)|\eta|^\alpha.\end{aligned} \tag{5.9}$$

By choosing $1 < p \leq 2$ so that $p\alpha\xi \in T$, we see

$$\begin{aligned}\mathcal{M}_2\{|\cdot|^\alpha\phi_{\alpha\xi}\}(x) &= \int_S \phi_{\alpha\xi}(x+y)|x+y|^\alpha G_2(dy) \\ &\leq 2\left[|x|^\alpha\mathcal{M}_2\phi_{\alpha\xi}(x) + \left(\int |y|^{\alpha q}G_2(dy)\right)^{1/q}(\mathcal{M}_2\phi_{p\alpha\xi}(x))^{1/p}\right] \\ &\leq \text{const}\ \phi_{\alpha\xi}(x)(|x|^\alpha + 1),\end{aligned} \tag{5.10}$$

where $1/q = 1 = 1/p$ and G_2 is a measure determined by $\mathcal{M}_2 f(x) = \int f(x+y)G_2(dy)$, and

$$\begin{aligned}E_0[((|\cdot|^\alpha + 1)\phi_{\alpha\xi})^{\check{}}(\mathbf{x}_t)] &\leq \text{const}\ E_0[(\check{\phi}_{\alpha\xi}(\mathbf{x}_t))^p]^{1/p}E_0[\sup_k|x_{t,k}|^{q\alpha}]^{1/q} + E_0[\check{\phi}_{\alpha\xi}(\mathbf{x}_t)] \\ &\leq \text{const}\ e^{A(\alpha\xi)t}(t^\alpha + 1)\end{aligned} \tag{5.11}$$

where we make use of Proposition 1 and Lemma 7 in the last inequality. From Lemma 8 and inequalities (5.9) to (5.11) we now obtain

$$\begin{aligned}E_0[|W_t^\lambda - W_t^\xi|^\alpha] &\leq \text{const}\int_0^t (s^\alpha + 1)^2 e^{-(\alpha\mathcal{R}A(\lambda) - A(\alpha\xi))s}\,ds\cdot|\eta|^\alpha \\ &= \text{const}\,|\eta|^\alpha \quad \text{for} \quad |\eta| < \delta,\end{aligned}$$

which deduces the lemma by applying the martingale inequality that was used in (5.8).

Coming back to the estimate of $II(t)$, we observe first that as in the case of $I(t)$

$$II(t) \le t^{d/2} \int \chi(t, \eta) e^{-J(\eta)t+o(1)} | W_t^\lambda - W_t^\xi || f_p^*(\eta) | \, d\eta$$

where $o(1) \to 0$ as $t \to \infty$ uniformly in η and $\chi(t, \eta)$ is the indicator function of $\{\eta : |\eta| \le t^{-1/3}\}$. Let $0 < t_1 < t_2 < \dots$. Then

(5.12)
$$E_0[\sup_{t_n < t \le t_{n+1}} \sup_{p \in S} |II(t)|^\alpha] \le (t_{n+1})^{\alpha d/2} |K^*(0)|^\alpha \left(\int e^{-J(\eta) t_n} \, d\eta \right)^{\alpha/\beta}$$
$$\times \int e^{-J(\eta)t_n + o(1)} \chi(t_n, \eta) E_0[\sup_{t>0} | W_t^\lambda - W_t^\xi |^\alpha] \, d\eta,$$

where $1/\beta = 1 - 1/\alpha$. By changing the variable according to $\eta \to \eta/\sqrt{t_n}$ and applying Lemma 9 the right-hand side above is bounded by

(5.13)
$$\text{const} \left(\int e^{-J(\eta)} \, d\eta \right)^{\alpha/\beta} \int e^{-J(\eta)} (\eta/\sqrt{t_n})^\alpha \, d\eta (t_{n+1}/t_n)^{\alpha d/2} \le \text{const } t_n^{-\alpha/2} (t_{n+1}/t_n)^{\alpha d/2}.$$

Consequently, taking $t_n = 2^n$, $\sum_{n=1}^\infty E_0[\sup_{t_n < t \le t_{n+1}, p \in S} |II(t)|^\alpha] < \infty$; in particular $\lim_{t \to \infty} \sup_p |II(t)| = 0$ a.s. This together with (5.5) and (5.6) proves (5.2). The proof of Theorem 1 is finished.

Remark 3. Let $A(r\xi) < \infty$ for an $r > 1$ and $f(x)$ be a Borel measurable function on S with $|f(x)| \le \phi_\xi(x)(1 + |x|)^{-2d}$. Then under the condition (A.2)

(5.14)
$$E_x[\check{f}(\mathbf{x}_t - tc)] \sim C_\xi \left(\int \phi_{-\xi}(x) f(x) \, dx \right) \phi_\xi(x) t^{-d/2} e^{(A(\xi) - c \cdot \xi)t}$$

as $t \to \infty$, where $c = \mathbf{D}A(\xi)$. For the proof it suffices to show the relation for $f(x) = K(x) e^{ip \cdot x} \phi_\xi(x)$. But for such a f the left hand side of (5.14) is equal to

$$\int K^*(\eta - p) \phi_\lambda(x) e^{(A(\lambda) - c \cdot \lambda)t} \, d\eta, \quad (\lambda = \xi + i\eta)$$

from which (5.14) can be deduced through the same calculations as leading (5.5). Theorem 1 combined with (5.15) yields

$$\lim_{t \uparrow \infty} Z_t(D + tc)/E_x[Z_t(D + tc)] = e^{-\xi \cdot x} W^\xi, \qquad \text{a.s. } (P_x)$$

if $c \in T^*$, D is a nonempty bounded open set with $\int_{\partial D} dx = 0$ and the hypothesis of Theorem 1 is satisfied.

Proof of Theorem 3. Let $\tilde{\mu}_t^c$ be defined by (2.9). By using the same notations as above and by writing $\lambda' = \xi + i\eta/\rho_t$, we have

(5.15)
$$\int f_p(x) \tilde{\mu}_t^c(dx) = \int f_p^*(\eta) W_t^{\lambda'} \exp\{(A(\lambda') - c \cdot \lambda')t - i(\eta \cdot a_t)\sqrt{t}/\rho_t\} \, d\eta \Xi(t)/\rho_t^d.$$

By the changes of variables according to $\eta \to \rho_t \eta$ and $\eta \to \rho_t \eta/\sqrt{t}$, the right-hand side of (5.15) becomes, respectively

(5.16)
$$\int f_p^*(\rho_t \eta) W_t^\lambda \exp\{(A(\lambda) - c \cdot \lambda)t - i(\eta \cdot a_t)\sqrt{t}\} \, d\eta \Xi(t)$$

and

(5.17)
$$\int f_p^*((\rho_t/\sqrt{t})\eta) W_t^{\lambda''} \exp\{(A(\lambda'') - c \cdot \lambda'')t - i(\eta \cdot a_t)\} \, d\eta \Xi(t)/\sqrt{t}^d,$$

where $\lambda'' = \xi + i\eta/\sqrt{t}$. First we assume the case (a). Then there exists a function σ_t such that $\lim \sigma_t\rho_t = \lim \sigma_t\sqrt{t} = \infty$ and $\lim \sigma_t = 0$. Let us consider the domain $|\eta| \le \sigma_t$ in the integral (5.16), which corresponds to $|\eta| \le \sigma_t\rho_t$ for (5.17) and to $|\eta| \le \sigma_t\sqrt{t}$ for (5.15), and define $I(t)$, $II(t)$ and $III(t)$ as in the proof of Theorem 1, but how to divide the domain of integration is decided by σ_t as just indicated. By (5.17) $I(t)$ converges (a.s.) to

$$f_p^*(0)\int \exp\left\{-\frac{1}{2}\mathbf{D}^2A(\xi)\eta\cdot\eta - i\eta\cdot a\right\}d\eta\, W^\xi$$

and by using (5.16) we can estimate $II(t)$ and $III(t)$ as before, proving the desired convergence.

For the case (b) take $t^{-1/3}$ for σ_t in above. $I(t)$ and $II(t)$ can be dealt with in the same way as in the proof of Theorem 1. To handle $III(t)$ divide further the domain of integration for it into two parts: $|\eta| > K$ and $t^{-1/3} < |\eta| \le K$ in the expression corresponding to (5.16), where K is chosen so that $[A(\xi) - \mathscr{R}A(\lambda)]/d > \lim\sup(1/t)\log(1/\rho_t)$ for $|\eta| \ge K$. Then by noting $\int |f^*(\rho_t\eta)|\,d\eta = O(\rho_t^{-d})$ and the Condition (2.10) you can discuss as before to show $\lim III(t) = 0$.

Proof of Theorem 4. Since $\tilde{\mu}_t^c(S) = (\sqrt{t}/\rho_t)^d W^\xi$, the weak convergences in (i) and (ii) follow from the corresponding vague convergences. For the proof of (i) let $\lim \rho_t/\sqrt{t} = \infty$. By letting $t \to \infty$ in the expression (5.15) we see formally that $(\rho_t/\sqrt{t})^d \int f_p\tilde{\mu}_t^c$ approaches to $f_p(0)\,W^\xi$. This is justified by dealing with the random factor $W_t^{\lambda'}$ carefully. To this purpose we substitute $W_t^{\lambda'} - W_t^\xi$ for $W_t^{\lambda'}$ in the integral and denote the resulting quantity by $II(t)$. Then by the expression corresponding to (5.16)

$$|II(t)| \le \int |f_p^*(\rho_t\eta)||\,W_t^\lambda - W_t^\xi|\,d\eta\rho_t^d,$$

and we can proceed as in (5.12) and (5.13); but this time we use the fact that $\sup_{|p|<N}|f_p^*(\cdot)|$ is zero outside a compact for each $N = 1, 2, \ldots$ rather than the integrability of $e^{-J(\cdot)}$, and take t_n so that $\rho_{t_n} = 2^n$, proving (i).

The proof of (ii) is the same as that of Theorem 1.

6. The lattice case. In this section we assume that the measure $G(dx)$ is supported by a lattice Γ generated by d independent vectors $e_1, \cdots, e_d \in S$:

$$\Gamma = \{\textstyle\sum_{k=1}^d n_ke_k : (n_1, \cdots, n_d) \in \mathbb{Z}^d\}.$$

To each $x \in S$ let us assign a point $[x] \in \Gamma$ determined by $x - [x] = \sum_{k=0}^d a_ke_k$ where $0 \le a_k < 1$ $(k = 1, \cdots, d)$, and we consider, instead of μ_t^c, a family of measures on Γ:

$$\mu_{t,\Gamma}^c(D) \equiv \sqrt{t}^d e^{-(A(\xi)t - [tc]\cdot\xi)}Z_t(D + [tc]),\quad D \subset \Gamma$$

where $c \in T^*$ and ξ is chosen so that $\xi \in T$ and $c = \mathbf{D}A(\xi)$ as before. The initial particle will be supposed to be located in Γ.

Theorem 1′. *Assume* (A.1) *and let* $c \in T^*$. *Then* $\lim_{t\to\infty}\mu_{t,\Gamma}^c(D) = C_\xi W^\xi\mu_\Gamma^\xi(D)$ *for every* $D \subset \Gamma$ a.s. (P_x), *where* C_ξ *is the same as before and* $\mu_\Gamma^\xi(D) = |\det(e_1, \cdots, e_d)| \cdot \sum_{x\in D}\phi_{-\xi}(x)$.

Proof. Let $\{e_j^*\}$ be the dual basis to $\{e_k\}$ of $S : e_k\cdot e_j^* = \delta_{k,j}$, $\Gamma^* = \{\sum(2\pi)n_je_j^* : (n_1, \cdots, n_d) \in \mathbb{Z}^d\}$ and

$$\Delta = \{\textstyle\sum_{j=1}^d a^je_j^* : -\pi \le a^j \le \pi,\ j = 1, \cdots, d\}.$$

Then any function $f(x)$ on Γ, which is zero outside a finite set, has the representation

$$f(x) = \frac{1}{|\Delta|}\int_\Delta f^*(\eta)e^{i\eta\cdot x}\,d\eta\ (x \in \Gamma)$$

where $f^*(\eta) = \sum_{x\in\Gamma} f(x)\exp(-i\eta\cdot x)(\eta \in S)$ and $|\Delta|$ denotes the Lebesgue measure of Δ, and we get as before

$$\int_\Gamma f(x)e^{\xi\cdot x}\mu_{t,\Gamma}^c(dx) = \frac{1}{|\Delta|}\int_\Delta W_t^\lambda e^{A(\lambda)t-[tc]\cdot\lambda} f^*(\eta)\ d\eta \cdot \sqrt{t}^d e^{-(A(\xi)t-[tc]\cdot\xi)}.$$

Since $\mathscr{R}A(\lambda) < A(\xi)$ for $\eta \in \Delta - \{0\}$, the same arguments for the proof of Theorem 1 leads us to

$$\lim_{t\to\infty} \int_\Gamma f(x)e^{\xi\cdot x}\mu_{t,\Gamma}^c(dx) = \frac{(2\pi)^d}{|\Delta|} C_\xi W^\xi f^*(0).$$

Now Theorem 1′ follows from $|\Delta|/(2\pi)^d = |\det(e_1^*, \cdots, e_d^*)| = |\det(e_1, \cdots, e_d)|^{-1}$.

7. The region of propagation. In this section we see that the region of propagation of the process $\mathbf{x}_t$ asymptotically agrees with $tM = \{tx : x \in M\}$ or $t\bar{M}$ where

$$M = \{c \in S : \nu(c) > 0\} \quad \text{and} \quad \nu(c) = \inf_{\xi\in S}(A(\xi) - c\cdot\xi),$$

and $\bar{M}$ is the closure of M. The results similar to what will be given here are found in Mollison's papers [19], [20], in which the problem is treated under a more general situation (see also [6]).

Because of (C.4) M is nonempty and bounded. M is open if (A.3) holds, but not in general. M is convex as is clear from

$$M = \cup_{n=1}^\infty \cap_{\xi\in S}\left\{c \in S : A(\xi) - c\cdot\xi > \frac{1}{n}\right\}.$$

It should be noted, though not needed here, that the definition of $\bar{M}$ in [19], [20] is $\cap_{\theta\in\Theta}\{c : \gamma(\theta) \geq c\cdot\theta\}$ where $\gamma(\theta)$ denotes the infimum appearing in (2.3). The agreement of two definitions can be verified without difficulty.

Throughout this section the process is supposed to *start with one particle at the origin.* First we assert the following.

i) *If* (A.2) *holds, c is an inner point of M and* $0 < \delta < \nu(c)$, *then* $e^{-\delta t}Z_t(D + tc) \to \infty$ a.s. *on* $\{Z_t \to \infty\}$ *for any nonempty open D.*

ii) *Let* $c \notin \bar{M}$ *and take* ξ *so that* $A(\xi) < c\cdot\xi$. *Then* $Z_t(D + tc) \to 0$ a.s. *for all D such that* $\inf_{x\in D}\xi\cdot x > -\infty$.

By observing $\bar{M} = \{c : \nu(c) \geq 0\}$, ξ in ii) always exists.

Let us deduce i) from Theorem 1. Given a positive number N, consider a particle, among those in $\mathbf{x}_t$, $t > 0$, which is born out of distance N from its parent and you discard every such one together with all its offspring. The resulting process, denoted by $\mathbf{x}_t^N$, is completely dominated by $\mathbf{x}_t$ and for the new process Theorem 1 can be applied if N is sufficiently large. Let T^{*N}, M^N and ν^N be defined for $\mathbf{x}_t^N$. Since $\nu^N \uparrow \nu$ as $N \uparrow \infty$, if M contains c as an inner point, then so does M^N for large N. In view of the next lemma, c also belongs to T^{*N} and, from Theorem 1, i) follows.

Lemma 10. *If* (A.4) *holds, the interior of M agrees with* T^*.

Proof. Let (A.4) be true. Then for each $c \in \partial T^*$ we have the following alternative: either (a) there exists a $\xi \in \partial T$ such that $c = \mathbf{D}A(\xi)$ or (b) there is an infinite sequence $\xi_n \in T$ such that $|\xi_n| \to \infty$, $\theta \equiv \lim \xi_n/|\xi_n|$ exists and $c = \lim \mathbf{D}A(\xi_n)$. (b) occurs only if $G(\{x : \theta\cdot x > 0\}) = 0$. Let us prove that if (b) is the case,

$$c\cdot\theta = 0 \quad \text{and} \quad \lim A(\xi_n)/|\xi_n| = 0. \tag{7.1}$$

For this purpose we write $\xi_n = r_n\theta + s_n\theta_n$, where $r_n > 0$, $s_n \in \mathbb{R}$ and $\theta_n \in \Theta$ with $\theta\cdot\theta_n = 0$.

Then $s_n/r_n \to 0$, $r_n \to \infty$, and by the dominated convergence theorem

$$\mathbf{D}A(\xi_n)\cdot\theta = \int_{\xi_n\cdot x\geq 0} \theta\cdot x e^{\xi_n\cdot x} G(dx) + o(1).$$

Since $\xi_n\cdot x \geq 0$ implies $-\theta\cdot x \leq (s_n/r_n)\theta_n\cdot x$ and $\mathbf{D}A(\xi_n)\cdot\theta_n$ is bounded, the integral on the right side above tends to zero. Thus $c\cdot\theta = 0$. The second relation of (7.1) follows from the first with the help of the inequality $0 < A(\xi) - \mathbf{D}A(\xi)\cdot\xi \leq A(0)$ valid for $\xi \in T$.

Now the proof of Lemma 10 is easy. Recall $T^* \subset M$ and M is convex. If the conclusion of the lemma is false, there is an inner point c of M which simultaneously belongs to the boundary of T^*. (a) cannot occur for this c, because it implies $c\cdot\xi = A(\xi)$, which contradicts $c \in M$. Let (b) occur. Since c is an inner point of M, there is an $\varepsilon > 0$ such that $c + \varepsilon\theta \in M$ and it follows that $A(\xi_n) > (c + \varepsilon\theta)\cdot\xi_n = |\xi_n|(c\cdot\theta + \varepsilon + o(1))$, which contradicts (7.1).

ii) is an easy consequence of $W^\xi < \infty$ a.s. In fact from the identity

$$\check{\phi}_\xi(\mathbf{x}_t - tc) = W_t^\xi \exp\{-(c\cdot\xi - A(\xi))t\} \tag{7.2}$$

it follows that if $c\cdot\xi > A(\xi)$,

$$\lim_{t\uparrow\infty}\check{\phi}_\xi(\mathbf{x}_t - tc) = 0 \quad \text{a.s.}, \tag{7.3}$$

which is stronger than the conclusion of ii).

Let B be the unit ball of S. To get a more visualized picture of the propagation we consider the random set $\{x \in S : Z_t(B + x) > 0\}$: the set of all points of S, within distance 1 from that there lives at least one particle at time t, or more generally

$$F_t \equiv \{x \in S : Z_t(B + x) > \rho(t)\}$$

where $\rho(t)$ is a positive function of $t \geq 0$ and is assumed to satisfy

$$\lim_{t\uparrow\infty} e^{-\delta t}\rho(t) = 0 \quad \text{for all} \quad \delta > 0.$$

i′) *If* (A.2) *holds and c is an inner point of M, then*

$$c \in \liminf_{t\uparrow\infty} F_t/t \quad \text{a.s. } on \quad \{Z_t \to \infty\}. \tag{7.4}$$

ii′) $\limsup_{t\uparrow\infty} F_t/t \subset \bar{M}$ a.s.

By i′) and ii′) the closures of $\liminf F_t$, $\limsup F_t$ and M agree with each other a.s. on $\{Z_t \to \infty\}$, provided (A.2) holds.

We shall also consider the following random variables

$$R_t^\theta = \max\{r \in \mathbb{R} : Z_t(B + r\theta) > \rho(t)\}, \quad \theta \in \Theta \tag{7.5}$$

and show that if (A.2) holds and M contains an inner point in the direction θ or $-\theta$, then

iii) $\lim_{t\uparrow\infty} R_t^\theta/t = s(\theta)$ a.s. *on* $\{Z_t \to \infty\}$
where

$$s(\theta) = \sup\{r \in \mathbb{R} : r\theta \in M\}.$$

For the proofs of i′), ii′) and iii) the previous results will be applied only via i) or ii).

i′) is ready from i): in fact if c is an inner point of M, then by i) we have $P_0[B + tc \subset F_t,\ t \uparrow \infty \mid Z_t \to \infty] = 1$ and hence (7.4). To prove ii′) let $\zeta = \mathbf{D}A(0)$ and

$$\tilde{A}(\xi) = A(\xi) - \zeta\cdot\xi. \tag{7.6}$$

Then $\tilde{A}(\xi)$ attains its minimum if and only if $\xi = 0$. Let $H^\delta = \{x : \operatorname{dist}(x, M) \geq \delta\}$ for $\delta > 0$. For each $c \in \partial H^\delta$ we can choose a point $\xi(c) \in S$ so that

$$c\cdot\xi(c) > A(\xi(c)) \tag{7.7}$$

which is the same as $(c - \zeta)\cdot\xi(c) > \tilde{A}(\xi(c))$. Thus $(c - \zeta)\cdot\xi(c) > 0$, which enables us to find a finite set of points $c_1, \cdots, c_n \in \partial H^\delta$ such that

$$H^{2\delta} - \zeta \subset \cup_{j=1}^n (D_j + (c_j - \zeta))$$

where $D_j = \{x \in S : x \cdot \xi(c_j) > 0\}$. Since $tD_j = D_j$, it follows $tH^{2\delta} \subset \cup_{j=1}^n (D_j + tc_j)$. By ii) with the help of (7.7) we consequently obtain that $Z_t(tH^{2\delta}) \to 0$ a.s. Hence lim sup $F_t/t \subset H^{2\delta}$ a.s., proving ii′).

For the proof of iii) let $c \in \partial M$ and $c = s(\theta)\theta$. By the assumption, $c - \theta/n = (s(\theta) - 1/n)\theta \in M$ for large n; hence from i) it follows $P_0[R_t^\theta \geq (s(\theta) - 1/n)t, t \uparrow \infty \mid Z_t \to \infty] = 1$, which proves the lower bound: lim inf $R_t^\theta/t \geq s(\theta)$ a.s. on $\{Z_t \to \infty\}$. The upper bound is clear from ii′), proving iii).

There is a gap between i′) and ii′). The fact would be lim $F_t/t = M$ a.s. on $\{Z_t \to \infty\}$, but the author could prove neither side of inclusion.

8. Proof of Theorem 2. For $f \in \mathbb{B}$ with the bounds $0 \leq f \leq 1$ we set

$$\hat{f}(\mathbf{x}) = \begin{cases} f(x_1) \cdots f(x_n) & \text{if } \mathbf{x} = (x_1, \cdots, x_n) \quad S^n, \quad n \geq 1 \\ 1 & \text{if } \mathbf{x} = \partial \end{cases}$$

and define

$$\mathbb{K}f(x) = \textstyle\sum_{n=0}^\infty \displaystyle\int_{S^n} \hat{f}(x + \mathbf{y})\pi_n(d\mathbf{y}), \quad x \in S$$

($x + \partial$ is understood as ∂). Then $u(t, x) = E_x[\hat{f}(\mathbf{x}_t)]$ is a unique bounded solution of the non-linear evolution equation

$$\frac{\partial u}{\partial t} = \kappa(\mathbb{K}u - u) \tag{8.1}$$

with an initial condition $u(0, x) = f(x)$. (In (8.1) $\mathbb{K}$ acts on x only). (See [14].)

The problem treated in the next proposition is one dimensional in nature and has been solved by Biggins (1977). For completeness we shall give a proof fitted to the present context.

PROPOSITION 2. *If $\xi \notin T$ and $A(\alpha\xi) < \infty$ for some $\alpha > 1$, then* $\lim_{t\uparrow\infty} W_t^\xi = 0$ a.s. (P_x).

PROOF. Let ξ satisfy the condition in the proposition. Following McKean [17] we set

$$u^*(t, x) = E_x[\exp(-W_t^\xi)].$$

Take $c \in S$ so that $A(\xi) = c \cdot \xi$ and let $f_\xi(x) = \exp(-\phi_\xi(x))$. Then $u^*(t, x) = E_x[\hat{f}_\xi(\mathbf{x}_t - tc)]$; or what is the same, if $u(t, x) = E_x[\hat{f}_\xi(\mathbf{x}_t)]$,

$$u^*(t, x) = u(t, x - tc). \tag{8.2}$$

Since $\exp(-W_t^\xi)$ is a sub-martingale, $u^*(t, x)$ is non-decreasing in t; in particular

$$1 - u^*(t, x) \leq 1 - \exp(-\phi_\xi(x)) \leq \min\{1, \phi_\xi(x)\}.$$

For the proof of Proposition 2 it suffices to show that the limit

$$w(x) = \lim_{t\to\infty}(1 - u^*(t, x))$$

is identically zero, for W_t^ξ has an a.s. limit. Let $\theta = \xi/|\xi|$ and define, for $0 < r < |\xi|$ and $t > 0$,

$$V(t, r) = \int_{-\infty}^\infty (1 - u^*(t, b\theta))e^{-rb}\, db$$

which is bounded in t if r is fixed. By changing the variable according to $b \to b + tc \cdot \theta$ and noting $u(t, x) = u(t, (x\cdot\theta)\theta)$, from (8.2) it follows that

$$V(t, r) = \int_{-\infty}^{\infty} (1 - u(t, b\theta))e^{-rb}\, db\cdot e^{-rtc\cdot\theta}.$$

Since $\mathbb{K}u(t, x) = \mathbb{K}u^*(t, x + tc)$ and u satisfies (8.1), we obtain

$$\frac{\partial V}{\partial t} = -\int_{-\infty}^{\infty} \kappa(\mathbb{K}u^* - u^*)(t, b\theta)e^{-rb}\, db - r(c\cdot\theta)V$$

and, letting t tend to infinity (along a sequence if necessary),

$$\int_{-\infty}^{\infty} \kappa(-\mathbb{K}\{1 - w\} + 1 - w)(b\theta)e^{-rb}\, db - r(c\cdot\theta)\int_{-\infty}^{\infty} w(b\theta)e^{-rb}\, db = 0.$$

From this and the following equation

$$\int_{-\infty}^{\infty} \mathbb{M}w(b\theta)e^{-rb}\, db = \int_{-\infty}^{\infty}\int_S w(b\theta + (y\cdot\theta)\theta)e^{-rb}G(dy)\, db$$

$$= (A(r\theta)/\kappa + 1)\int_{-\infty}^{\infty} w(b\theta)e^{-rb}\, db,$$

it follows that if we set

$$Q(x) = \mathbb{M}w(x) - (1 - \mathbb{K}\{1 - w\}(x)),$$

then

$$(8.3)\qquad (A(r\theta) - rc\cdot\theta)\int_{-\infty}^{\infty} w(b\theta)e^{-rb}\, db = \kappa\int_{-\infty}^{\infty} Q(b\theta)e^{-rb}\, db.$$

By induction it is easily shown that $\sum_{k=1}^{n} a^k - 1 + \prod_{k=1}^{n}(1 - a^k) > 0$ if $0 \le a^k \le 1$ $(k = 1, \cdots, n)$ and at least two a^k's are not zero $(n \ge 2)$. Since $Q(x) = \sum_n \int (\check{w}(x + \mathbf{y}) - 1 + (1 - w)\hat{}(x + \mathbf{y}))\pi_n(d\mathbf{y})$ and $w(b\theta)$ is monotone in b,

$$(8.4)\qquad \int_{-\infty}^{\infty} Q(b\theta)e^{-rb}\, db > 0 \quad \text{for all}\quad r > 0 \quad\text{or}\quad w \equiv 0.$$

Now let $\xi \in \partial T$. Then we can take $\mathbf{D}A(\xi)$ for c and it follows that $A(r\theta) - rc\cdot\theta = A(r\theta) - A(\xi) - \mathbf{D}A(\xi)\cdot(r\theta - \xi) = O((|r - |\xi||)^2)$ as $r \to |\xi|$. By this relation and the inequality $w(b\theta) \le \min\{1, \exp(|\xi| b)\}$, the left-hand side of (8.3) tends to zero as $r \uparrow |\xi|$, and so the right-hand side is zero. Thus by (8.4) $w \equiv 0$. When $\xi \notin T \cup \partial T$, there is $0 < s < |\xi|$ such that $A(s\theta) = sc\cdot\theta$. By (8.3) we have again $\int Qe^{-sb}\, db = 0$ and hence $w \equiv 0$. The proof of Proposition 2 is complete.

In view of Proposition 2, Theorem 2 is clear from the identity (7.2) and the first assertion in the next lemma, which we arrange for quotation in the next section.

LEMMA 11. *Assume the hypothesis of Theorem* 2 *and let* $\zeta = \mathbf{D}A(0)$. *Then* (a) $M = T^*$ *and for each* $c \notin T^*$ *there exists* $\xi \in \partial T$ *and* $r \ge 1$ *such that* $(c - \zeta)\cdot\xi > 0$, $c - \zeta = r(\mathbf{D}A(\xi) - \zeta)$ *and* $c\cdot\xi \ge A(\xi)$; (b) *for each* $c \notin T^* \cup \partial T^*$ *there exists* $\xi \in T$ *such that* $c\cdot\xi = A(\xi)$.

PROOF. First we observe that by (A.3) T is bounded and by (A.4) $\bar{T} \subset X$ (see Section 1 for the definition of X). Hence for each $c \in \partial T^*$ there is a $\xi \in \partial T$ such that $c = \mathbf{D}A(\xi)$ and $A(\xi) - c\cdot\xi = 0$; in particular $\nu(c) = 0$ and in view of Lemma 10 $M = T^*$. Let $\tilde{A}(\xi) = A(\xi) - \zeta\cdot\xi$ as in (7.6). Since $\zeta \in T^*$, for $c \notin T^*$ there exists a $c_1 \in \partial T^*$ which lies between

c and ζ. If we take $\xi_1 \in \partial T$ so that $c_1 = \mathbf{D}A(\xi_1)$, then $(c_1 - \zeta)\cdot \xi_1 = \tilde{A}(\xi_1) > 0$. It is clear that this ξ_1 satisfies the requirement for ξ in (a). If not only $c \notin T^*$, but also $c \notin \partial T^*$, we have $c\cdot\xi_1 > A(\xi_1)$ with ξ_1 taken above. By definition of T, there exists $0 < s < 1$ such that if $\xi = s\xi_1$, then $\xi \in T$ and $c\cdot\xi = A(\xi)$, proving (b).

9. Applications to a non-linear equation. In this section we study the following non-linear evolution equation

$$\frac{\partial v}{\partial t} = \kappa(1 - v - \mathbb{K}\{1 - v\}), \qquad (v = v(t, x),\ t \geq 0,\ x \in S) \tag{9.1}$$

which is obtained from (8.1) by the simple change of dependent variables $v = 1 - u$. We consider only solutions with values in the unit interval: $0 \leq v \leq 1$. For simplifying the exposition we *assume* $\pi_0 = 0$ *and that the hypotheses of Theorems 1 and 2 are valid* throughout this section, unless otherwise specified. When v is independent of the space variable x, (9.1) is reduced to an ordinary differential equation:

$$\frac{dv(t)}{dt} = \omega(v(t)) \tag{9.2}$$

where

$$\omega(s) = \kappa(1 - s - \textstyle\sum_{n=1}^{\infty} \pi_n(S^n)(1 - s)^n), \quad 0 \leq s \leq 1.$$

The function ω has two zeros 0 and 1, which are equilibrium states of (9.1) as well as (9.2). We discuss below the nature of the stability of these two states concerning the equation (9.1).

Let us first recall that the solution of (9.1) has a following representation

$$v(t, x) = 1 - E_x[\hat{g}(\mathbf{x}_t)] = 1 - E_0[\hat{g}(\mathbf{x}_t + x)]$$

where $g(x) = 1 - v(0, x)$. Then from Theorem 1 or, what interests us, its crude consequence that if $c \in T^*$, for any open set D which is not empty,

$$\mathbf{x}_t(D + tc) \to \infty \ (t \to \infty) \quad \text{a.s.} \quad (P_x),$$

it follows that if $v(0, x)$ is continuous and not identical to zero, then for each $c \in T^*$, $v(t, x - tc) \to 1$ $(t \to \infty)$ locally uniformly. This can be made a little more precise, though we need an additional condition of

$$\limsup_{|\eta|\to\infty, \eta \in S} \mathcal{R}A(i\eta) < A(0). \tag{A.2)'$$

This implies the condition (A.2) and is satisfied if the absolutely continuous part of the measure $G(dx)$ is not zero. We shall prove the following:

i) *if* (A.2)′ *is further assumed, then for any closed set* $F \subset S$ *which is contained in* T^* *and not empty,*

$$\inf_{x \in -tF} v(t, x) \to 1 \quad \text{as} \quad t \to \infty \tag{9.3}$$

$(-tF = \{-tc : c \in F\})$, *provided* $v(0, x)$ *is continuous and not identical to zero*; *and*

ii) *if* $v(0, x)$ *is zero outside a compact set,*

$$\sup_{x \notin -tT^*} v(t, x) = O(t^{-d/2}) \quad \text{as} \quad t \to \infty. \tag{9.4}$$

REMARK 4. (i) The results stated above are suggested by very similar ones observed for the semi-linear diffusion equation

$$\frac{\partial v}{\partial t} = \Delta v + v(1 - v) \tag{9.5}$$

(cf. Aronson and Weinberger [1]). In a special case (1.5) some partial results are obtained by several authors (cf. [4], [12], [19], or [20]; in this special case with $d = 1$ the present author [27] has obtained a direct proof of i)). (ii) Let $d = 1$. Denote by y_t the position of the rightmost particle at time t of the process $\mathbf{x}_t$, and let $u(t, x)$ be its distribution function $P_0[y_t < x]$. If χ is the indicator function of negative real axis, we have $u(t, x) = E_{-x}[\chi(\mathbf{x}_t)]$, so the function $v(t, x) \equiv 1 - u(t, -x) = P_0[y_t \geq -x]$ is a solution of (9.1) with $v(0, \cdot) = 1 - \chi$ and the function $m(t) \equiv \inf\{x : P_0[y_t \geq -x] > ½\}$ would determine in a sense the position of $v(t, \cdot)$. Since $\lim y_t/t = c^* \equiv \sup T^*$ a.s. as we have known, $\lim m(t)/t = -c^*$. For the one-dimensional branching Brownian motion to which (9.5) corresponds it has been known that $v(t, x + m(t))$ converges to a proper distribution function as $t \to \infty$ [16] and an accurate asymptotic expansion of $m(t)$ has been obtained [25], [9] (see also [26]). Any corresponding result to the present case is not known.

For the proof of ii) we make use of the relation (5.14) which holds, as easily verified, uniformly for $f(x) = K(x)\phi_\xi(x)$, $\xi \in \partial T$ (note T is bounded because of (A.3)). Let the initial function $f(x) = v(0, x)$ vanish off of a compact set. Then there exist a constant C and measurable functions $f_\xi(x)$, $\xi \in \partial T$ such that for all $x \in S$

$$f_\xi(x) \leq K(x)\phi_\xi(x) \tag{9.6}$$

$$f(x) \leq C \cdot \inf_{\xi \in \partial T} f_\xi(x) \tag{9.7}$$

$$f_\xi(x + b(c_\xi - \zeta)) \quad \text{is non-decreasing in} \quad b \in \mathbb{R}^1 \tag{9.8}$$

where $c_\xi = \mathbf{D}A(\xi)$ and $\zeta = \mathbf{D}A(0)$. For each $c \notin T^*$ we can find $\xi \in \partial T$ and $r \geq 1$ so that $c - \zeta = r(c_\xi - \zeta)$ by virtue of Lemma 11 (a). Then by (9.7) and (9.8)

$$f(x - tc) \leq Cf_\xi(x - tc) = Cf_\xi(x - tr(c_\xi - \zeta) - t\zeta) \leq Cf_\xi(x - tc_\xi).$$

Since $E_0[\check{f}_\xi(\mathbf{x}_t - tc_\xi)] = O(t^{-d/2})$ uniformly in $\xi \in \partial T$ by virtue of (9.6), we conclude

$$\sup_{c \notin T^*} E_0[\check{f}(\mathbf{x}_t - tc)] = O(t^{-d/2}).$$

The required relation now follows from

$$1 - E_x[(1 - f)\hat{}(\mathbf{x}_t)] \leq E_x[\check{f}(\mathbf{x}_t)].$$

For the proof of i) we show a lemma and a theorem, which are interesting by themselves.

LEMMA 9′. *Assume* (A.1) *and* (A.4). *Then for any closed set* $\Gamma \subset S$ *which is contained in* T, *there exist real numbers* C, $\delta > 0$ *and* $1 < \alpha \leq 2$ *such that*

$$E_0[|W_t^{\xi'} - W_t^{\xi}|^\alpha] \leq C|\xi' - \xi|^\alpha \quad \textit{if} \quad |\xi' - \xi| < \delta \quad \textit{and} \quad \xi, \xi' \in \Gamma.$$

THEOREM 5. *Assume* (A.2)′ *in addition to the assumption of Lemma* 9′. *Then for any closed set* $F \subset S$ *which is contained in* T^* *there is a constant* $1 < \alpha \leq 2$ *such that if* f *is continuous and vanishing off of a compact set,*

$$\lim_{t\to\infty} \sup_{c \in F} E_x\left[\left|\int_S f(x)\mu_t^c(dx) - C_\xi\left(\int_S f(x)\mu^\xi(dx)\right)W^\xi\right|^\alpha\right] = 0.$$

(Here notations μ_t^c, etc. are the same as in Theorem 1.)

By assuming, instead of (A.2)′, that $G(dx)$ is supported by a centered lattice as in Section 6, we also have the conclusion of Theorem 5 with a minor modification of the proof.

Before proceeding into proofs of Lemma 9′ and Theorem 5, we show i) by applying them. Let $g(x) = 1 - v(0, x)$, $f(x) = -\log g(x)$ and write simply $\mu_t^c(f)$ for $\int_S f(x)\mu_t^c(dx)$. We can assume that $g(x)$ is positive, continuous and equal to 1 outside a compact set, reflecting on f so that Theorem 5 can be applied to f. Since

$$\begin{aligned} 1 - v(t, -tc) &= E_0[\hat{g}(\mathbf{x}_t - tc)] = E_0[\exp(-\check{f}(\mathbf{x}_t - tc))] \\ &= E_0[\exp(-\mu_t^c(f)/\Xi(t))] \end{aligned}$$

where $\Xi(t)$ is one appearing in (5.3), it suffices to show

$$\limsup_{t\to\infty}\sup_{c\in F} P_0[\mu_t^c(f) < \varepsilon] \to 0 \quad \text{as } \varepsilon \downarrow 0,$$

for $F \subset S$ closed and contained in T^*. Assign $\xi \in T$ to $c \in T^*$ so that $c = \mathbf{D}A(\xi)$ and observe

$$P_0[\mu_t^c(f) < \varepsilon] \le P_0[C_\xi \mu^\xi(f) W^\xi < 2\varepsilon] + P_0[C_\xi \mu^\xi(f) W^\xi - \mu_t^c(f) > \varepsilon].$$

By Theorem 5 the second term on the right side vanishes as $t \to \infty$ uniformly for $c \in F$. By using the analogous inequality and applying Lemma 9′, we can verify that the first term also vanishes as $\varepsilon \downarrow 0$ uniformly for $c \in F$. Thus we obtain i).

PROOF OF LEMMA 9′. Let Γ be a closed subset of S which is contained in T. Then there exist real numbers $\varepsilon > 0$, $\delta > 0$ and $1 < \alpha \le 2$ such that

$$\alpha A(\xi) - A(\alpha\xi') \ge \varepsilon \quad \text{if} \quad |\xi - \xi'| < \delta \quad \text{and} \quad \xi, \xi' \in \Gamma. \tag{9.9}$$

Recall the proofs of Lemmas 8 and 9 and the definition of $\Phi_t(x)$ there, and replace $\lambda = \xi + i\eta$ by $\xi' \in S$ in them. Then instead of (5.9) we have

$$|\Phi_t(x)|^\alpha \le \text{const}[e^{-\alpha A(\xi)t}|x|^\alpha(\phi_{\alpha\xi}(x) + \phi_{\alpha\xi'}(x)) + |t|^\alpha e^{-\alpha A(\xi')t}\phi_{\alpha\xi}(x)]|\xi - \xi'|^\alpha.$$

The rest goes through as in Lemma 9, by noting (9.9).

PROOF OF THEOREM 5. We shall follow the proof of Theorem 1. Let $\tilde{I}(t)$'s be the same as $I(t)$'s except that $f_p(x)$ in $I(t)$'s is replaced by $f(x)e^{-\xi\cdot x}$ where $f(x)$ is a smooth function with compact support. Let $F \subset T^*$ and $\Gamma \subset T$ be closed sets and $\mathbf{D}A(\Gamma) = F$, and choose δ and α as in the proof of Lemma 9′. Our task is to show that the following three expectations

$$E_0[|\widetilde{II}(t)|^\alpha]; \qquad E_0[|\widetilde{III}(t)|^\alpha]; \qquad E_0\left[\left|\tilde{I}(t) - C_\xi \int_S f(x)\mu^\xi(dx) W^\xi\right|^\alpha\right]$$

converge to zero uniformly in $c \in F$. The assertion for the first one is easily seen without any cost. To estimate the second expectation we note that the condition (A.2)′ implies that if $A(\xi) < \infty$

$$j(\xi) \equiv A(\xi) - \limsup_{|\eta|\to\infty, \eta\in S} \mathscr{R}A(\xi + i\eta) > 0.$$

Since Γ is compact and $A(\xi + i\eta)$ is equi-continuous in $\xi \in \Gamma$ for $\eta \in S$, $\inf_{\xi\in\Gamma} j(\xi) > 0$. This enables us to obtain the uniform convergence for $E_0|\widetilde{III}(t)|^\alpha$. As for the third one, it suffices to show that the convergence of $\lim_{t\to\infty} E_0[|W_t^\xi - W^\xi|^\alpha] = 0$ is uniform in $\xi \in \Gamma$, but this is easily verified in view of Lemma 9′. The proof of Theorem 5 is finished.

When $v(0, x)$ is smooth, the solution v of (9.1) is differentiable in x and if we let $\tilde{v}(t, x) = v(t, x - tc)$,

$$\frac{\partial \tilde{v}}{\partial t} = \kappa(1 - \tilde{v} - \mathbb{K}\{1 - \tilde{v}\}) - c\cdot\partial_x \tilde{v},$$

where $\partial_x = ((\partial/\partial x^1), \cdots, (\partial/\partial x^d))$. If we take $v(0, x) = 1 - \exp(-\phi_\xi(x))$ and $c \in S$ such that $c\cdot\xi = A(\xi)$, then $\tilde{v}(t, x) = 1 - E_x[\exp(-W_t^\xi)]$ and its limit

$$w(x) = 1 - E_x[\exp(-W^\xi)] \tag{9.10}$$

satisfies

$$\kappa(1 - w - \mathbb{K}\{1 - w\}) - c\cdot\partial_x w = 0. \tag{9.11}$$

By Propositions 1 and 2 $w(x)$ in (9.10) is not constant if $\xi \in T$ and is identical to zero if $\xi \notin T$. A nonconstant solution of (9.11) (with values in [0, 1]) is called a *traveling wave* for

(9.1) with velocity c. If w is a traveling wave with velocity c, we have a plain wave solution $w(x + tc)$ for (9.1). In the diffusion case (9.5) it is known that there is a traveling wave with velocity c (which is analogously defined for (9.5)) if and only if $|c| \geq 2$ (cf. [1]). An analogue is

iii) *a traveling wave for* (9.1) *with velocity c exists if and only if* $c \notin T^*$.

(This result is not new; it is obtained by applying the fixed point theorem [3], [11].) It is plain to see from i) that there is no traveling wave for $c \in T^*$. To see the existence, first we take $c \notin T^* \cup \partial T^*$, and choose $\xi \in T$ so that $c \cdot \xi = A(\xi)$, which is possible by Lemma 11 (b). Then $w(x)$ in (9.10) is a traveling wave with velocity c as is remarked just after (9.11). Let us shift this w so that the resultant, say $w_c(x)$, takes value ½ at $x = 0 : w_c(0) = ½$. Now let $c_0 \in \partial T^*$. Since by (9.11) $\partial_x w_c$ is bounded uniformly for c, by the Arzela-Ascoli theorem there is a sequence c_n such that $c_n \to c_0$ and w_{c_n} is locally uniformly convergent. By using (9.11) again $\partial_x w_c$ is also convergent and so $w^* = \lim w_{c_n}$ is a solution of (9.11) with $c = c_0$. Since $w^*(0) = ½$ and $\mathbb{K}\{½\} < ½$, w^* is not constant. Thus w^* is a traveling wave with velocity c_0.

Finally let us prove (2.5) by making use of (9.11). When $\xi = 0$, (2.5) is nothing but the well-known result for the Galton Watson processes. Let w be defined by (9.10) for $\xi \in T$, $\xi \neq 0$. Noting that $w(x) = 1 - E_0[\exp(-e^{\xi \cdot x} W^\xi)]$, we have $q \equiv 1 - P_0[W^\xi = 0] = \lim_{r \to \infty} w(r\xi)$ and by (9.11) $1 - q = \mathbb{K}\{1 - q\} = \omega(1 - q)$. Therefore q is 0 or 1, but by Proposition 1 $q \neq 0$. Hence $P_x[W^\xi > 0] = 1$. (A similar argument proves (2.5) without restriction of $\pi_0 = 0$.)

The following two examples illustrate typical cases of our process and of the equation (9.1).

EXAMPLE 1. Let $\pi_n = 0$ for $n \neq 1, 2$ and $\pi_2(\cdot) = p_2\delta_0(\cdot)$ where $0 < p_2 < 1$ and δ_0 denotes the delta measure carrying unit mass at $0 \in S^2$. Then our process $\mathbf{x}_t$ becomes a binary splitting branching Poisson process in which each particle moves through its lifetime obeying the law of a compound Poisson process in S whose infinitesimal generator is

$$\mathscr{L}f(x) = \kappa_1\left(\int_S f(x + y)H(dy) - f(x)\right)$$

where $\kappa_1 = \kappa\pi_1(S)$ and $H(\cdot) = \pi_1(\cdot)/\pi_1(S)$, and the rate of exponential holding time for splitting is $\kappa_2 = \kappa - \kappa_1 = \kappa p_2$. The equation (9.1) then reduces to

$$\frac{\partial v}{\partial t} = \mathscr{L}v + \kappa_2 v(1 - v)$$

and $A(\xi) = \kappa_1(\int e^{\xi \cdot x}H(dx) - 1) + \kappa_2$.

EXAMPLE 2. Let $\pi_n = 0$ for $n \neq 2$ and $\pi_2(dy^1 dy^2) = \delta_0(dy^1)H(dy^2)$ where $H(dy)$ is a probability measure on S. Then $A(\xi) = \kappa \int e^{\xi \cdot x}H(dx)$ and (9.1) becomes

$$\frac{\partial v}{\partial t} = \kappa\left(\int_S v(t, x + y)H(dy)\right)(1 - v).$$

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