# SPECIAL INVITED PAPER

## PERCOLATION THEORY[1]

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An introduction is provided to the mathematical tools and problems of percolation theory. A discussion of Bernoulli percolation models shows the role of graph duality and correlation inequalities in the recent determination of the critical probability in the square, triangular, and hexagonal lattice bond models. An introduction to first passage percolation concentrates on the problems of existence of optimal routes, length of optimal routes, and conditions for convergence of first passage time and reach processes.

**1. Introduction.** Percolation processes were introduced by Broadbent and Hammersley (1957) to model the random flow of a fluid through a medium. The percolation approach contrasts with conventional diffusion theory, which treats fluid flow as a random movement in a structureless medium, by considering fluid flow to be deterministic movement in a randomly structured medium. The connectivity properties of the random medium, rather than the properties of the fluid, provide the explanation for the characteristics of the fluid flow.

Recent research interest in percolation theory in mathematical, physical science, and engineering circles may be attributed to a variety of reasons. Percolation models are appealing for their simplicity of description and ease of visualization, and yet attractive for their display of critical behavior, where a model's behavior changes abruptly as a parameter value crosses a threshold.

The percolation approach provides insight in such diverse applications as petroleum flow in sandstone, spread of a blight disease in an orchard, conductive transport in rock or alloys, traffic flow in city street networks, and critical phenomena in statistical mechanics.

Percolation processes have yielded few results to standard probabilistic techniques due to their complex stochastic dependence. New tools such as subadditive processes and correlation inequalities were developed in percolation research. Mathematical interest in percolation research was stimulated by an intriguing non-rigorous approach of Sykes and Essam (1964) in Bernoulli percolation and by the conjectures of Hammersley and Welsh (1965) in first-passage percolation. Recent results resolve several long-standing open questions. The principal aim of this paper is to summarize the current state of knowledge in percolation theory.

The structure of the medium in a percolation model is represented by an infinite regular crystal lattice graph $\mathscr{G}$, consisting of a set $V(\mathscr{G})$ of sites (vertices) and a set $E(\mathscr{G})$ of bonds (edges) which connect pairs of sites in $V(\mathscr{G})$. There is no standard definition of "regular crystal lattice", which generally indicates a structure that is homogeneous in the large but may have local variations. In such a lattice, the vertices and edges of $\mathscr{G}$ are imbedded in $\mathbb{R}^d$ and there exist $d$ linearly independent vectors $U_1, \cdots, U_d \in \mathbb{R}^d$ such that the set of vertices and edges of $\mathscr{G}$ is invariant when translated by any $U_i$. A wide range of behaviors,

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which allows percolation models to be adapted to such diverse applications, is obtained by virtue of the choice of lattice graph as medium and the manner in which randomness is introduced in the model.

Percolation models are classified as Bernoulli percolation models or first-passage percolation models according to the type of random mechanism employed. Bernoulli percolation models are classified further as bond or site models. In a Bernoulli bond percolation model, each bond $e$ is assigned a Bernoulli random variable $X_e$. $\{X_e : e \in E(\mathscr{G})\}$ is an independent family of random variables on a probability space $(\Omega, \mathscr{F}, P)$ with common distribution $P(X_e = 0) = p = 1 - P(X_e = 1)$. For a configuration $\omega \in \Omega$, fluid may flow along any bond $e$ with $X_e(\omega) = 0$, but may not pass through a bond $e$ with $X_e(\omega) = 1$. Standard terminology refers to bonds assigned 0 as *open*, and bonds assigned 1 as *closed* (or *blocked*). The Bernoulli site percolation model is described similarly, with sites declared open or closed independently with probabilities $p$ and $1 - p$ respectively.

The class of site models is more general than the class of bond models. A bond-to-site transformation (Fisher, 1961) converts any bond percolation model to an equivalent site percolation model on a different lattice. However, site percolation models exist which may not be converted into an equivalent bond model. The major problem of Bernoulli percolation involves determining the probability that fluid may flow infinitely far from a single fluid source site.

The first-passage percolation model on a lattice $\mathscr{G}$ is defined by an independent family $\{X_e, e \in E(\mathscr{G})\}$ of identically distributed random variables with common distribution function $U$. (Since $U$ may be a Bernoulli distribution, the class of first-passage models includes the class of Bernoulli bond models.) The random variable $X_e$ is usually assumed to be nonnegative with finite mean, interpreted as the time required for fluid to travel the length of bond $e$. Although less intuitive, allowing $X_e$ to assume negative values or to have infinite mean has produced useful results. The questions of interest in first-passage models relate to the rate of spread of fluid, existence of optimal routes between sites, and the asymptotic shape of the fluid-filled region from a single source site.

The remainder of the paper provides separate discussions of Bernoulli and first-passage models, each concentrating on recent solutions to major conjectures. For additional discussion of previous results, applications, and numerical studies, the reader is referred to surveys by Essam (1972) and Shante and Kirkpatrick (1971), and the monograph by Smythe and Wierman (1978a).

**2. Bernoulli Percolation.** In a Bernoulli percolation model, imagine fluid flowing from a single fluid source site. The fluid spreads only locally if $p$ is near zero, but may spread throughout the medium if $p$ is near one. One suspects that there exists a threshold value of $p$, a critical probability, which separates intervals of $p$ corresponding to local spread and extensive spread throughout the lattice. Different definitions of critical probability exist, arising from various concepts of penetration of the medium.

A *path* from $x$ to $y$, $x, y \in V(\mathscr{G})$, is an alternating sequence of distinct sites and bonds, of the form $v_0, e_1, v_1, e_2, v_2, \cdots, e_n, v_n$ where $v_0 = x$ and $v_n = y$ and $e_{i+1}$ joins $v_i$ to $v_{i+1}$. A *circuit* is a path $v_0, e_1, v_1, \cdots, e_n, v_n$ together with an additional bond $e_{n+1}$ which joins $v_n$ to $v_0$. A path or circuit is *open* (*closed*) if all of its bonds or sites are open (closed). For any sets $A$ and $B$ of sites, $\{A \rightarrow B\}$ denotes the event that there exists an open path connecting some pair of sites $a$ and $b$, where $a \in A$, $b \in B$. The number of bonds in $A$ is denoted by $|A|$. The *open cluster* $C_x$ containing $x$ is the set of sites and bonds connected to $x$ by open paths. The distance $d(x, y)$ between sites $x$ and $y$ is the number of bonds in the shortest path connecting $x$ and $y$. The radius of cluster $C_x$ is $\rho(C_x) = \max\{d(x, y) : y \in C_x\}$.

Let $\mathscr{P}_n(p, x) = P_p(|C_x| \geq n)$, where the subscript $p$ refers to the parameter of the Bernoulli model. The *percolation probability* $\mathscr{P}(p, x) = \lim_{n \rightarrow \infty} \mathscr{P}_n(p, x)$ represents the probability that $x$ is contained in an infinite open cluster. There are few theoretical results concerning the percolation probability, although it is known that in the square lattice bond problem it is continuous in $p$.

Monte Carlo simulations (Frish, Hammersley and Welsh, 1962) suggest that the

percolation probability, as a function of $p$, has a characteristic shape independent of the lattice or of choice of site or bond model.

Broadbent and Hammersley (1957) proved the existence of a *cluster size critical probability* $p_H = \inf\{p : \mathscr{P}(p, x) > 0\}$ which is independent of the initial site $x$. The *mean cluster size critical probability* is defined by $p_T = \inf\{p : E_p | C_x | = \infty\}$.

A critical probability was introduced in the square lattice bond model by Seymour and Welsh (1978) for technical reasons. The discussion here will deal with the square lattice, but may be adapted to the triangular and hexagonal lattices and possibly others. Let $T(m, n)$ denote the $m \times n$ *sponge*, which is the portion of the square lattice lying in $\{(x, y) : 1 \leq x \leq n, 1 \leq y \leq m\}$. Let $\{\to T(m, n)\}$ denote the event that an open path lying entirely in $T(m, n)$ connects a site on $x = 1$ to a site on $x = n$. The *sponge crossing probability* is $S_p(n, m) = P(\to T(n, m))$. The *sponge crossing critical probability* is defined by $p_S = \inf\{p : \limsup_{n \to \infty} S_p(n, n + 1) > 0\}$.

A fourth concept of critical probability was proposed in the controversial paper by Sykes and Essam (1964). Consider a finite region $R$ of the lattice. Let $\theta_R(p)$ denote the mean number of open clusters per bond in $R$. Grimmett (1976) showed that as $R$ expands uniformly in all directions, a limit $\theta(p) = \lim \theta_R(p)$ exists and is continuous everywhere. Sykes and Essam assumed $\theta(p)$ to have a unique singularity, which is the critical probability $p_E$.

The first problem of Bernoulli percolation theory is the determination of the values of the critical probabilities for various lattices. A program proposed by Sykes and Essam (1964) provided conjectured values of $P_E = \frac{1}{2}$ for the square lattice bond problem, $p_E = 2 \sin \pi/18$ for the triangular lattice bond problem, $p_E = 1 - 2 \sin \pi/18$ for the hexagonal lattice bond problem, and $p_E = \frac{1}{2}$ for the triangular lattice site problem. These values became the conjectured values for $p_H$, $p_T$, and $p_S$ also. The Sykes and Essam result stimulated research which recently led to rigorous verification of their critical probability values (Kesten, 1980b; Wierman, 1980), and which may be expected to solve the problem for additional lattices in the near future.

Percolative behavior is most clearly understood for the square lattice bond percolation model. Three critical probabilities share the common value $p_H = p_T = p_S = \frac{1}{2}$. If the singularity of $\theta(p)$ in the definition of $p_E$ is interpreted as a point where $d^n\theta/dp^n = \infty$ for some $n$, then $p_E = \frac{1}{2}$ if a singularity exists at all. At $p = \frac{1}{2}$, $\theta(p)$ has a continuous second derivative (Kesten, 1980). It is unknown if the third derivative of $\theta(p)$ exists at $p = \frac{1}{2}$. $\theta(p)$ is analytic elsewhere.

If $p < \frac{1}{2}$, the tail of the open cluster radius distribution declines exponentially (Kesten, 1980b): There exists a constant $C_1(p) > 0$ such that for all $n$,

$$(2.1) \qquad\qquad P_p(\rho(C_{(0,0)}) \geq n) \leq 2e^{-C_1(p)n}.$$

At $p = \frac{1}{2}$, the common values of the critical probabilities, the model's behavior is considerably different. Although $E_{1/2} | C_{(0,0)} | = \infty$, the percolation probability remains at zero, so the probability that an infinite open cluster exists is zero. ($\mathscr{P}(\frac{1}{2}) = 0$ is due to Harris (1960).) The sponge crossing probability limit is discontinuous at $p = \frac{1}{2}$, since $\lim_{n \to \infty} S_{1/2}(n, n + 1) = \frac{1}{2}$, while obviously from (2.1), $S_p(n, n + 1) \to 0$ as $n \to \infty$ for $p < \frac{1}{2}$. The open cluster radius distribution satisfies, for all $n$

$$(2.2) \qquad\qquad P_{1/2}(\rho(C_{(0,0)}) \geq n) \geq \frac{1}{2n}.$$

By symmetry when $p = \frac{1}{2}$, the results above apply to closed clusters as well.

If $p > \frac{1}{2}$, there exists a unique open cluster with probability one. (Uniqueness is due to Harris, 1960). The sponge crossing probabilities $S_p(n, n + 1)$ converge to one. The distribution of distance from the origin to the infinite cluster is bounded by

$(2.3)$   $P_p$(the infinite open cluster contains no vertices within distance $n$ of the origin)

$$\leq 2\{1 - e^{-C_1(q)}\}^{-1}e^{C_1(q)n},$$

with $q = 1 - p$ and $C_1$ as in the $p < \frac{1}{2}$ case.

Similar behavior occurs in the triangular and hexagonal lattice bond models. Wierman (1980) proved that $p_H = p_T = p_S$ for each lattice, for an appropriate definition of $p_S$.

Exact values of the critical probability are not known for other lattices which contain circuits. Values are·known for Bethe trees (see Essam, 1972), which can be obtained by a simple branching process argument.

Lower bounds are obtained by a path-counting method, which applies to lattices of any dimension. Let $f_n$ denote the number of self-avoiding paths of length $n$ starting at a specified site. $f_n$ typically increases exponentially, with rate $\lambda = \lim \sup_{n \to \infty} (f_n)^{1/n}$ called the *connectivity constant* of the lattice. Then, $P(\rho(C_x) \geq n) \leq \sum_{k=n}^{\infty} f_n p^n$, which converges to zero if $p < 1/\lambda$. The lower bound provided is $p_H \geq 1/\lambda$. Path-counting bounds are relatively crude, because interactions between paths are ignored. The square lattice connectivity constant is approximately 2.6395, giving the lower bound $p_H \geq .397$ for comparison with $p_H = \frac{1}{2}$. Although the method is crude, its usefulness is its generality, and it currently provides the best values for three-dimensional lattices. For example, for the simple cubic lattice $\lambda = 4.6826$ providing the lower bound .214, which may be compared with the Monte Carlo estimate .254.

If the connectivity constant is unknown, the bound may be expressed in terms of the *coordination number* for some lattices. The coordination number $z$ is the number of bonds leaving a single site in a lattice where this number is identical for all sites. The number of $n$-step paths with no immediate reversals is $z(z - 1)^{n-1}$, so $1/(z - 1) \leq 1/\lambda \leq p_H$.

An approximate relationship appearing in the physics literature and supported by Monte Carlo simulations is $p_H \approx (d/(d - 1)) \, 1/z$, where $d$ is the dimension of the lattice. Although no exact critical probabilties are known for higher dimensions, the values for the three known planar lattices agree well with this estimate. For a summary of bounds and Monte Carlo estimates of critical probabilities of various lattices, consult Shante and Kirkpatrick (1971).

A crucial tool in the program for determining critical probabilities is Whitney duality of planar graphs. For any planar graph $L$, there exists a dual graph $L^*$ constructed as follows: Place a site of $L^*$ in each face of $L$. Connect two sites of $L^*$ with a bond of $L^*$ if the sites lie in the faces of $L$ which share a common edge. Each bond of $L$ is crossed by exactly one bond of $L^*$, providing a one-to-one correspondence. Thus, the dual lattice of $L^*$ is again $L$, so planar graphs occur in dual pairs. Note that the square lattice is self-dual, and that the triangular and hexagonal lattices form a dual pair.

A bond percolation model on $L$ induces a bond percolation model on $L^*$. Assign each bond of $L^*$ the character (open or closed) of the corresponding bond of $L$. The induced model on $L^*$ has the same parameter $p$ as the original model on $L$.

The usefulness of duality in bond percolation stems from the fact that a closed path in $L^*$ may not be crossed by an open path in $L$, and thus serves as a barrier. The existence of a closed circuit in $L^*$ surrounding the origin then implies that $C_{(0,0)}$ is finite, and conversely.

Duality is particularly helpful when applied to sponge crossing problems. Let $T^*(m, n)$ denote the portion of the dual square lattice contained in $\{(x, y) : \frac{3}{2} \leq x \leq n - \frac{1}{2}, \frac{1}{2} \leq y \leq m + \frac{1}{2}\}$. Although $T^*(m, n)$ is not precisely the dual graph of $T(m, n)$, duality theory can be applied to show that either an open path crosses $T(m, n)$ from right to left or a closed path crosses $T^*(m, n)$ from top to bottom. Noticing that $T^*(m, n)$ is a copy of $T(n - 1, m + 1)$ yields

(2.4) $$P_p(\to T(m, n)) + P_{1-p}(\to T(n - 1, m + 1)) = 1.$$

In particular,

(2.5) $$P_{1/2}(\to T(n, n + 1)) = \frac{1}{2},$$

showing that $p_S \leq \frac{1}{2}$ for the square lattice bond problem.

Another useful tool is an intuitive correlation inequality introduced by Harris (1960). Although subsequently generalized by Fortuin, Kasteleyn, and Ginibre (1971), the original version is sufficient for the present discussion.

Let $G$ be a finite graph, and consider the bond percolation model on $G$. Let $A$ and $B$ be collections of paths in $G$. Let $\{\to A\}$ denote the event that one of the paths in $A$ is open. Then

$$(2.6) \qquad P(\to A, \to B) \geq P(\to A)P(\to B).$$

Harris (1960) employed both the correlation inequality and the self-duality of the square lattice to prove that $p_H \geq \frac{1}{2}$ for the square lattice bond problem.

The insightful method for determining critical probabilities proposed by Sykes and Essam (1964) has been a focus of much recent mathematical percolation research. The method is described for site models, since a bond percolation model may be transformed by the bond-to-site transformation to an equivalent site model on the *covering lattice* of the original lattice.

The *covering lattice* is constructed by placing a site of the covering lattice at the midpoint of each bond of the original lattice, then connecting each pair of sites of the covering lattice for which the corresponding bonds share an endpoint by a bond of the covering lattice.

A planar graph $G$ is "decorated" by drawing in all possible diagonal bonds in some collection of its faces. The graph $M$ constructed in this manner is typically not planar. The matching graph $M^*$ of $M$ is formed by decorating $G$ by inserting all diagonals in all faces which were not decorated to form $M$. Thus the matching graph of $M^*$ is $M$. The covering lattices of a dual lattice pair form a matching lattice pair.

Sykes and Essam (1964) used Euler's Law to show that the mean number of open clusters per site on a finite graph and the mean number of closed clusters per site on the matching graph differ by a polynomial function of $p$. By neglecting edge effects in expanding finite regions to the infinite lattice, this property is found to hold for the lattices themselves. Let $\theta$ and $\theta^*$ denote the mean number of open clusters per site in the site percolation models on $M$ and $M^*$ respectively. Then for some polynomial $f(p)$,

$$(2.7) \qquad \theta(p) + f(p) = \theta^*(1-p).$$

Assume that each of $\theta$ and $\theta^*$ have a unique singularity. If so, since $f$ is a polynomial, the singularities must occur only at values $p_E(M)$ and $p_E(M^*)$ satisfying $p_E(M) + p_E(M^*) = 1$. As a consequence, the self-duality of the square lattice implies $p_E = \frac{1}{2}$ for the square lattice bond model (since the covering lattice is self-matching). For the bond models on the triangular and hexagonal lattices, the star-triangle transformation is employed to determine that $p_E$ for the triangular lattice is a root of $1 - 3p + p^3 = 0$, i.e., $p_E = 2 \sin \pi/18$. The star-triangle relation contributes to the rigorous solution also.

The program that led to the rigorous determination of critical probabilities for the square, triangular, and hexagonal lattices will now be summarized in the remainder of this section. Half the program follows the method of Seymour and Welsh (1978) and Russo (1978), who proved that $p_H + p_T = 1$ for the square lattice bond and site percolation models. The apparent general relationship for bond percolation models is $p_T(L) + p_H(L^*) = 1$, where $L$ and $L^*$ are a dual pair of lattices, which has been verified for the triangular and hexagonal lattices (Wierman, 1980).

The definitions of critical probabilities imply that $p_T(L) < p_H(L)$. Evidently, $p_T(L) \leq p_S(L)$ holds in general also. In the square lattice bond model, suppose $p \leq p_T(L)$ and consider the $n \times (n+1)$ sponge. If $\{\to T(n, n+1)\}$ occurs, one of the sites $(1, i)$, $1 \leq i \leq n$, is in an open cluster of at least $n$ bonds. Thus

$$S_p(n, n+1) \leq nP(|C_{(0,0)}| \geq n) \leq E(|C_{(0,0)}|; |C_{(0,0)}| \geq n) \to 0 \quad \text{as } n \to \infty,$$

which implies $p \leq p_S(L)$.

Similar reasoning provides the easier half of the Seymour and Welsh result, that

$$(2.8) \qquad\qquad p_T(L) + p_H(L^*) \leq 1.$$

Consider the sponge $T(2n, n)$. An open crossing from left to right occurs only if one of $2n$ sites is in an open cluster of at least $n$ bonds, so $S_p(2n, n) \leq 2nP(|C_{(0,0)}| \geq n)$. Consider copies of the sponges $T(2^{k+1}, 2^k)$ placed with alternating orientation so that for each $k$ the length of the $k$th sponge crosses the width of the $(k + 1)$st sponge. If $p < p_T(L)$, $E|C_{(0,0)}| < \infty$, which implies that

$$2 \sum_{k=1}^{\infty} 2^k P(|C| \geq 2^k) < \infty.$$

Thus by the Borel-Cantelli Lemma, with probability one, only finitely many of these sponges are crossed by open paths. By duality, with probability one, closed paths cross the dual sponges $T^*(2^{k+1}, 2^k)$ for all $k$ sufficiently large. Such closed paths link up to form an infinite closed cluster in the dual lattice. Hence $p < p_T(L)$ implies $1 - p > p_H(L^*)$. Since $1 > p + p_H(L^*)$ for all $p < p_T(L)$, conclude $1 \geq p_T(L) + p_H(L^*)$.

The more difficult half of the Seymour-Welsh method, that $p_T(L) + p_H(L^*) \geq 1$ involves the construction of open circuits surrounding the origin from open crossings of sponges. The key step relates the probability of crossing a sponge to the probability of crossing a sponge twice as long:

LEMMA 2.1. *If $S_p(2n, 2n) = \tau$, then $S_p(2n, 4n) \geq \tau[1 - \sqrt{1 - \tau}]^8$.*

The proof (see Seymour and Welsh, 1978) depends heavily on the use of the correlation inequalities and considerable symmetry in the sponge. The extent of symmetry required is unknown.

Once the sponge length may be doubled, it is a simple matter to triple the length with control of the sponge crossing probability. Considering $T(2n, 6n)$, open paths cross the leftmost and rightmost copies of $T(2n, 4n)$ horizontally each with probability at least $\tau[1 - \sqrt{1 - \tau}]^8$, while the central copy of $T(2n, 2n)$ is crossed vertically by an open path with probability at least $\tau$, linking the two paths. By inclusion and the correlation inequality (2.6), conclude

$$(2.9) \qquad\qquad P(\rightarrow T(2n, 6n)) \geq \tau^3[1 - \sqrt{1 - \tau}]^{16}.$$

Similarly, four copies of $T(2n, 6n)$ may be placed inside a $6n \times 6n$ square to form an annular region, which contains an open circuit around the center with probability at least

$$(2.10) \qquad\qquad \tau^{12}[1 - \sqrt{1 - \tau}]^{64}.$$

Harris' (1960) pioneering result that $p_H \geq \frac{1}{2}$ for the square lattice bond model is now a simple consequence of (2.10). Consider a nested sequence of annuli in the dual lattice centered at $(\frac{1}{2}, \frac{1}{2})$ with side length increasing by a factor of three (to obtain disjointness). Since $S_{1/2}(n, n) \geq \frac{1}{2}$ for all $n$, for $p = \frac{1}{2}$ apply the Borel-Cantelli Lemma to closed circuits with the lower bound (2.10). Thus, almost surely, there exists a closed circuit in the dual lattice surrounding the origin. The origin is in a finite open cluster almost surely, so $p_H \geq \frac{1}{2}$.

A modification of this argument applies when $\limsup S_p(n, n) > 0$. Consider a sequence of annuli with sizes determined by the subsequence $n_i$ for which $S_p(n_i, n_i) > \delta$ for some $\delta > 0$. Then $p > p_S(L)$ implies that a closed cluster in the dual lattice is finite a.s. so $1 - p \leq p_H(L^*)$. Conclude that

$$(2.11) \qquad\qquad 1 \leq p_S(L) + p_H(L^*).$$

The remaining step is to show that $p_T(L) \geq p_S(L)$.

LEMMA 2.2 *If $\varepsilon > 0$ and $p > p_T$, then for infinitely many $k$,*

$$(2.12) \qquad\qquad [1 - S_p(2k, 2k)]^{12}[1 - \sqrt{S_p(2k, 2k)}]^{64} \leq \frac{8}{9} + \varepsilon.$$

PROOF. Suppose not. There exists $N$ such that for all $k \geq 3^N$ the inequality is reversed. By monotonicity and duality, with $q = 1 - p$,

$$S_q(2k, 2k) \geq S_q(2k - 1, 2k + 1) = 1 - S_p(2k, 2k).$$

Thus, for $i \geq N$, the probability that there exists a closed circuit in the annulus of outer edge $3^{i+1}$ is at least $\frac{8}{9} + \varepsilon$. Letting $D_i$ be the event that there exists an open path connecting the inner and outer boundary of the annulus of outer edge $3^{i+1}$, $P(D_i) \leq \frac{1}{9} - \varepsilon$. Counting the number of bonds inside each annulus,

$$E \, | \, C_{(0,0)} | \leq (4 \times 3^{2N}) + \sum_{n \geq N} (4 \times 3^{2n}) \prod_{i=N}^{n-1} P(D_i)$$

$$\leq (4 \times 3^{2N}) + \sum_{n \geq N} (4 \times 3^{2n}) \left( \frac{1}{9} - \varepsilon \right)^{n-N} < \infty,$$

contradicting $p > p_T$.

The desired interpretation of Lemma 2.2 is that $p > p_T$ implies $\lim \sup_{n \to \infty} S_p(2n, 2n) \geq \delta$ for some $\delta > 0$, so $p \geq p_S$. Therefore $p_T \geq p_S$. To complete the Seymour and Welsh argument,

$$p_T(L) + p_H(L^*) \leq 1 \leq p_S(L) + p_H(L^*) \leq p_T(L) + p_H(L^*),$$

by (2.8) and (2.11), so equality holds throughout.

Modification of the method is required for application to other lattices. Hexagonal sponges may be used for the triangular and hexagonal lattice dual pair. One type of sponge crossing was insufficient; it was necessary to consider crossings between a pair of opposite sides, but also between a pair of opposite pairs of sides. A crucial fact is that the two types of sponge crossing probabilities provide a common sponge critical probability $p_S(L)$.

Note that the method applies to site percolation models as well, in which case the matching lattice plays the role of the dual lattice. Russo (1978) independently considered the square lattice site problem. Kesten adapted the method to the triangular lattice site problem.

Kesten (1980b) recently contributed the second half of the program for determining critical probabilities. The program is completed by showing that if $p < 1 - p_S(L^*) = p_H(L)$, then $\lim_{n \to \infty} S_p(n, n) = 0$, so $p_S(L) \geq p_H(L)$. This is accomplished by producing a mechanism by which closed crossings in the dual sponge eliminate open crossings of the sponge with increasing probability as the sponge size increases, when $1 - p \geq p_S(L^*)$.

Two additional tools are employed in Kesten's argument. The first is a simple observation. If there exists an open self-avoiding path across a sponge horizontally, then there exists a "lowest" open self-avoiding path across the sponge. (Lowest in the sense that the path divides the sponge into two regions, with it being the only open crossing in the closure of the lower region.)

The second tool is a step-wise method for closing bonds. For a bond $e$, let $X(e) = \prod_{i=0}^{N} X_i(e)$, where

$$X_i(e) = \begin{cases} 1 & \text{with probability} \quad p_i \\ 0 & \text{with probability} \quad 1 - p_i, \, 0 < p_i < 1. \end{cases}$$

The bond $e$ is $\ell$-open if $X_0(e) = \cdots = X_\ell(e) = 1$. A bond is *open* if it is $N$-open, so the resulting percolation model is a bond percolation model with $p = \prod_{i=0}^{N} p_i$. (Note that the previous interpretation of $X_i = 0$ or $1$ is now reversed.) A bond open after $\ell$ stages may become closed at a later stage, but a closed bond remains closed in later stages.

The method will be described in the square lattice bond model. Let $p_0 = \frac{1}{2}$, so $p = \prod_{i=0}^{N} p_i < \frac{1}{2}$. Consider the sponge $T = T(2^{k+1}, 2^{k+1})$.

Fix $\ell$, and let $R_\ell$ denote the lowest $\ell$-open path crossing $T$ from left to right, if such a path exists. Note that for a self-avoiding path $r$, the event $\{R_\ell = r\}$ is independent of bonds in the region strictly above $r$.

Suppose $R_\ell = r$, where $r$ lies entirely within the lower half of $T$. Consider the event that

the left half of $T^*$ contains a closed path connecting the top of $T$ to a site $v^*$ lying $\frac{1}{2}$ unit distant from $r$. This event is independent of $\{R_\ell = r\}$, with probability at least $(\frac{1}{2})[1 - \sqrt{\frac{1}{2}}]^8 = \alpha$ which is the probability of a 0-closed vertical crossing of the left half of $T^*$. If such a closed path exists, there is a leftmost such closed path $r^*$ with its "foot" at a site $v^*$ one-half unit from $r$.

Consider an increasing set of disjoint annuli centered at $v^*$. Each annulus has probability at least $(\frac{1}{2})^{12}[1 - \sqrt{\frac{1}{2}}]^{64}$ of containing a 0-closed path from $r^*$ to a "foot" one-half unit from $r$, which lies in the portion above $r$ and right of $r^*$.

For any $\tau$, for $k$ sufficiently large, $T^*(2^{k+1}, 2^{k+1})$ contains $\tau$ such closed paths in annuli with probability at least one-half. From the foot of each closed path, a bond crosses a bond of $r$. With probability $1 - p_{\ell+1}^\tau$, one of these bonds is $(\ell + 1)$ closed. Thus, an $\ell + 1$ closed path extends from the upper boundary across $r$. Since $R_\ell = r$ is the lowest $\ell$-open crossing, $R_{\ell+1}$ does not exist. Hence,

$$(2.13) \qquad P(R_{\ell+1} \text{ exists} \mid R_\ell \text{ exists}) \leq 1 - \alpha\left(\frac{1}{2}\right)(1 - p_{\ell+1}^\tau).$$

By choice of $\tau$, depending on $\ell$, the right side of (2.13) may be made less than a constant $\gamma < 1$ independently of $\ell$, $N$, and $\{p_\ell\}$. Thus,

$$P(R_N \text{ exists}) \leq \gamma^N \quad \text{for} \quad k \quad \text{sufficiently large.}$$

$N$ may be chosen arbitrarily large for a given $p < \frac{1}{2}$ by choosing $p_\ell$ sufficiently near one. Conclude

$$S_p(2^k, 2^{k+1}) \to 0 \quad \text{as} \quad k \to \infty$$

which implies $S_p(2^k, 2^k) \to 0$ by Lemma 2.1.

The choice of $p_0 = \frac{1}{2}$ is due to the self-duality of the square lattice which produces the lower bound (2.5) for sponge crossing probabilities. For the triangular and hexagonal lattices, the choice of $p_0$ depends on sponge crossing probabilities in a more complicated manner. For each lattice $L$, choose $p_0$ so open sponge crossing probabilities are uniformly bounded below, so $p_S(L) \leq p_0$, but also closed sponge crossing probabilities in the dual lattice are uniformly bounded below, so by Kesten's argument $p_S(L) \geq p_0$. By the Seymour and Welsh result, conclude $p_H(L) = p_0$ and $p_H(L^*) = 1 - p_0$.

The star-triangle relationship, used by Sykes and Essam (1978), provides a way to find the desired value for $p_0$. Consider a face in the triangular lattice. A portion of the hexagonal lattice is superimposed by placing a site in the center of the triangular face, with bonds connecting it to each of the three triangular lattice sites. Let the triangular lattice bonds be open with probability $p$, and the hexagonal lattice bonds be open with probability $1 - p$. If $p$ is a root of $1 - 3p + p^3 = 0$, the connectivity of the triangular lattice sites is the same in both lattices. This relationship extends to sponges in the lattices, so an open sponge crossing in one lattice occurs with the same probability as a closed crossing of the corresponding sponge in the dual lattice. Applied with the basic duality relationship for sponges, this fact implies that sponge crossing probabilities are uniformly bounded below for both lattices when $p$ satisfies $1 - 3p + p^3 = 0$. The resulting solution for the critical probability is the root $2 \sin \pi/18$ for the triangular lattice, and $1 - 2 \sin \pi/18$ for the hexagonal lattice.

The determination of critical probabilities for each additional dual lattice pair may require a different technique to find the correct $p_0$ for bounded sponge crossing probabilities.

To restore the connection with the Sykes and Essam definition of critical probability $p_E$, let $a(n, \ell)$ denote the number of open clusters of $n$ bonds containing the origin which have a boundary of $\ell$-closed bonds. Then

$$P_p(|C_{(0,0)}| = n) = \sum_\ell a(n, \ell) p^n (1 - p)^\ell$$

and the number of clusters per bond equals

$$(2.14) \qquad \theta(p) = E_p \left[ \frac{1}{|C_{(0,0)}|} ; |C_{(0,0)}| \geq 1 \right] = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\ell} a(n, \ell) p^n (1 - p)^{\ell}$$

(see Grimmett, 1976, and Wierman, 1978). With this representation, Kesten (1980b) showed that $\theta(p)$ is analytic in [0,1] except at $p = \frac{1}{2}$, where $\theta(p)$ has two continuous derivatives.

**3. First-passage percolation.** Let $(\Omega, \mathscr{F}, P)$ be a fixed probability space. Let $L$ denote the square lattice, with $\{e_i\}_{i=1}^{\infty}$ an enumeration of the bonds of $L$. Let $\{X_i\}_{i=1}^{\infty}$ be an independent identically distributed sequence of random variables on $(\Omega, \mathscr{F}, P)$ with common distribution function $U$. The random variable $X_i$ represents the *travel time* of the bond $e_i$, and the sequence $\{X_i(\omega)\}_{i=1}^{\infty}$ defines a configuration of travel times on $L$ called the time state under $\omega$.

The travel time under $\omega$ of a path $r$ containing the bonds $e_{i_1}, \cdots, e_{i_n}$ is defined as

$$t(r, \omega) = X_{i_1}(\omega) + \cdots + X_{i_n}(\omega).$$

If $R$ is a nonempty set of paths on $L$, the *first-passage time* of $R$ under $\omega$ is

$$t_R(\omega) = \inf\{t(r, \omega) : r \in R\}.$$

If there exists $r_0 \in R$ such that $t(r_0, \omega) = t_R(\omega)$, $r_0$ is a *route* for $t_R(\omega)$. Note that routes need not be unique.

The processes of primary interest are the "unrestricted" first-passage processes. For $m < n$, let $R_{mn}$ be the set of self-avoiding paths from the site $(m, 0)$ to the site $(n, 0)$. Define the unrestricted point-to-point first-passage time from $(m, 0)$ to $(n, 0)$ by

$$a_{mn}(\omega) = t_{R_{mn}}(\omega).$$

Let $\tilde{R}_{mn}$ be the set of self-avoiding paths from the point $(m, 0)$ to some point on the line $x = n$, which are contained entirely, except for the final endpoint, in $\{(x, y) : x < n\}$. The unrestricted point-to-line first-passage time is

$$b_{mn}(\omega) = t_{\tilde{R}_{mn}}(\omega).$$

The restriction to self-avoiding paths, and to $\{(x, y) : x < n\}$ in the case of $b_{mn}$, are unnecessary if the travel times are nonnegative random variables, but crucial if negative travel times are allowed.

In the pioneering work by Hammersley and Welsh (1965), little information was obtained regarding $a_{mn}$, and $b_{mn}$ presented a complete mystery. A restriction of paths to remain inside the region $m < x < n$, except for endpoints, produced the corresponding "cylinder-restricted" point-to-point process $t_{mn}$ and point-to-line process $s_{mn}$. Independence introduced by this restriction simplified analysis of the asymptotic behavior of the processes. Other path restrictions were employed for technical reasons by Smythe (1976), Wierman and Reh (1978), Reh (1979), and Wierman (1980).

A convenient fact follows from the Bernoulli percolation results concerning the existence of open circuits.

**THEOREM 3.1.** *Routes exist for $a_{mn}$ and $b_{mn}$ for all $m$, $n$ almost surely.*

**PROOF.** Two cases are distinguished by the size of the atom at zero of the travel time distribution.

Suppose $U(0) \geq \frac{1}{2}$. Convert the first-passage model to a Bernoulli percolation model by declaring a bond with zero travel time to be open, and a bond with positive travel time to be closed. For fixed $m$ and $n$, take a square subgraph containing both $(m, 0)$ and $(n, 0)$. Since $U(0) \geq \frac{1}{2}$, with probability one there exists an open circuit surrounding the square. The set of paths from $R_{mn}$ or $\tilde{R}_{mn}$ contained inside the open circuit is finite. Any path which leaves the region enclosed by the circuit may have its travel time reduced by

substituting a portion of the circuit, which has zero travel time, for a portion of the path. The path among the finite collection $R_{mn}$ (or $\tilde{R}_{mn}$) enclosed by the circuit which has minimum travel time is a route for $a_{mn}$ (or $b_{mn}$, resp.).

Suppose $U(0) < \frac{1}{2}$. Then there exists $a > 0$ for which $U(a) \leq \frac{1}{2}$. Convert to a Bernoulli percolation model by declaring a bond open if its travel time is less than $a$, and closed if the travel time is at least $a$. Take a square in the dual lattice which encloses $(m, 0)$ and $(n, 0)$. Since $1 - U(a) \geq \frac{1}{2}$, there exist infinitely many disjoint closed circuits in the dual lattice around the square. Let $S_{m,n}$ denote the travel time along the $x$-axis from $(m, 0)$ to $(n, 0)$. On the event $\{S_{m,n} \leq ka\}$, routes exist for $a_{mn}$ and $b_{mn}$ and are contained in the region surrounded by the $k$th closed circuit. Letting $k \to \infty$, routes for $a_{mn}$ and $b_{mn}$ exist a.s.

By countability of the pairs $(m, n)$, routes exist for $a_{mn}$ and $b_{mn}$ for all $m, n$ simultaneously with probability one.

The concept of subadditive process, introduced by Hammersley and Welsh (1965), is indispensible for determining the asymptotic behavior of first-passage times. A more restrictive definition than the original, due to Kingman (1968), is given here.

DEFINITION. Let $(\Omega, \mathscr{F}, P)$ be a probability space; $N$ the set of nonnegative integers. A collection of random variables $\{X_{mn}; m, n \in N, m < n\}$ defined on $(\Omega, \mathscr{F}, P)$ is a *subadditive process* if it satisfies the following conditions:

  (i)  If $m < n < p$, $X_{mp}(\omega) \leq X_{mn}(\omega) + X_{np}(\omega)$ for all $\omega \in \Omega$.
  (ii)  The process $\{X_{m+1,n+1}\}$ has the same joint distributions as the process $\{X_{mn}\}$.
  (iii)  $E(X_{0n}) < \infty$ for all $n \in N$, and for some constant $A$, $\inf_n E(X_{0n}/n) \geq A$.

Let $g_n = E(X_{0n})$. From (i) and (ii) it follows that

$$g_{m+n} \leq g_n + g_m$$

for all $m, n \in N$. A standard result, which is easy to prove, implies that

$$\lim_{n \to \infty} \frac{g_n}{n} = \inf \frac{g_n}{n} = \gamma$$

exists, and is finite by (iii). The constant $\gamma$ is the *time constant* of the subadditive process.

The conditions imposed by Kingman's definition enabled him to prove the powerful ergodic theorem for subadditive processes.

THEOREM 3.2 [Kingman, 1968]. *If $\{X_{mn}\}$ is a subadditive process, $\xi = \lim_{n \to \infty} X_{0n}/n$ exists a.s. and in $L^1$, and $E(\xi) = \gamma$.*

Kingman's ergodic theorem applies directly to the point-to-point first-passage time process $a_{mn}$. For $m < p < n$, a path from $(m, 0)$ to $(n, 0)$ may, but need not, pass through $(p, 0)$, so

$$a_{mp} + a_{pn} \geq a_{mn}.$$

The integrability condition (iii) is satisfied with $A = 0$ if the travel time random variables are nonnegative and have finite mean. (Relaxation of these assumptions will be discussed later.) Application of a zero-one law implies that $\lim_{n \to \infty} a_{0n}/n$ is actually constant almost surely, and thus is equal to the time constant. Since the time constant depends on the travel time distribution $U$ only, denote it by $\mu(U)$.

On the other hand, the point-to-line first-passage process $b_{mn}$ is not subadditive, and has proved much more difficult to handle. Smythe (1976) showed that $\lim_{n \to \infty} b_{0n}/n = \mu(U)$ in probability, by approximating a route for $b_{0n}$ by combinations of routes of suitable subadditive first-passage processes. The conclusion was improved to almost sure convergence by Wierman and Reh (1978). These basic convergence results are summarized in the following.

THEOREM 3.3. *Let $U(x) = 0$ for $x < 0$, and $\int_0^\infty x \, dU(x) < \infty$. Then $\lim_{n \to \infty} a_{0n}/n = \lim_{n \to \infty} b_{0n}/n = \mu(U)$ a.s.*

The reach processes

$$x_t^u(\omega) = \sup\{n : a_{0n}(\omega) \leq t\}, \qquad y_t^u(\omega) = \sup\{n : b_{0n}(\omega) \leq t\}$$

describe the maximum distance in the $x$-axis direction that fluid reaches from the origin by time $t$. The asymptotic behavior of the reach processes is determined from Theorem 3.3. Letting $a(0, n)$ denote $a_{0n}$, notice that

$$\frac{x_t^u}{t} = \frac{x_t^u}{a(0, x_t^u)} \frac{a(0, x_t^u)}{t},$$

and that by definition

$$\frac{a(0, x_t^u)}{t} \to 1 \quad \text{as} \quad t \to \infty.$$

Since $x_t^u \to \infty$ a.s. as $t \to \infty$,

$$\frac{x_t^u}{t} \to \frac{1}{\mu(U)} \quad \text{a.s.,}$$

where $1/\mu(U)$ is interpreted as $+\infty$ if $\mu(U) = 0$. $y_t^u$ is handled identically.

An argument of Hammersley (1966) yields uniform integrability of

$$\left\{ \left( \frac{x_t^u}{t} \right)^p \right\}_{t \geq a} \quad \text{and} \quad \left\{ \left( \frac{y_t^u}{t} \right)^p \right\}_{t \geq a}, \qquad (a > 0) \quad \text{for all} \quad p > 0,$$

when $U(0) < 1/\lambda$. In this case, a weak renewal theorem holds. The reach process results are summarized in the following.

THEOREM 3.4.   *If $U(x) = 0$ for $x < 0$, then*

$$\lim_{t \to \infty} \frac{x_t^u}{t} = \lim_{t \to \infty} \frac{y_t^u}{t} = \frac{1}{\mu(U)} \quad \text{a.s.}$$

If, in addition $U(0) < \frac{1}{2}$, then

$$\lim_{t \to \infty} \frac{x_t^u}{t} = \lim_{t \to \infty} \frac{y_t^u}{t} = \frac{1}{\mu(U)} \quad \text{in} \quad L^p \quad \text{for all} \quad p > 0,$$

and particularly, $\lim_{t \to \infty} E(x_t^u/t) = \lim_{t \to \infty} E(y_t^u/t) = 1/\mu(U)$.

A major outstanding problem is the determination of the time constant, or of sharp bounds for the time constant. Indirect means are necessary, since Kingman's ergodic theorem provides no method for evaluating $\mu(U)$. By definition $\gamma = \inf g_n/n$, so crude upper bounds are easily obtained, but finding lower bounds presents difficulty.

The simplest upper bound for $\mu(U)$ is the mean of the travel time distribution $U$. Let $S_n$ denote the travel time of the path along the positive $x$-axis from $(0, 0)$ to $(n, 0)$. $S_n$ is a sum of $n$ i.i.d. random variables. Since $a_{0n} \leq S_n$, by the Strong Law of Large Numbers,

(3.1) $$\mu(U) \leq E(X_1) = \int_0^\infty x \, dU(x).$$

If the distribution $U$ is degenerate, $a_{0n} = S_n$ for all $n$, and equality holds in (3.1). A simple two-path argument of Hammersley and Welsh (1965) shows that strict inequality holds if $U$ is nondegenerate. Other upper bounds are provided by Hammersley and Welsh; however, the bounds appear crude. For example, for the Uniform $(0, 1)$ distribution, compare $\mu(U) \leq .425$ with Monte Carlo simulations which indicate that $\mu(U) \leq .328$ (Welsh, 1965).

A modification of (3.1) incorporates information from the size of the atom at zero of the travel time distribution.

THEOREM 3.5.   (Wierman, 1977).   *Let $U(0) = p$, and let $\mu(p)$ denote the time constant for the Bernoulli distribution with $P(X_t = 0) = p$. Then*

(3.2)
$$\mu(U) \le \frac{\mu(p)}{1-p} \int_0^\infty x\, dU(x).$$

PROOF.   Define the distribution function $F$ by

$$F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \dfrac{U(x) - p}{1-p} & \text{if } x > 0 \end{cases}$$

which is the conditional distribution of a single travel time given that it is strictly positive. Let $\{B_i\}$ be a sequence of i.i.d. Bernoulli random variables with $P(B_i = 0) = p$, and $Y_i$ be i.i.d. from $F$, so $\{B_i Y_i\}$ have distribution $U$. Consider the time state determined by the sequence $\{B_i\}$, find a route for the point-to-point first passage time, denoted $a_{0n}^B$. Letting $N_n$ denote the number of bonds in the route for $a_{0n}^B$ with travel time $B_i = 1$, note that $N_n = a_{0n}^B$, so $\lim_{n\to\infty} E(N_n/n) = \mu(p)$. In the time state given by $\{B_i Y_i\}$, the conditional mean travel time of each such bond is $E(Y_1) = E(X_1)/(1-p)$, yielding the conclusion.

Although little is known of the dependence of the time constant upon the travel time distribution, it has been determined when $\mu(U) = 0$. The controlling quantity, as suggested by Theorem 3.5, is the atom $U(0)$.

THEOREM 3.6.   *Let $U(x) = 0$ if $x < 0$, and $\int_0^\infty x\, dU(x) < \infty$. Then $\mu(U) = 0$, if and only if $U(0) \ge \frac{1}{2}$.*

REMARKS.   If $U(0) > \frac{1}{2} = p_H$, there is positive probability that the origin is in an infinite cluster of bonds with zero travel times, in which case reach processes $x_t^u$ and $y_t^u$ are infinite. By Theorem 3.4, $\mu(U) = 0$.

If $p = \frac{1}{2}$, transform the model into a Bernoulli percolation model by declaring a bond open if its travel time is zero. Recall that for the $n \times (n+1)$ sponge, $S_{1/2}(n, n+1) = \frac{1}{2}$. Thus there exists a site $(1, i)$, $1 \le i \le n$, which is connected by an open path in the sponge to the line $x = n + 1$ with probability at least $1/2n$. Thus $P(y_0^u \ge n) \ge 1/2n$ for all $n$, so $E(y_0^u) = +\infty$. Since $y_t^u \ge y_0^u$ for $t > 0$, $E(y_t^u/t) = +\infty$ for all $t$. Apply Theorem 3.4 to obtain $\mu(U) = 0$.

In a recent major development, Kesten (1980a) proved the remaining half in the form $\mu(U) > 0$ if $U(0) < p_T$, confirming a conjecture of Smythe and Wierman (1978a). Kesten employs a recursion relation to develop an exponential bound on the tail of the reach distribution. Since it is now known that $p_T = \frac{1}{2}$ for the square lattice bond problem, $\frac{1}{2}$ is identified as the "critical probability" for first-passage percolation on the square lattice also. Prior to Kesten's result, it was known that $\mu(U) > 0$ if $U(0) < 1/\lambda$, and in this case a lower bound was provided by Hammersley (1966).

It seems plausible that $\mu(U)$ should be reasonably smooth as a function of the underlying distribution. Some properties of $\mu(U)$ are discussed in the next few paragraphs.

An intuitive monotonicity property holds: If two travel time distribution functions $U_1$ and $U_2$ satisfy $U_1(x) \le U_2(x)$ for all $x$, then $\mu(U_1) \ge \mu(U_2)$.

Under certain conditions, $\mu(U)$ varies continuous in $U$ with respect to weak convergence. Initial continuity theorems of Hammersley and Welsh (1965) and Smythe and Wierman (1978a) were improved by the following result of Cox and Kesten (1981).

THEOREM 3.7.   *Let $F_n$, $n \ge 1$, be travel time distribution functions. If $F_n \to_w F$, then $\mu(F_n) \to \mu(F)$.*

A concavity property holds which becomes useful in determining the asymptotic behavior of the length of optimal routes. We introduce the following notation for "shifted"

time states. If $r$ is a real number, let $\omega \oplus r$ denote the time state obtained by adding the constant $r$ to the travel time of each bond. Let $U \oplus r$ denote the distribution function of the shifted travel times. Notice that if the original travel times are nonnegative, the shifted travel times may assume negative values. The fundamental convergence results for first-passage times may be extended to such travel time distributions under certain conditions. The route existence proof is also complicated in this case (See Smythe and Wierman, 1978a).

Let $q$ be any path, and let $N(q)$ denote the number of bonds in $q$. Then

$$t(q; \omega \oplus r) = t(q; \omega) + rN(q).$$

Therefore

$$a_{0n}(\omega \oplus r) \leq a_{0n}(\omega) + rN_n(\omega),$$

where $N_n(\omega)$ is the number of bonds in a route of $a_{0n}(\omega)$. Taking expected values,

(3.3) $$\tau_t(0, n) \equiv E(a_{0n}(\omega \oplus r)) \leq \tau_0(0, n) + rE(N_n(\omega)).$$

Apply 3.3 to $r$ and $-r$, to obtain

$$\tfrac{1}{2}[\tau_t(0, n) + \tau_{-r}(0, n)] \leq \tau_0(0, n).$$

Divide by $n$, and pass to the limit, yielding

(3.4) $$\tfrac{1}{2}\mu(U \oplus r) + \tfrac{1}{2}\mu(U \oplus -r) \leq \mu(U).$$

Thus, since $\mu(U \oplus r)$ is nondecreasing in $r$, $\mu(U \oplus r)$ is a concave function of $r$.

Define the route length for $a_{0n}$ by

$$N_n^a(\omega) = \min\{k : \exists \text{ a route of } a_{0n}(\omega) \text{ containing } k \text{ bonds}\}.$$

Fix $\varepsilon > 0$ and $r > 0$. For almost all $\omega$, and $n$ sufficiently large,

$$\frac{a_{0n}(\omega \oplus r)}{n} \geq \mu(U \oplus r) - \varepsilon r$$

and

$$\frac{a_{0n}(\omega)}{n} \leq \mu(U) + \varepsilon r.$$

Thus, from

$$\mu(U \oplus r) - \mu(U) - 2\varepsilon r \leq r \liminf_{n \to \infty} \frac{N_n^a(\omega)}{n} \quad \text{a.s.}$$

Dividing by $r$, and letting $r \to 0$,

$$\mu^+(0) - 2\varepsilon \leq \liminf \frac{N_n^a(\omega)}{n} \quad \text{a.s.}$$

where $\mu^+$ denotes the right derivative of $\mu(U \oplus r)$. Letting $\varepsilon \to 0$,

(3.5) $$\mu^+(0) \leq \liminf \frac{N_n^a(\omega)}{n} \quad \text{a.s.}$$

Similarly,

(3.6) $$\limsup \frac{N_n^a(\omega)}{n} \leq \mu^-(0) \quad \text{a.s.}$$

The procedure is valid at values of $r$ other than zero. Note that $\mu^+(r) = \mu^-(r)$ except possibly at countably many points $r$. The condition under which the argument above is valid is a bound on the atom at zero, as a consequence of work by Kesten (1980) and Cox and Durrett (1980).

THEOREM 3.8.  *Let $U(0) < \frac{1}{2}$. Then*

$$(3.7) \qquad \mu^+(0) \leq \lim \inf \frac{N_n^a(\omega)}{n} \leq \lim \sup \frac{N_n^a(\omega)}{n} \leq \mu^-(0) \quad \text{a.s.}$$

Results concerning uniform integrability and numerical estimates for route lengths are presented in Smythe and Wierman (1978a).

An asymptotic upper bound holds when $U(0) \geq \frac{1}{2}$ (Smythe and Wierman, 1978a):

$$(3.8) \qquad \lim \sup \frac{N_n(\omega)}{n} \leq E |C| + 2 - U(0) \quad \text{a.s.}$$

where $|C|$ is the number of bonds in the closed cluster containing the origin in the Bernoulli percolation model with parameter $p = U(0)$.                  \

A more difficult problem is to determine the asymptotic behavior of the maximum height of optimal routes from the $x$-axis. Few results have been obtained on this problem.

The problem of finding weaker conditions under which the basic convergence theorem for first-passage times is valid was addressed by Reh (1978). He weakened the moment hypothesis to $\int_0^\infty [1 - U(t)]^4 \, dt < \infty$, which arises from the fact that four bonds exit from each site of the square lattice, implying that four disjoint paths may be. constructed between any pair of sites. Under this condition, $E(a_{0n}) < \infty$ for all $n$, so Kingman's theorem applies.

Modification of Reh's argument to consider interval-to-interval first passage times with appropriate connecting paths allowed Wierman (1980) to obtain convergence in probability when $\int_0^\infty x^\alpha \, dU(x) < \infty$ for some $\alpha > 0$.

Cox and Durrett (1979) discovered that in fact no conditions are required at all:

THEOREM 3.9.  *Let $U(x) < 0$ for $x < 0$. There is a finite constant $\gamma$ such that*

$$\lim a_{0n}/n = \gamma \text{ in probability},$$

$$\lim \inf \frac{a_{0n}}{n} = \gamma \quad \text{a.s.} \qquad \text{and} \quad \lim \frac{b_{0n}}{n} = \gamma \quad \text{a.s.}$$

In addition, Cox and Durrett (1979) provided a description of the asymptotic shape of the wetted region at time $t$. (This problem was previously considered by Richardson (1974) and Schurger (1979).) Let $t(0, x)$ denote the first-passage time between the origin and $x \in \mathbb{Z}^2$. Extend the definition of $t$ to $x \in \mathbb{R}^2$ by assigning $x$ the passage time to the nearest site in $\mathbb{Z}^2$. Then, in probability,

$$\lim_{n \to \infty} \frac{t(0, nx)}{n} = \varphi(x) < \infty$$

exists without any assumption on $U$. Let $A_t = \{ y : t(0, y) \leq t \}$ denote the wetted region at time $t$.

Cox and Durrett (1979) show that for $\varepsilon > 0$,

$$(3.9) \qquad P(\{x : \varphi(x) \leq 1 - \varepsilon\} \subseteq t^{-1}A_t \subseteq \{x : \varphi(x) \leq 1 + \varepsilon\} \; \forall \, t \text{ suff. large}) = 1$$

if and only if $\gamma > 0$ and $E[(\min_{1 \leq i \leq 4} X_i)^2] < \infty$, providing necessary and sufficient conditions for convergence in the sense of Richardson.

Without assumptions on $U$,

$$(3.10) \qquad P(t^{-1}A_t \subseteq \{x : \varphi(x) < 1 + \varepsilon\} \text{ for all } t \text{ suff. large}) = 1,$$

and if $\gamma > 0$

$$(3.11) \qquad P(| \{x : \varphi(x) \leq 1\} - t^{-1}A_t | < \varepsilon \text{ for all } t \text{ suff. large}) = 1$$

where $| \cdot |$ denotes Lebesgue measure. The last two results show that $t^{-1}A_t$ grows like $\{x : \varphi(x) \leq 1\}$, and covers most (but not all) of the interior.

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