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INVARIANCE PRINCIPLES FOR MIXING SEQUENCES OF RANDOM VARIABLES

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In this note we prove weak invariance principles for some classes of mixing sequences of L_2-integrable random variables under the condition that the variance of the sum of n random variables is asymptotic to $\sigma^2 n$ where $\sigma^2 > 0$. One of the results is simultaneously an extension to nonstationary case of a theorem of Ibragimov and an improvement of the φ-mixing rate used by McLeish in his invariance principle for nonstationary φ-mixing sequences.

1. Introduction. The aim of this paper is to prove some invariance principles for mixing sequences of random variables under L_2 moment conditions, without assumptions of stationarity. First we give an invariance principle for nonstationary ψ-mixing sequences (Corollary (2.2)) that improves Theorem 10 of [5], showing that the condition $\sum_n \psi(n) < \infty$ may be dropped.

Another result (Corollary (2.4)) is an extension to the nonstationary case of the invariance principle for φ-mixing sequences proved by Ibragimov (1975). This result improves the mixing rate used by McLeish (1975) in his invariance principles for nonstationary φ-mixing sequences (Theorem (3.8)); (for example McLeish's condition $\varphi(n) = O[1/n(\log n)^{2+\varepsilon}]$, $\varepsilon > 0$ may be replaced by $\varphi(n) = O[1/(\log n)^{2+\varepsilon}]$, $\varepsilon > 0$. This also improves the mixing rate used in Theorem (3.4) of [12].)

Finally, we obtain an invariance principle for nonstationary ρ-mixing sequences of random variables, the mixing coefficient ρ being the maximal coefficient of correlation. A central limit theorem for such sequences of random variables in the stationary case has been proved by Ibragimov (1975), the mixing rate being $\sum_i \rho(2^i) < \infty$. Our theorem gives a functional form for Ibragimov's result; however our mixing rate is more restrictive, namely $\sum_i \rho^{1/2}(2^i) < \infty$. In Section 2 we summarize the invariance principles. Proofs of these theorems are given in Section 3.

Our invariance principles are obtained for mixing sequences, under the condition that the variance of the sum of n random variables is asymptotic to $\sigma^2 n$ where $\sigma^2 > 0$. But it is known that in the stationarity case this is a consequence of the condition $\sum_i \rho(2^i) < \infty$ (see Ibragimov, 1975). In Section 4 we give some conditions that are automatically satisfied for stationary sequences of random variables, and these imply the condition quoted above.

Definitions, notations and auxiliary results. Let (Ω, K, P) be a probability space and K_1 and K_2 two σ-algebras contained in the σ-algebra K. Define the following measures of dependence between K_1 and K_2 by

$$\rho(K_1 K_2) = \sup_{\{X\in L_2(K_1), Y\in L_2(K_2)\}} \frac{|E(X-EX)(Y-EY)|}{E^{1/2}(X-EX)^2 E^{1/2}(Y-EY)^2}$$

$$\alpha(K_1, K_2) = \sup_{\{A\in K_1, B\in K_2\}} |P(A\cap B) - P(A)P(B)|$$

$$\varphi(K_1, K_2) = \sup_{\{A\in K_1, P(A)\neq 0, B\in K_2\}} |P(B \mid A) - P(B)|$$

and

$$\psi(K_1, K_2) = \sup_{\{A\in K_1, B\in K_2, P(A)P(B)\neq 0\}} \left| \frac{P(A\cap B)}{P(A)P(B)} - 1 \right|.$$

Let $(X_i;\ i \geq 1)$ be a sequence of real valued random variables on (Ω, K, P), denote $F_n^m =$

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$\sigma(X_i;\, n \le i < m)$ and put:

$$\rho(n) = \sup_{p\in N} \rho(F_1^p, F_{n+p}^\infty)$$

$$\alpha(n) = \sup_{p\in N} \alpha(F_1^p, F_{n+p}^\infty)$$

$$\varphi(n) = \sup_{p\in N} \varphi(F_1^p, F_{n+p}^\infty)$$

and

$$\psi(n) = \sup_{p\in N} \psi(F_1^p, F_{n+p}^\infty).$$

The sequence $(X_n;\, n \ge 1)$ is said to be ρ-mixing, α-mixing, φ-mixing or ψ-mixing according as $\rho(n) \to 0$, $\alpha(n) \to 0$, $\varphi(n) \to 0$ or $\psi(n) \to 0$ as $n \to \infty$.

The following lemma (Lemma 1.17 of [6]) relates the concept of ρ-mixing to that of φ-mixing.

(1.1) LEMMA. *Suppose X is a random variable K_1 measurable and Y a random variable K_2 measurable, and $E^{1/2}X^2 < \infty$, $E^{1/2}Y^2 < \infty$. Then*

$$| E(X - EX)(Y - EY)| \le 2\varphi^{1/2}(K_1, K_2)E^{1/2}(X - EX)^2E^{1/2}(Y - EY)^2.$$

By this lemma $\rho(K_1, K_2) \le 2\varphi^{1/2}(K_1, K_2)$. Clearly a ψ-mixing sequence is φ-mixing, a φ-mixing sequence is ρ-mixing and a ρ-mixing sequence is α-mixing.

We shall also use the following lemma:

(1.2) LEMMA. *Suppose X is a random variable K_2 measurable and $E\,|X| < \infty$. Then*

$$| E(X \mid K_1) - EX | \le \psi(K_1, K_2)E \mid X \mid \quad \text{a.s.}$$

If Y is a random variable K_1 measurable and $E \mid Y \mid < \infty$ then

$$| EXY - EXEY | \le \psi(K_1, K_2)E \mid X \mid E \mid Y \mid.$$

The proof of this lemma is immediate by using simple functions. We denote by $S(n) = \sum_{i=1}^n X_i$ and by $S_k(n) = \sum_{i=k+1}^{k+n} X_i$. We assume that

$$\text{(1.3)} \qquad \lim_{n\to\infty} \frac{ES^2(n)}{n} = \sigma^2 > 0 \quad \text{and} \quad EX_i = 0 \quad \text{for every} \quad i.$$

For each $t \in [0,1]$ put

$$W_n(t) = \frac{\sum_{i=1}^{[nt]} X_i}{n^{1/2}\sigma}$$

where $[x]$ is the greatest integer $\le x$. The function $\omega \to W_n(t, \omega)$ is a measurable map from (Ω, F) into (D, B) where D is the set of all functions on the interval $[0, 1]$ which have left hand limits and are continuous from the right at every point, and B the Borel σ-algebra on D induced by the Skorohod topology. We shall give sufficient conditions for the weak convergence of W_n to the standard Brownian process on D, denoted by W in the sequel.

We shall denote the L_p norm by $\| \cdot \|_p$ and the weak convergence by $\Rightarrow$.

2. The invariance principles.

(2.1) THEOREM. *Let $(X_i;\, i \ge 1)$ be an α-mixing sequence of random variables satisfying* (1.3) *and*

a) $(X_i^2;\, i \ge 1)$ *is uniformly integrable*
b) $\sup_k ES_k^2(n) = O(n)$
c) $\lim_p \psi(p) < \infty$

then $W_n \Rightarrow W$.

This Theorem implies immediately

(2.2). COROLLARY. *Assume* $(X_i;\ i \geq 1)$ *is a* ψ*-mixing sequence of random variables satisfying* (1.3) *and the conditions* a) *and* b) *of the above theorem. Then* $W_n \Rightarrow W$.

(2.3) THEOREM. *Let* $(X_i;\ i \geq 1)$ *be a* ρ*-mixing sequence of random variables satisfying* (1.3) *and*

a) $(X_i^2;\ i \geq 1)$ *is uniformly integrable*
b) $\sum_i \rho(2^i) < \infty$
c) $\lim_n \varphi(n) < 1$.

Then $W_n \Rightarrow W$.

From this we deduce

(2.4) COROLLARY. *Assume* $(X_i;\ i \geq 1)$ *is a* φ*-mixing sequence of random variables satisfying* (1.3) *and*

a) $(X_i^2;\ i \geq 1)$ *is uniformly integrable*
b) $\sum_i \varphi^{1/2}(2^i) < \infty$

Then $W_n \Rightarrow W$.

(2.5) THEOREM. *Let* $(X_n;\ n \geq 1)$ *be a* ρ*-mixing sequence of random variables satisfying* (1.3) *and*

a) $(X_i^2;\ i \geq 1)$ *is uniformly integrable*
b) $\sum_i \rho^{1/2}(2^i) < \infty$

Then $W_n \Rightarrow W$.

3. Proofs. To prove these theorems we shall apply the following theorem which is a consequence of Theorem 19.2 of [1] taking into account (1.3) and the fact that α-mixing condition implies by induction that $W_n(t)$ has asymptotically independent increments (See the proof of Theorem 20.1 of [1]).

(3.1) THEOREM. *Suppose that* $(X_i;\ i \geq 1)$ *is an* α*-mixing sequence of random variables satisfying* (1.3) *and*

a) $(W_n^2(t);\ n \geq 1)$ *is uniformly integrable for each* t
b) *for each* $\varepsilon > 0$, *there exists* λ, $\lambda > 1$, *and an integer* n_0 *such that* $n \geq n_0$ *implies*

$$P(\{\max_{i \leq n} |S_k(i)| \geq \lambda \sigma n^{1/2}\}) \leq \frac{\varepsilon}{\lambda^2}$$

for all k. *Then* $W_n \Rightarrow W$.

To verify the tightness condition (3.1, b), we shall use the following.

(3.2) THEOREM. *Suppose that* $(X_i;\ i \geq 1)$ *is a sequence of random variables such that the families* $(X_n^2;\ n \geq 1)$ *and* $(S_k^2(n)/n;\ k \geq 1, n \geq 1)$ *are uniformly integrable. If one of the following conditions holds,*

a) $\lim_n \varphi(n) < 1$

or

b) $\lim_n \psi(n) < \infty$

then the condition (3.1.b) *is satisfied.*

The proof follows the same lines as the proof of tightness condition in Theorem 20.1 of [1]. We indicate only the changes to be made in that proof.

If condition a) holds, there exist $r \geq 1$ and $0 < \alpha < 1$, such that for $p \geq r$, $\varphi(p) < \alpha$. Therefore under the condition of this theorem, for n large enough,

$$\sum_{i=1}^{n-p-1} P(\{\max_{j<i} | S_k(j)| < 3\lambda n^{1/2} \leq | S_k(i)|\} \cap \{| S_k(n) - S_k(i+p)| \geq \lambda n^{1/2}\})$$
$$\leq \sum_{i=1}^{n} P(\{\max_{j<i} | S_k(j)| < 3\lambda n^{1/2} \leq | S_k(i)|\})\left(\alpha + \frac{2\varepsilon}{\lambda^2}\right)$$
$$\leq \alpha P(\{\max_{i\leq n} | S_k(i)| > 3\lambda n^{1/2}\}) + \frac{2\varepsilon}{\lambda^2} \quad \text{for all } k \text{ and } i \leq n,$$

whence by [1], pages 175–176 it follows that for large n

$$(1-\alpha)P(\{\max_{i\leq n} | S_k(i)| \geq 3\lambda n^{1/2}\}) \leq \frac{3\varepsilon}{\lambda^2}.$$

If the Condition b) holds, there exist $r \geq 1$ and $\beta > 0$ such that for $p > r$, $\psi(p) < \beta < \infty$. Therefore for n large

$$P(\{\max_{j<i} | S_k(j)| < 3\lambda n^{1/2} \leq | S_k(i)|\} \cap \{| S_k(n) - S_k(i+p)| \geq \lambda n^{1/2}\})$$
$$\leq P(\{\max_{j<i} | S_k(j)| < 3\lambda n^{1/2} \leq | S_k(i)|\}) \frac{\varepsilon}{\lambda^2}(1+\beta) \quad \text{for all } k \text{ and } i \leq n.$$

Note that, to prove Theorem (2.1) and Theorem (2.3) it is enough, because of Theorem (3.1) and (3.2), to establish the uniform integrability of the family $(S_k^2(n)/n;\, k \geq 1, n \geq 1)$. The Theorem (2.1) will follow from the following lemma.

(3.3) Lemma. *Suppose that $(X_i;\, i \geq 1)$ is a sequence of centered real valued random variables satisfying the conditions* a), b) *and* c) *of Theorem* (2.1). *Then the family*

$$\left(\frac{S_k^2(n)}{n};\, k \geq 1, n \geq 1\right)$$

is uniformly integrable.

Proof. Since $\lim_p \psi(p) < \infty$, there is an integer p such that $\psi(p) < \infty$. Let us denote by $F_n = \sigma(X_i, 1 \leq i < n)$. For $k > p$ let us put

$$\bar{S}_k(n) = \sum_{i=k+1}^{k+n} E(X_i \mid F_{i-p})$$

and for $k > p$ and $j < p$

$$Z_k^j(n) = \sum_{i=k+1}^{k+n} (E(X_i \mid F_{i-j}) - E(X_i \mid F_{i-j-1})).$$

Obviously for $k > p$ and $j < p$ fixed, $(Z_k^j(n), F_{n+k-j};\, n \geq 1)$ is a martingale and we can write

$$S_k(n) = \sum_{j=0}^{p-1} Z_k^j(n) + \bar{S}_k(n)$$

whence

$$ES_k^2(n) \leq 2p \sum_{j=0}^{p-1} E(Z_k^j(n))^2 + 2E\bar{S}_k^2(n).$$

Since the square of the differences of the martingale $(Z_k^j(n);\, n \geq 1)$ are uniformly integrable it follows as in Theorem 23.1 of [1], that for every $j < p$ the family $((Z_k^j(n))^2/n;\, k > p, n \geq 1)$ is uniformly integrable. It remains to prove that the family $(\bar{S}_k^2(n)/n;\, n \geq 1, k \geq 1)$ is uniformly integrable. To prove this, we shall show that $\sup_k E\bar{S}_k^3(n) = O(n^{3/2})$. By Lemma (1.2) it follows that for every $k > p$

$$| E(X_k \mid F_{k-p})| \leq \psi(p)E \mid X_k \mid \quad \text{a.s.}$$

Since the sequence $(X_k^2;\, k \geq 1)$ is uniformly integrable, it follows that there exists a constant C such that for every $k > p$

$$|E(X_k \mid F_{k-p})| < C \quad \text{a.s.}$$

From the equation

$$\bar{S}_k(2n) = \bar{S}_k(n) + \bar{S}_{k+n}(n)$$

we obtain:

$$E\mid \bar{S}_k^3(2n)| \leq E\mid \bar{S}_k^3(n)| + E\mid \bar{S}_{k+n}^3(n)| + 3E\bar{S}_k^2(n)|\bar{S}_{k+n}(n)|$$
$$+ 3E\mid \bar{S}_k(n)|\bar{S}_{k+n}^2(n).$$

By Cauchy-Schwartz's inequality we have

$$E\bar{S}_k^2(n)|\bar{S}_{k+n}(n)| \leq (E\bar{S}_k^2(n))^{1/2}(E\bar{S}_k^2(n)\bar{S}_{k+n}^2(n))^{1/2}$$

and

$$E\mid \bar{S}_k(n)|\bar{S}_{k+n}^2(n) \leq (E\bar{S}_{k+n}^2(n))^{1/2}(E\bar{S}_k^2(n)\bar{S}_{k+n}^2(n))^{1/2}.$$

For every $k > p$

$$E\bar{S}_k^2(n) \leq 2ES_k^2(n) + 2p\sum_{j=0}^{p-1} E(Z_k^j(n))^2.$$

On account of (2.1.b) and of the fact that $Z_k^j(n)$ is a martingale we have:

$$\sup_{k>p} E\bar{S}_k^2(n) = O(n).$$

We also have

$$E\bar{S}_k^2(n)\bar{S}_{k+n}^2(n) = E\bar{S}_k^2(n)(S_{k+n}(n) - \sum_{j=0}^{p-1} Z_{k+n}^j(n))^2$$
$$\leq 2E\bar{S}_k^2(n)S_{k+n}^2(n) + 2p\sum_{j=0}^{p} E\bar{S}_k^2(n)(Z_{k+n}^j(n))^2.$$

By Lemma (1.2) it follows that

$$E\bar{S}_k^2(n)S_{k+n}^2(n) \leq (1 + \psi(p))E\bar{S}_k^2(n)ES_{k+n}^2(n) = O(n^2).$$

For $j < p$ we have successively

$$E(\bar{S}_k(n)Z_{k+n}^j(n))^2 = \sum_{i=k+n+1}^{k+2n} E[\bar{S}_k(n)(E(X_i \mid F_{i-j}) - E(X_i \mid F_{i-j-1}))]^2$$
$$= \sum_{i=k+n+1}^{k+2n} [E\bar{S}_k^2(n)E^2(X_i \mid F_{i-j}) - E\bar{S}_k^2(n)E^2(X_i \mid F_{i-j-1})]$$
$$\leq E\bar{S}_k^2(n)\sum_{i=k+n+1}^{k+2n} X_i^2.$$

But by Lemma (1.2)

$$E\bar{S}_k^2(n)\sum_{i=k+n+1}^{k+2n} X_i^2 \leq (1 + \psi_p)E\bar{S}_k^2(n)E\sum_{i=k+n+1}^{k+2n} X_i^2 = O(n^2),$$

so we have

$$E\bar{S}_k^2(n)\bar{S}_{k+n}^2(n) = O(n^2) \quad \text{uniformly in } k.$$

Therefore

$$E\mid \bar{S}_k(n)|\bar{S}_{k+n}^2(n) = O(n^{3/2}) \quad \text{uniformly in } k$$

and

$$E\bar{S}_k^2(n)|\bar{S}_{k+n}(n)| = O(n^{3/2}) \quad \text{uniformly in } k.$$

From what was proved, it follows that there exists a constant K_1 such that for every $k > p$ and for every $n \geq 1$,

$$E\mid \bar{S}_k^3(2n)| \leq E\mid \bar{S}_k^3(n)| + E\mid \bar{S}_{k+n}^3(n)| + K_1 n^{3/2}.$$

If we denote by $a_n = \sup_{k>p} E \mid \bar{S}_k^3(n)\mid$ we have

$$a_{2n} \le 2a_n + K_1 n^{3/2}$$

whence it follows that for every integer r

$$a_{2^r} \le 2^r \sup_{k>p} E \mid E(X_k \mid F_{k-p})\mid^3 + K_1 \textstyle\sum_{j=1}^r 2^{j-1} 2^{3(r-j)/2}.$$

Therefore there is a constant K_2 such that for every r

$$a_{2^r} \le K_2 2^{3r/2}.$$

Now let n, $2^r \le n \le 2^{r+1}$ and write n in binary form,

$$n = \nu_0 2^r + \nu_1 2^{r-1} + \cdots + \nu_r (\nu_j = 0, \text{ or } 1).$$

We write $\bar{S}_k(n)$ in the form

$$\bar{S}_k(n) = \bar{S}_k(\nu_0 2^r) + \bar{S}_{k+\nu_0 2^r}(\nu_1 2^{r-1}) + \cdots + \bar{S}_{k+\nu_0 2^r+\cdots+\nu_{r-1}2}(\nu_r).$$

Using Minkowski's inequality

$$a_n^{1/3} \le \textstyle\sum_{j=0}^r a_{2^{r-j}}^{1/3} = O(2^{r/2}).$$

Therefore $a_n = O(n^{3/2})$. This relation implies that $(S_k^2(n)/n;\ k > p;\ n \ge 1)$ is uniformly integrable. Now if we write that

$$\frac{S_k^2(n)}{n} \le 2\frac{S_k^2(p)}{n} + 2\frac{S_{k+p}^2(n-p)}{n}$$

it follows that $(S_k^2(n)/n;\ k \ge 1,\ n \ge 1)$ is uniformly integrable.

(3.4) LEMMA. *Assume $(X_i;\ i \ge 1)$ is a ρ-mixing sequence of random variables satisfying*

a) $\sup_i \| X_i \|_2 < \infty$

and

b) $\sum_i \rho(2^i) < \infty$.

Then, there exists a constant K depending only on the $\rho(n)$'s, such that for every n

$$\sup_k E S_k^2(n) \le K(n \sup_k E X_k^2 + n^2 \sup_k \mid E X_k \mid^2).$$

PROOF. We denote by $\sigma_m = \sup_k \| S_k(m) \|_2$ and by $a = \sup_k \mid EX_k \mid$. From the equation

$$S_k(2m) = S_k(m) + S_{k+m}([m^{1/3}]) + S_{k+m+[m^{1/3}]}(m) - S_{k+2m}([m^{1/3}]),$$

we find that

$$\| S_k(2m)\|_2 \le \| S_k(m) + S_{k+[m^{1/3}]+m}(m)\|_2 + 2\sigma_1 [m^{1/3}].$$

Using the definition of ρ,

$$E(S_k(m) + S_{k+[m^{1/3}]+m}(m))^2 \le (1 + \rho([m^{1/3}]))(ES_k^2(m) + ES_{k+[m^{1/3}]+m}^2(m)) + 2 \mid ES_k(m)\mid \mid ES_{k+[m^{1/3}]+m}(m) \mid$$

and we find that:

$$\sigma_{2m} \le 2^{1/2}(1 + \rho([m^{1/3}]))^{1/2}\sigma_m + 2ma + 2[m^{1/3}]\sigma_1$$

whence for any integer r

$$\sigma_{2^r} \le 2^{r/2} \textstyle\prod_{i=0}^{r-1} (1 + \rho([2^{i/3}]))^{1/2}\sigma_1 + \sum_{i=2}^r 2^{(i-1)/2}(2^{r-i+1}a + 2^{(r-i)/3+1}\sigma_1)$$
$$\times \textstyle\prod_{k=r-i+1}^{r-1} (1 + \rho([2^{k/3}]))^{1/3} + 2^r a + 2^{(r-1)/3+1}\sigma_1.$$

Therefore

$$\sigma_{2^r} \le 18 \prod_{i=0}^{r-1} (1 + \rho([2^{i/3}]))^{1/2}(2^{r/2}\sigma_1 + 2^r a).$$

Since $\sum_i \rho(2^i) < \infty$, we have $\prod_{i=1}^{\infty} (1 + \rho([2^{i/3}])) < \infty$. Writing n in binary form, we obtain that for every n we have

$$\sigma_n^2 \le K(n\sigma_1^2 + n^2a^2)$$

where $K = 8{,}000 \prod_{i=1}^{\infty} (1 + \rho([2^{i/3}]))$.

The following lemma together with Theorem (3.1) and Theorem (3.2) will prove Theorem (2.3).

(3.5) LEMMA. *Suppose $(X_i;\ i \ge 1)$ is a centered ρ-mixing sequence satisfying*

a) *$(X_i^2;\ i \ge 1)$ is uniformly integrable*
b) $\sum_i \rho(2^i) < \infty$

Then $(S_k^2(n)/n;\ n \ge 1,\ k \ge 1)$ is uniformly integrable.

PROOF. Let N be a positive number and put

$$X_i^N = X_i I_{\{|X_i|<N\}} - EX_i I_{\{|X_i|<N\}}$$

and

$$\bar{X}_i^N = X_i I_{\{|X_i|\ge N\}} - EX_i I_{\{|X_i|\ge N\}}.$$

We denote

$$S_k^N(n) = \sum_{i=k+1}^{k+n} X_i^N \quad \text{and} \quad \bar{S}_k^N(n) = \sum_{i=k+1}^{k+n} \bar{X}_i^N.$$

Obviously $S_k(n) = S_k^N(n) + \bar{S}_k^N(n)$, and if we denote $\int_{\{U>\alpha\}} U$ by $E_\alpha U$ we have

$$E_\alpha \frac{S_k^2(n)}{n} \le 4E_{\alpha/4} \frac{(S_k^N(n))^2}{n} + 4E\frac{(\bar{S}_k^N(n))^2}{n}.$$

By Lemma (3.4) for every n we obtain

$$\sup_k \frac{E(\bar{S}_k^N(n))^2}{n} \le K \sup_k E(\bar{X}_k^N)^2$$

and by the uniform integrability of $(X_n^2;\ n \ge 1)$ we may choose and fix N such that $\sup_i E(X_i^N)^2 \le \varepsilon/8K$, $(\varepsilon > 0)$ and therefore

$$\frac{E(\bar{S}_k^N(n))^2}{n} \le \frac{\varepsilon}{8} \quad \text{for every } k \ge 1 \text{ and } n \ge 1.$$

With this fixed value N, replacing in the proof of Lemma (2.1) of [2], $a_n = ES_{(n)}^{2+\delta}$ by $a_n' = \sup_k E\, S_k^{2+\delta}(n)$, where $\delta > 0$, we obtain the existence of a positive constant K_1 such that

$$\sup_k \frac{E(S_k^N(n))^{2+\delta}}{n^{1+\delta/2}} \le K_1 \quad \text{for every } n,$$

whence, choosing α sufficiently large, $E_{\alpha/4}\dfrac{(S_k^N(n))^2}{n} \le \dfrac{\varepsilon}{8}$. Therefore

$$\left(\frac{(S_k(n))^2}{n};\ k \ge 1,\ n \ge 1\right)$$

is uniformly integrable.

In order to prove Theorem (2.5) we need the following lemmas.

(3.6) LEMMA. *Suppose* $(X_i; i \geq 1)$ *is a* ρ*-mixing sequence of centered random variables satisfying*

a) $\sup_i \| X_i \|_4 < \infty$

b) $\sum_i \rho^{1/2}(2^i) < \infty$.

Then there exists a constant K_1 *depending only on the* $\rho(n)$'s, *such that for every* $m \geq 1$,

$$\sup_k E(S_k(m))^4 \leq K_1 (m^{1/4} \sup_i \| X_i \|_4 + m^{1/2} \sup_i \| X_i \|_2).^4$$

PROOF. Let us denote by $a_m = \sup_k \| S_k(m) \|_4$ and by $\sigma_m = \sup_k \| S_k(m) \|_2$. Obviously we have

$$\| S_k(2m) \|_4 \leq \| S_k(m) + S_{k+m+[m^{1/5}]}(m) \|_4 + 2[m^{1/5}]a_1.$$

We will show that

$$\| S_k(m) + S_{k+m+[m^{1/5}]}(m) \|_4 \leq 2^{1/4}(1 + 7\rho^{1/2}([m^{1/5}]))^{1/4} a_m + 2\,\sigma_m.$$

In fact, by Cauchy-Schwartz's inequality:

$$E(S_k(m) + S_{k+m+[m^{1/5}]}(m))^4$$
$$\leq 2a_m^4 + 6E(S_k(m)S_{k+m+[m^{1/5}]}(m))^2 + 8a_m^2[E(S_k(m)S_{k+m+[m^{1/5}]}(m))^2]^{1/2}.$$

Using the definition of ρ-mixing

$$E(S_k(m)S_{k+m+[m^{1/5}]}(m))^2 \leq \sigma_m^4 + \rho([m^{1/5}])a_m^4$$

whence

$$E(S_k(m) + S_{k+m+[m^{1/5}]}(m))^4 \leq 2(1 + 7\rho^{1/2}([m^{1/5}]))a_m^4 + 8a_m^2\sigma_m^2 + 6\sigma_m^4$$
$$\leq (2^{1/4}(1 + 7\rho^{1/2}([m^{1/5}]))^{1/4}a_m + 2\sigma_m)^4.$$

Therefore

$$a_{2m} \leq 2^{1/4}(1 + 7\rho^{1/2}([m^{1/5}]))^{1/4}a_m + 2\sigma_m + 2[m^{1/5}]a_1.$$

By Lemma (3.4) there exists a constant K depending only on $\rho(n)$'s, such that $\sigma_m \leq Km^{1/2}\sigma_1$, and from the preceding inequality we obtain for every integer r

$$a_{2^r} \leq 2^{r/4} \prod_{i=0}^{r-1} (1 + 7\rho^{1/2}([2^{i/5}]))^{1/4}a_1 + 2 \sum_{i=2}^{r} 2^{(i-1)/4}$$
$$\cdot (K\sigma_1 2^{(r-i)/2} + [2^{(r-i)/5}]a_1) \prod_{k=r-i+1}^{r-1} (1 + 7\rho^{1/2}([2^{k/5}]))^{1/4} + 2K\sigma_1 2^{(r-1)/2} + 2[2^{(r-1)/5}]a_1.$$

This inequality and the condition b) imply the existence of a constant K_1 which depends only on $\rho(n)$'s such that

$$a_{2^r} \leq K_1(2^{r/4}a_1 + 2^{r/2}\sigma_1).$$

Writing n now in the binary form, we get the conclusion of the Lemma.

The following lemma is a particular case of Theorem 5 of [10], or of Corollary 3 of [9]. Here $\log m$ means $\log_2 m$.

(3.7) LEMMA. *If* $(X_i; i \geq 1)$ *is a sequence of random variables such that* $E(X_i)^4 < \infty$ *for every* i, *and if there exists a positive and nondecreasing sequence* $b(m)$ *such that for every* $k \geq 1$ *and* $m \geq 1$ *we have*

$$E(S_k(m))^4 \leq mb^4(m)$$

then for every $k \geq 1$ and $m \geq 1$ we have

$$E(\max_{1\leq i\leq m} S_k(i))^4 \leq 27\, m\left[\sum_{k=1}^{\log m} b\left(\left[\frac{m}{2^k}\right]\right)\right]^4.$$

(3.8) LEMMA. *Suppose $(X_i;\, i \geq 1)$ is a ρ-mixing sequence of random variables. Then for any integers k, n and r such that $n/r \geq 2$, and for every $a > 0$;*

$$P(\{\max_{1\leq j\leq n}|S_k(j)| \geq 3a\}) \leq \max_{0\leq i\leq n-1} 2P(\{|S_{k+i}(n-i)| \geq a\}) + \left[\frac{n}{r}\right]^{1/2}\rho(r) + \left[\frac{n}{r}\right]\max_{k\leq i\leq k+n-2r} P(\{\textstyle\sum_{j=1}^{i+2r}|X_j| \geq a\}).$$

PROOF. Although the argument below is similar to that of Remark 1 in [11] we give it here because of some differences that occur. Let $p = \left[\frac{n}{r}\right]$. If we put $E_i = \{\max_{j\leq i}|S_k(j)| < 3a \leq |S_k(i)|\}$ we obtain

$$\begin{aligned}P(\{\max_{i\leq n}|S_k(i)| \geq 3a\}) \leq{}& P(\{|S_k(n)| \geq a\})\\ &+ \textstyle\sum_{i=0}^{p-2} P((\cup_{j=1}^{r} E_{ir+j}) \cap \{|S_k(n) - S_k(i+2)r| \geq a\})\\ &+ \textstyle\sum_{i=0}^{p-2} P(\cup_{j=1}^{r} E_{ir+j} \cap \{|S_k(i+2)r - S_k(ir+j)| \geq a\})\\ &+ \textstyle\sum_{j=(p-1)r+1}^{n} P(E_j \cap \{|S_k(n) - S_k(j)| \geq 2a\}).\end{aligned}$$

Taking into account that the sequence is ρ-mixing we obtain

$$\begin{aligned}P(\{\max_{i\leq n}|S_k(i)| \geq 3a\}) \leq{}& P(\{|S_k(n)| \geq a\})\\ &+ \max_{2r\leq i\leq n-1} P(\{|S_k(n) - S_k(i)| \geq a\}) \textstyle\sum_{i=1}^{n} P(E_i)\\ &+ \rho(r) \textstyle\sum_{i=0}^{p-2} P^{1/2}(\cup_{j=1}^{r} E_{ir+j}) P^{1/2}(\{|S_k(n) - S_k(i+2)r| \geq a\})\\ &+ \textstyle\sum_{i=0}^{p-2} P(\{\max_{1\leq j\leq r}|S_k(i+2)r - S_k(ir+j)| \geq a\})\\ &+ P(\{\max_{(p-1)r+1\leq j<n}|S_k(n) - S_k(j)| \geq a\})\end{aligned}$$

and the lemma follows by applying the Cauchy Schwartz's inequality to the third term in the last sum.

(3.9) PROOF OF THEOREM (2.5). Without losing generality we assume $\sigma^2 = 1$ in (1.3). To prove this theorem, we verify the conditions of Theorem (3.1). By Lemma (3.5) the family $(S_k^2(n)/n;\, n \geq 1,\, k \geq 1)$ is uniformly integrable and thus condition a) of this theorem is satisfied. To verify the tightness condition b) let us define for every $i \geq 1$ and $n \geq 1$,

$$X_i^n = X_i I_{\left\{|X_i| < \frac{n^{1/2}}{(\log n)^2}\right\}} - EX_i I_{\left\{|X_i| < \frac{n^{1/2}}{(\log n)^2}\right\}}$$

and

$$\bar{X}_i^n = X_i I_{\left\{|X_i| \geq \frac{n^{1/2}}{(\log n)^2}\right\}} - EX_i I_{\left\{|X_i| \geq \frac{n^{1/2}}{(\log n)^2}\right\}}$$

and put

$$S_k^n(i) = \textstyle\sum_{j=k+1}^{k+i} X_j^n \quad\text{and}\quad \bar{S}_k^n(i) = \sum_{j=k+1}^{k+i} \bar{X}_j^n.$$

Obviously $S_k(i) = S_k^n(i) + \bar{S}_k^n(i)$ and

$$P(\{\max_{1\leq i\leq n}|S_k(i)| \geq 4\lambda n^{1/2}\}) \leq P(\{\max_{1\leq i\leq n}|S_k^n(i)| \geq \lambda n^{1/2}\}) + P(\{\max_{1\leq i\leq n}|\bar{S}_k^n(i)| \geq 3\lambda n^{1/2}\}).$$

By Lemma (3.6) we have

$$\sup_k (ES_k^n(m))^4 \leq K_1 (m^{1/4} \sup_k (E(X_k^n)^4)^{1/4} + m^{1/2} \sup_k \|X_k^n\|_2)^4$$

$$\leq K_1 m \sup_k \|X_k\|_2^4 \left(\frac{2^{1/2} n^{1/4}}{\log n} + m^{1/4}\right)^4$$

and from Lemma (3.7) it follows that

$$\sup_k E \max_{1 \leq i \leq n} (S_k^n(i))^4 \leq 27 K_1 n \sup_k \|X_k\|_2^4 \left(\sum_{k=1}^{\log n} \frac{2^{1/2} n^{1/4}}{\log n} + \left(\left[\frac{n}{2^k}\right]\right)^{1/4}\right)^4 = O(n^2).$$

Therefore the family $\{\max_{1 \leq i \leq n} (S_k^n(i))^{2/n};\ n \geq 1,\ k \geq 1\}$ is uniformly integrable and for n sufficiently large and $\varepsilon > 0$

$$P(\{\max_{1 \leq i \leq n} | S_k^n(i) | \geq \lambda n^{1/2}\}) \leq \frac{\varepsilon}{2\lambda^2}.$$

Now, Lemma (3.8) implies for $r = [n/(\log n)^4]$ and $a = 3\lambda n^{1/2}$ that

$$P(\{\max_{1 \leq j \leq n} | \bar{S}_k^n(i) | \geq 3\lambda n^{1/2}\}) \leq \max_{0 \leq i \leq n-1} 2P(\{ | \bar{S}_{k+i}^n(n-i) | \geq \lambda n^{1/2}\})$$

$$+ \left[\frac{n}{r}\right]^{1/2} \rho(r) + \left[\frac{n}{r}\right] \max_{k \leq i \leq k+n-2r} \frac{E(\sum_{j=i}^{i+2r} | \bar{X}_j^n |)^2}{\lambda^2 n}.$$

By Lemma (3.4) we first obtain that

$$\max_{0 \leq i \leq n-1} P(\{ | \bar{S}_{k+i}^n(n-i) | \geq \lambda n^{1/2}\}) \leq \frac{K \sup_k E(\bar{X}_k^n)^2}{\lambda^2}$$

and therefore the uniform integrability of X_i^2 implies that for large n

$$\max_{0 \leq i \leq n-1} P(\{ | \bar{S}_{k+i}^n(n-i) | \geq \lambda n^{1/2}\}) \leq \frac{\varepsilon}{12\lambda^2}.$$

Lemma (3.4) also implies that

$$E(\sum_{j=i}^{i+2r} | \bar{X}_j^n |)^2 \leq K\left(2r \sup_k \|\bar{X}_k^n\|_2^2 + 4r^2 \frac{(\log n)^4}{n} \|\bar{X}_k^n\|_2^4\right)$$

whence for n sufficiently large we have

$$\left[\frac{n}{r}\right] \max_{k \leq i \leq k+n-2r} \frac{E(\sum_{j=i}^{1+2r} | \bar{X}_j^n |)^2}{\lambda^2 n} \leq \frac{\varepsilon}{6\lambda^2}.$$

Finally, from the condition $\sum_i \rho^{1/2}(2^i) < \infty$ it follows that $(\log n)^2 \rho(n) \to 0$ as $n \to \infty$. This implies that $[n/r]^{1/2} \rho(r) \to 0$ as $n \to \infty$. Therefore $P(\{\max_{1 \leq i \leq n} | \bar{S}_k^n(i) | \geq 3\lambda n^{1/2}\}) \leq \varepsilon/2\lambda^2$ for n sufficiently large which completes the proof.

4. The variance of S_n.

(4.1) THEOREM. *Suppose $(X_i;\ i \geq 1)$ is a ρ-mixing sequence of centered real valued random variables satisfying*

a) $\sup_i \|X_i\|_2 = \sigma_1 < \infty$

b) $ES^2(n) \to \infty$

c) $\lim_{n \to \infty} \frac{ES_{kn}^2(n)}{ES^2(n)} = 1$ *uniformly in k.*

Then $ES^2(n) = nL(n)$ where $L(n)$ is a slowly varying function of the integral variable n

which has a slowly varying extension to the whole real line. If in addition we have

d) $\sum_n \rho(2^n) < \infty$

then $E\, S^2(n)/n$ *converges to a positive constant.*

To prove this theorem we need the following Lemmas:

(4.2) LEMMA. *Let* $(X_i, i \geq 1)$ *be a* ρ *– mixing sequence of centered real valued random variables satisfying* a). *Then if* m, p, q, k *and* i *are natural numbers such that* $p + q = m$ *we have,*

$$(4.3)\quad \begin{aligned}(1-\rho(i))(ES_{km}^2(p) + ES_{km+p}^2(q)) - C_1 \leq ES_{km}^2(m)\\ \leq (1+\rho(i))(ES_{km}^2(p) + ES_{km+p}^2(q)) + C_1\end{aligned}$$

where

$$C_1 = C_1(m, p, i) \leq 20\sigma_1^2 i^2 + 12\,\sigma_1 i(\|S_{km}(p)\|_2 + \|S_{km+p}(q)\|_2)$$

and

$$(4.4)\quad (1-\rho(i))^{1/2}\|S_{km}(p)\|_2 \leq \|S_{km}(m)\|_2 + C_2$$

where $C_2 \leq 2\sigma_1 i$.

PROOF. By definition of $\rho(n)$ we have

$$(4.5)\quad \begin{aligned}|E(S_{km}(p) + S_{km+p+i}(q))^2 - (ES_{km}^2(p) + ES_{km+p+i}^2(q))|\\ \leq \rho(i)(ES_{km}^2(p) + ES_{km+p+i}^2(q)).\end{aligned}$$

From the equation

$$S_{km+p+i}(q) = S_{km+p}(q) - S_{km+p}(i) + S_{(k+1)m}(i)$$

we find that

$$(4.6)\quad \|S_{km+p+i}(q)\|_2 = \|S_{km+p}(q)\|_2 + \theta_1$$

where $|\theta_1| \leq 2\sigma_1 i$. Therefore by (4.5) and (4.6) we get

$$(4.7)\quad \begin{aligned}(1-\rho(i))(ES_{km}^2(p) + ES_{km+p}^2(q) + \theta_2) \leq E(S_{km}(p) + S_{km+p+i}(q))^2\\ \leq (1+\rho(i))(ES_{km}^2(p) + ES_{km+p}^2(q) + \theta_2)\end{aligned}$$

where $|\theta_2| \leq 4\sigma_1^2 i^2 + 4\sigma_1 i\|S_{km+p}(q)\|_2$. Now if we write

$$S_{km}(m) = S_{km}(p) + S_{km+p+i}(q) + S_{km+p}(i) - S_{(k+1)m}(i),$$

we obtain

$$(4.8)\quad \|S_{km}(m)\|_2 = \|S_{km}(p) + S_{km+p+i}(q)\|_2 + \theta_3$$

where $|\theta_3| \leq 2\sigma_1 i$, whence we deduce

$$ES_{km}^2(m) = E(S_{km}(p) + S_{km+p+i}(q))^2 + \theta_4$$

where by (4.6), $|\theta_4| \leq 12\sigma_1^2 i^2 + 4\sigma_1 i(\|S_{km}(p)\|_2 + \|S_{km+p}(q)\|_2)$. Introducing this in (4.7) we obtain the relation (4.3) where

$$C_1 = \max(|(1-\rho(i))\theta_2 + \theta_4|, |(1+\rho(i))\theta_2 + \theta_4|)$$

does not exceed $20\sigma_1^2 i^2 + 12\,\sigma_1 i(\|S_{km}(p)\|_2 + \|S_{km+p}(q)\|_2)$. To prove (4.4) we use (4.5) and get

$$(1 - \rho(i))ES_{km}^2(p) \le E(S_{km}(p) + S_{km+p+i}(q))^2$$

and by (4.8) we have

$$(1 - \rho(i))^{1/2} \| S_{km}(p) \|_2 \le \| S_{km}(m) \|_2 + |\theta_3|.$$

(4.9) LEMMA. *If* $(X_i;\ i \ge 1)$ *is a* ρ*-mixing sequence of centered random variables satisfying* a), b) *and* c) *then*

$$\lim_{n\to\infty} \frac{ES^2(kn)}{ES^2(n)} = k.$$

PROOF. The relation (4.3) yields for $k = 0$, $m = kn$, $p = (k-1)n$, $q = n$ and $i = i(n) = [(ES_n^2)^{1/3}]$ the following inequalities

$$(1 - \rho(i(n)))(ES^2(k-1)n + ES_{(k-1)n}^2(n)) - \mathscr{C} \le ES^2(kn)$$
$$\le (1 + \rho(i(n)))(ES^2(k-1)n + ES_{(k-1)n}^2(n)) + \mathscr{C}$$

where $\mathscr{C} \le 20\sigma_1^2 i^2(n) + 12i(n)\sigma_1(\| S(k-1)n \|_2 + \| S_{(k-1)n}(n) \|_2)$. Dividing by $ES^2(n)$ and taking into account the conditions a), b) and c) and the definition of $i(n)$, we obtain the desired result by induction on k.

(4.10) PROOF OF THEOREM (4.1). Let us define

$$L(n) = \frac{ES^2(n)}{n}.$$

By the preceding lemma it follows that $L(n)$ is a slowly varying function of the integral variable n. To prove that this function has a slowly varying extension to the whole real line defined by $L(t) = ES^2([t])/t$, we shall establish that

$$\lim_{n\to\infty} \frac{L(n(1-\varepsilon_n))}{L(n)} = 1 \tag{4.11}$$

where $\varepsilon_n \downarrow 0$ such that $\varepsilon_n n$ is an integer. First we note that Ibragimov's Lemmas (1.5), (1.6) and (1.7) in [3] are still valid in our setting, therefore for $\varepsilon_n \downarrow 0$ and $\delta > 0$ we have

$$\lim_{n\to\infty} \frac{L(n\,\varepsilon_n)}{L(n)}\,\varepsilon_n^\delta = \lim_{n\to\infty} \frac{L(n)}{L(n\,\varepsilon_n)}\,\varepsilon_n^\delta = 0. \tag{4.12}$$

Let $k_n = \max\{k,\ kn\varepsilon_n < n - n\varepsilon_n\}$. Because

$$S(n - n\varepsilon_n) = S(n) - S_{k_n n\varepsilon_n}(\varepsilon_n n) + S_{k_n n\varepsilon_n}(p) - S_{k_n n\varepsilon_n + n\varepsilon_n}(p)$$

where $p = n - (k_n + 1)n\,\varepsilon_n \le n\varepsilon_n$, using (4.4) with $m = \varepsilon_n n$ and $k = k_n$ (and also a second time with $k = k_n + 1$) we find

$$|\| S(n - n\,\varepsilon_n) \|_2 - \| S(n) \|_2 | \le \| S_{k_n n\varepsilon_n}(\varepsilon_n n) \|_2$$
$$+ \frac{1}{(1-\rho(i))^{1/2}} (\| S_{k_n n\varepsilon_n}(\varepsilon_n n) \|_2 + \| S_{(k_n+1)n\varepsilon_n}(\varepsilon_n n) \|_2 + 4\sigma_1 i)$$

where i is such that $\rho(i) < 1$. Dividing this inequality by $\| S(n) \|_2$ and using c) and (4.12), we obtain the equality (4.11). The validity of (4.11) implies that we may now use the proof of the last part of Theorem 2 in [13] page 75, thus completing the proof of the first part of our Theorem.

To prove the second part of the Theorem, we write the inequalities (4.3) for $m = 2n$, p

$= q = n$ and $i = [n^{1/3}]$ and we obtain for every integer k and n,

$$\begin{aligned}(1 - \rho([n^{1/3}]))(ES^2_{2kn}(n) + ES^2_{(2k+1)n}(n))(1 - a(n)) &\le ES^2_{2kn}(2n) \\ &\le (1 + \rho([n^{1/3}]))(ES^2_{2kn}(n) + ES^2_{(2k+1)n}(n))(1 + a_n)\end{aligned} \tag{4.13}$$

where

$$a_n = \sup_k \frac{20\sigma_1^2[n^{2/3}] + 12\sigma_1[n^{1/3}](\| S_{2kn}(n) \|_2 + \| S_{(2k+1)n}(n) \|_2)}{(1 - \rho([n^{1/3}]))(ES^2_{2kn}(n) + ES^2_{(2k+1)}(n))}$$

and $n \ge n_0$ such that $\rho([n_0^{1/3}]) < 1$.

Using conditions a) and c) it follows that:

$$a_n = O\left(\frac{n^{2/3} + n^{1/3} \| S(n) \|_2}{\| S(n) \|^2}\right).$$

Because Lemma (18.2.2) of [4] page 327 is valid in our setting, we have for every $\varepsilon > 0$,

$$n^{\varepsilon} \frac{ES^2(n)}{n} \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty.$$

Therefore $(ES^2(n))^{-1} = O(n^{-1+\varepsilon})$ and

$$a_n = O\left(\frac{1}{n^{1/6-\varepsilon}}\right) \tag{4.14}$$

where $\varepsilon < 1/6$.

From the inequalities (4.13) we deduce for every integer $r \ge p \le n_0$ such that $\rho([2^{n_0/3}]) < 1$,

$$\begin{aligned}\textstyle\prod_{i=p}^{r-1} (1 - \rho([2^{i/3}]))(1 - a(2^i)) \sum_{i=0}^{2^{r-p}-1} ES^2_{i2^p}(2^p) &\le ES^2(2^r) \\ &\le \textstyle\prod_{i=p}^{r-1} (1 + \rho([2^{i/3}]))(1 + a(2^i)) \sum_{i=0}^{2^{r-p}-1} ES^2_{i2^p}(2^p).\end{aligned} \tag{4.15}$$

By d) $\sum_i \rho([2^{i/3}]) < \infty$ and by (4.14) $\sum_i a(2^i) < \infty$. Therefore (4.15) implies that

$$\lim_{r>p\to\infty} \frac{ES^2(2^r)}{\sum_{i=0}^{2^{r-p}-1} ES^2_{i2^p}(2^p)} = 1$$

whence by c) we deduce

$$\lim_{r>p\to\infty} \frac{L(2^r)}{L(2^p)} = 1$$

and therefore $L(2^r)$ is convergent to a positive constant. Using now the Consequence (3) of Theorem A.11 of [4, page 397] (applied to both of the functions $L(t)$ and $1/L(t)$) we deduce that $L(n)$ converges to the same limit as $L(2^r)$, which completes the proof of Theorem (4.1).

(4.16) Remark. It is easy to see, using Theorem (4.1) and the results of Section 3, that Theorems (2.3) and (2.5) remain valid when replacing $W_n(t)$ by $W_n(t) = S([nt])/\sigma_n$, $\sigma_n^2 = ES^2(n)$, and considering besides the conditions of these theorems, the conditions b) and c) of Theorem (4.1) instead of $ES_n^2/n \to \sigma^2 > 0$.

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