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LARGE DEVIATIONS FOR BOUNDARY CROSSING PROBABILITIES[1]

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For random walks s_n, $n = 1, 2, \cdots$ whose distribution can be imbedded in an exponential family, a method is described for determining the asymptotic behavior as $m \to \infty$ of

$$P\{s_n > m\, c(n/m) \quad \text{for some} \quad n < m \mid s_m = m\,\mu_0\}, \quad \mu_0 < c(1).$$

Applications are given to the distribution of the Smirnov statistic and to modified repeated significance tests.

1. Introduction. Let $x_1, x_2, \cdots$ be independent, identically distributed random variables and $s_n = x_1 + \cdots + x_n$. Given a positive function $c(t)$, $t \geq 0$, and $m > 0$, define the first passage time

$$T = T_m = \inf\{n : s_n > m\, c(n/m)\}. \tag{1}$$

The purpose of this paper is to describe a method for studying the asymptotic behavior of the conditional probabilities

$$P\{T < m \mid s_m = m\,\mu_0\}, \quad \mu_0 < c(1), \tag{2}$$

which under conventional assumptions on the function $c(\cdot)$ and the distribution of x_1 converge to 0 exponentially fast as $m \to \infty$.

A number of methods have been developed for approximating the unconditional probabilities $P\{T \leq m\}$ under various conditions on $c(\cdot)$ and the distribution of x_1, cf. Borovkov (1962, 1964), Daniels (1974), Ferebee (1981), Jennen and Lerche (1981), Lai and Siegmund (1977), Lalley (1980), Siegmund (1978), and Woodroofe (1976b, 1978). Some of these methods seem adaptable to an investigation of the conditional probabilities (2). In principle, knowledge about the conditional probabilities (2) can be translated into knowledge about $P\{T \leq m\}$ by integrating out μ_0, although a rigorous justification of this approach leads to questions of uniformity in μ_0 which may involve additional technical difficulties.

This paper gives a new technique for studying (2). Although the problems of approximating $P\{T \leq m\}$ and $P\{T < m \mid s_m = m\,\mu_0\}$ differ in important respects, there seems to be enough similarity to warrant an informal comparison of the method introduced here with the techniques of the papers mentioned above. The method of this paper permits a fairly broad class of functions $c(\cdot)$ and has the aesthetically pleasing feature of making a minimal distinction between random walk and Brownian motion. In contrast, the methods of Borovkov and Woodroofe are directly applicable only to random walk; those of Daniels, of Ferebee, and of Jennen and Lerche apply to an extremely broad class of functions $c(\cdot)$ but seem limited to the intrinsically simpler case of Brownian motion. The Lai-Siegmund method is general with regard to processes but limited with regard to functions $c(\cdot)$. Like the methods of Woodroofe, Lai-Siegmund, and Lalley, the method described below seems to adapt readily to certain multidimensional problems, although no results in this direction have been developed in detail. For linear $c(\cdot)$ it is particularly simple. In a very special case it was used by Siegmund and Yuh (1981) to give an easy derivation of Anderson's (1960) results for Brownian motion. In addition to these technical aspects, the method provides a different perspective towards boundary crossing problems, which will become apparent during the development of the paper.

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A glance at the literature mentioned above shows that systematic theory for problems of this sort is exceedingly technical. Hence the following discussion is restricted to two examples which are important in applications and which seem to indicate the scope of the method. The case of linear $c(\cdot)$ receives a reasonably general and rigorous treatment in Section 2. As an application one obtains the large deviation probabilities of the Smirnov statistic, which are shown numerically to provide excellent approximations even for "small deviations."

A simple but illuminating non-linear example is provided by normal random walk and the stopping boundary $c(t) = \beta t^{1/2}$. The stopping rule T is closely related to Armitage's (1975) repeated significance test. This example is discussed in Section 3, and the results are applied to give an asymptotic approximation to the error probabilities of a modification of this test suggested by Peto *et al.* (1976) and Siegmund (1978).

2. The linear case. Assume that the distribution F of x_1 can be imbedded in an exponential family, i.e. for all θ in some neighborhood of 0 $\exp[\psi(\theta)] =: \int \exp(\theta x)F(dx)$ is finite, so $\exp[\theta x - \psi(\theta)]F(dx)$ defines a family of probability distributions indexed by θ. It is well known (and easy to see) that the mean and variance of these distributions are respectively $\psi'(\theta)$ and $\psi''(\theta) \geq 0$. Hence $\mu = \psi'(\theta)$ is a one to one function of θ (unless F is degenerate). It will be convenient to regard this family of distributions as indexed by μ and write $F_\mu(dx) = \exp[\theta x - \psi(\theta)]F(dx)$. To emphasize that θ is a function of μ, the notation $\theta(\mu)$ is occasionally used. Let P_μ denote the probability according to which $x_1, x_2, \cdots$ are independent with $P_\mu\{x_k \in dx\} = F_\mu(dx)$, $(k = 1, 2, \cdots)$.

An additional technical assumption is required to insure that conditional probabilities are well defined and that local limit theorems apply. This assumption can be either that F is arithmetic or that it has a well-behaved density. Only the latter case is explicitly considered here. A convenient assumption is that for all μ there exists an n such that

$$\int_{-\infty}^{\infty} | E_\mu \exp(i\lambda x_1) |^n \, d\lambda < \infty. \tag{3}$$

This implies that the P_μ distribution of s_n has a bounded density $f_{\mu,n}$ which obeys a standard local central limit theorem (Feller, 1966, page 489). To avoid some uninteresting calculations, it is also convenient to assume that the P_μ distribution of x_1 has a bounded density.

Let $\mathscr{F}_n = \mathscr{B}(x_1, \cdots, x_n)$ and $P_{\mu,n}$ denote the restriction of P_μ to $\mathscr{F}_n$, so

$$dP_{\mu,n}/dP_{\tilde{\mu},n} = f_{\mu,n}/f_{\tilde{\mu},n} = \exp\{(\theta - \tilde{\theta})s_n - n[\psi(\theta) - \psi(\tilde{\theta})]\}, \tag{4}$$

where $\tilde{\theta} = \theta(\tilde{\mu})$. For $m = 1, 2, \cdots$ and $A \in \mathscr{F}_m$ let $P_\xi^{(m)}(A) = P_\mu(A \mid s_m = m\xi)$, and for $n < m$ let $P_{\xi,n}^{(m)}$ denote the restriction of $P_\xi^{(m)}$ to $\mathscr{F}_n$. By sufficiency of $s_m, P_\xi^{(m)}$ does not depend on μ.

The main result of this section is the following.

THEOREM 1. *Let $\zeta > 0$ and $\mu_0 \in (-\infty, \zeta)$. Define $\tau = \inf\{n : s_n > m\zeta\}$ and $\tau_+ = \inf\{n : s_n > 0\}$. Assume that there exist $\mu_2 < 0 < \mu_1$ (necessarily unique) such that*

$$\psi(\theta(\mu_2)) = \psi(\theta(\mu_1)) \tag{5}$$

and

$$1 = \mu_1^{-1}\zeta + |\mu_2|^{-1}(\zeta - \mu_0). \tag{6}$$

Let $\theta_i = \theta(\mu_i)$ and $\sigma_i^2 = \psi''(\theta_i)$, $i = 0, 1, 2$. Then as $m \to \infty$

$$P_{\mu_0}^{(m)}\{\tau < m\} \sim K(\zeta, \mu_0)\exp\{-m[(\theta_1 - \theta_2)\zeta + (\theta_2 - \theta_0)\mu_0 - \psi(\theta_2) + \psi(\theta_0)]\}, \tag{7}$$

where

$$K(\zeta, \mu_0) = \frac{\sigma_0 |\mu_2|^{1/2} P_{\mu_2}\{\tau_+ = \infty\}}{\sigma_2 \mu_1 (\theta_1 - \theta_2)(\zeta - \mu_0)^{1/2} E_{\mu_1} \tau_+} \left\{1 + \frac{\sigma_1^2 |\mu_2|^3 \zeta}{\sigma_2^2 \mu_1^3 (\zeta - \mu_0)}\right\}^{-1/2}. \tag{8}$$

REMARKS.

(i) It is usually routine to verify the existence of μ_1 and μ_2 and to compute them. For example, if $\psi(\theta)$ and $\psi'(\theta)$ both diverge (continuously) to $+\infty$ as θ approaches the endpoints of its interval of definition, a simple picture demonstrates the existence of μ_1 and μ_2 satisfying (5) and (6).

(ii) The quantities $P_{\mu_2}\{\tau_+ = \infty\}$ and $E_{\mu_1}\tau_+$ must usually be computed numerically. See Woodroofe (1979) for a Fourier inversion technique to obtain the ratio $P_{\mu_2}\{\tau_+ = \infty\}/E_{\mu_1}\tau_+$ and for several examples.

(iii) It is not difficult to generalize the Theorem to allow ζ and μ_0 to depend on m and converge at suitable rates. The details of this generalization have been omitted, although the example of the Smirnov statistic given below uses such a result.

EXAMPLES.

(a) Suppose that the P_μ distribution of x_1 is normal with mean μ and variance 1. By symmetry $\mu_1 = -\mu_2$, and it is easy to see that $\mu_1 = 2\zeta - \mu_0$. The proof below shows that this value of μ_1 is tantamount to the standard reflection principle. Easy algebra and known random walk theory (Feller, 1966, Chapter XVIII) show that the right hand side of (7) becomes

$$[2(2\zeta - \mu_0)^2]^{-1}\exp\{-2\textstyle\sum_1^\infty n^{-1}\Phi[-n^{1/2}(2\zeta - \mu_0)]\}\exp\{-2m\zeta(\zeta - \mu_0)\}.$$

The final exponential factor in this expression is the well known, exact probability for the corresponding problem with Brownian motion instead of random walk. The first two factors account for excess over the boundary.

(b) If $F_n(x)$ denotes the uniform empirical distribution function, the well known representation of the uniform order statistics in terms of sums of independent exponential random variables (e.g. Breiman, 1968, page 285) shows that

$$P\{\sup_{0<x<1}(x - F_n(x)) > \zeta\} = P\{\max_{1\le j\le n}(W_j - j) \ge n\zeta - 1 \mid W_{n+1} - (n+1) = -1\},$$

where $W_j = y_1 + \cdots + y_j$ and $y_1, y_2, \cdots$ are independent standard exponential. This is almost in the form required by Theorem 1 with $m = n + 1$ and $s_k = W_k - k$, except that $m\zeta$ has been replaced by $(m-1)\zeta - 1$ and $\mu_0 = -1/m$ depends on m. Minor changes in the calculation which yields Theorem 1 give as $n \to \infty$

$$P\{\sup_{0<x<1}(x - F_n(x)) > \zeta\}$$

$$\sim \frac{\exp\{-n[(\theta_1 - \theta_2)\zeta + \theta_2 + \log(1 - \theta_2)]\}}{\{\zeta|\theta_2|^{-1}(1-\theta_2)[1 + (|\theta_2|\theta_1^{-1})^3(1-\theta_1)(1-\theta_2)^{-1}]\}^{1/2}}, \tag{9}$$

where $\theta_2 < 0 < \theta_1$ satisfy $\theta_1 - \theta_2 = \log[(1-\theta_2)/(1-\theta_1)]$ and $\theta_1^{-1} + |\theta_2|^{-1} = \zeta^{-1}$. Bahadur (1971, page 15) has determined the exponent on the right hand side of (9). It is not immediately obvious that his answer is the same as that given here, but a simple calculation shows that the two agree.

The exact distribution of $\sup_x(x - F_n(x))$ is known (Birnbaum and Tingey, 1951), although it is inconvenient for numerical calculation when n is large. For small values of n, Table 1 compares some exact probabilities with approximations obtained from (9). The classical Smirnov approximation, $\exp(-2n\zeta^2)$, is also included. Once can easily see that (9) provides a very good approximation even for small ζ, for which the probability in (9) is not close to 0.

PROOF OF THEOREM 1. Let f_n denote the density function of s_n. Recall that $f_{\mu,n}$ denotes the P_μ density of s_n and $\theta_i = \theta(\mu_i)$. (Hence $f_n = f_{\tilde\mu,n}$, where $\theta(\tilde\mu) = 0$.) For arbitrary $\mu_1 > 0$ and $n < m$

TABLE 1.
Exact and Approximate Tail Probabilities for the Smirnov Statistic. In each cell the first entry is the exact probability. The second and third are approximations given by (9) *and by* $e^{-2n\zeta^2}$ *respectively.*

n			
4	ζ = .04395	.2555	.5652
	P = .950	.500	.050
	.957	.510	.049
	.985	.593	.078
9	ζ = .03730	.1804	.4796
	P = .950	.500	.010
	952	.499	.010
	975	.557	.016

$$dP^{(m)}_{\mu_0,n}/dP_{\mu_1,n} = f_{m-n}(m\mu_0 - s_n)\exp[-\theta_1 s_n + n\psi(\theta_1)]/f_m(m\mu_0). \tag{10}$$

By (4)

$$f_m(m\mu_0) = f_{\mu_0,m}(m\mu_0)\exp[-m\theta_0\mu_0 + m\psi(\theta_0)] \tag{11}$$

and for a yet to be specified μ_2

$$f_{m-n}(m\mu_0 - s_n) = f_{\mu_2,m-n}(m\mu_0 - s_n)\exp[-\theta_2(m\mu_0 - s_n) + (m-n)\psi(\theta_2)]. \tag{12}$$

If $\mu_2 < 0 < \mu_1$ are chosen so that (5) holds, substitution of (11) and (12) into (10) leads to the basic identity

$$\begin{aligned} P^{(m)}_{\mu_0}\{\tau < m\} &= \exp\{-m[(\theta_2 - \theta_0)\mu_0 + \psi(\theta_0) - \psi(\theta_2)]\}\cdot \\ &\int_{\{\tau<m\}} \{f_{\mu_2,m-\tau}(m\mu_0 - s_\tau)\exp[-(\theta_1 - \theta_2)s_\tau]/f_{\mu_0,m}(m\mu_0)\}\, dP_{\mu_1}. \end{aligned} \tag{13}$$

The condition (6) can now be understood as putting $m\mu_0 - s_\tau$ at approximately the center of the distribution $f_{\mu_2,m-\tau}$. Since $P_{\mu_1}\{\tau/m \to \zeta/\mu_1\} = 1$ and $s_\tau \approx \zeta m$ on $\{\tau < m\}$, the proper centering is determined by $\mu_2(m - m\zeta/\mu_1) = m(\mu_0 - \zeta)$, which is equivalent to (6).

Let $R_m = s_\tau - \zeta m$. Then (13) may be rewritten

$$\begin{aligned} &P^{(m)}_{\mu_0}\{\tau < m\}\exp\{m[(\theta_1 - \theta_2)\zeta + (\theta_2 - \theta_0)\mu_0 + \psi(\theta_0) - \psi(\theta_2)]\} \\ &= \int_{\{\tau<m\}} \{\exp[-(\theta_1 - \theta_2)R_m]f_{\mu_2,m-\tau}[m(\mu_0 - \zeta) - R_m]/f_{\mu_0,m}(m\mu_0)\}\, dP_{\mu_1}. \end{aligned} \tag{14}$$

By the assumption (3)

$$f_{\mu_0,m}(m\mu_0) \sim (2\pi m\sigma_0^2)^{-1/2}, \quad m \to \infty. \tag{15}$$

Since $f_{\mu_2,m-\tau}$ is bounded (uniformly in $m - \tau$) by assumption, Lemmas 1 and 2 given below imply that the integral in (14) may be replaced by an integral over

$$\{\tau < m\zeta\mu_1^{-1}(1 + \varepsilon), \qquad R_m < (\log m)^2\} \tag{16}$$

plus terms converging to 0 as $m \to \infty$. (Here $\varepsilon > 0$ is arbitrary subject to $\zeta\mu_1^{-1}(1 + \varepsilon) < 1$.) It follows from (3) and (6) that uniformly on the event (16)

$$\begin{aligned} &f_{\mu_2,m-\tau}[m(\mu_0 - \zeta) - R_m] \\ &= \sigma_2^{-1}(m - \tau)^{-1/2}\varphi\{[\mu_2(\tau - m\zeta/\mu_1) - R_m]/\sigma_2(m - \tau)^{1/2}\} + o(m^{-1/2}). \end{aligned}$$

Hence the integral in (14) has the same limit as $m \to \infty$ as

$$(2\pi)^{1/2}\sigma_0\sigma_2^{-1}\int_{\{\tau<m\zeta\mu_1^{-1}(1+\epsilon),R_m<(\log m)^2\}} \exp[-(\theta_1-\theta_2)R_m] \cdot(1-\tau/m)^{-1/2}\varphi\{\mu_2(\tau-m\zeta/\mu_1)/\sigma_2(m-\tau)^{1/2}\}\,dP_{\mu_1}.$$

Keeping in mind that $P_{\mu_1}\{m^{-1}\tau\to\zeta\mu_1^{-1}\}=1$ and using the known limiting joint distribution of R_m and $(\tau-m\zeta/\mu_1)/m^{1/2}$ (e.g. Siegmund, 1975), one may evaluate the limit of this integral as $m\to\infty$ and hence complete the proof of Theorem 1.

LEMMA 1. *Let μ, $\varepsilon>0$. Then*

$$P_\mu\{\tau>m\zeta\mu^{-1}(1+\varepsilon)\}=o(m^{-1/2}).$$

PROOF. Let n denote the least integer greater than $m\zeta\mu^{-1}(1+\varepsilon)$, so $\{\tau>m\zeta\mu^{-1}(1+\varepsilon)\}\subset\{s_{n-1}<n\mu/(1+\varepsilon)\}$. Standard exponential Chebyshev inequalities show that the probability of this event is actually exponentially small.

LEMMA 2. *Let $\mu>0$. Then $P_\mu\{\tau<m, R_m>(\log m)^2\}=o(m^{-1/2})$.*

PROOF. The proof follows easily from

$$P_\mu\{\tau<m, R_m>(\log m)^2\}\le mP_\mu\{x_1>(\log m)^2\}$$

and standard estimates.

3. Repeated significance tests – a non-linear example. Assume now that $x_1, x_2, \cdots$ are independent $N(\mu, 1)$ random variables and that for given $b>0$

$$T=\inf\{n:|s_n|>bn^{1/2}\}. \tag{17}$$

For $m=1,2,\cdots$ let $T'=\min(T,m)$. The stopping rule T' defines the repeated significance test of Armitage (1975): to test $H_0:\mu=0$ against $H_1:\mu\neq0$ stop sampling at T' and reject H_0 if and only if $T\le m$. The power function of this test is $P_\mu\{T\le m\}$, for which asymptotic approximations have been given as $b\to\infty$, $m\to\infty$ and $bm^{-1/2}=\beta$ (e.g. Siegmund, 1977, 1978). Peto *et al.* (1976) and Siegmund (1978) suggested a modification of this test in which there is given a number c, $0<c\le b$, and one rejects H_0 if either $T\le m$ or $T>m$ and $|s_m|>cm^{1/2}$. The power function of the modified test is

$$P_\mu\{T\le m\}+P_\mu\{T>m, |s_m|>cm^{1/2}\} = P_\mu\{|s_m|>cm^{1/2}\}+P_\mu\{T<m, |s_m|<cm^{1/2}\}. \tag{18}$$

The second probability on the right hand side of (18) may be rewritten

$$\int_{|\mu_0|\le cm^{-1/2}} P_{\mu_0}^{(m)}\{T<m\}\varphi(m^{1/2}(\mu_0-\mu))m^{1/2}\,d\mu_0. \tag{19}$$

Letting c have the same asymptotic normalization as b, i.e. $cm^{-1/2}=\gamma$ for some $0<\gamma\le\beta$ and appealing to the following theorem gives asymptotic expressions for (19).

THEOREM 2. *Assume $b\to\infty$, $m\to\infty$ and $bm^{-1/2}=\beta>0$. For each compact subinterval K of $(0,\beta)$, uniformly for $\mu_0\in K$*

$$P_{\mu_0}^{(m)}\{T<m\}\sim \exp\{-\tfrac12 m(\beta^2-\mu_0^2)\}\beta\mu_0^{-1}\nu(\beta^2\mu_0^{-1}),$$

where $\nu(x)=2x^{-2}\exp\{-2\sum_1^\infty n^{-1}\Phi(-\tfrac12 xn^{1/2})\}$.

COROLLARY 1. *Suppose $m \to \infty$ and $bm^{-1/2} = \beta \geq \gamma = cm^{-1/2}$. Then*

$$P_0\{T < m, |s_m| < cm^{1/2}\} \sim \beta m^{1/2} e^{-\beta^2 m/2}(2/\pi)^{1/2} \int_{\beta^2\gamma^{-1}}^{\infty} x^{-1}\nu(x)\,dx.$$

COROLLARY 2. *Suppose $m \to \infty$ and $bm^{-1/2} = \beta > \gamma = cm^{-1/2}$. Then for $\mu \neq 0$*

$$P_\mu\{T < m, |s_m| < cm^{1/2}\} \sim \frac{\varphi[m^{1/2}(\beta - |\mu|)]}{|\mu| m^{1/2}} \nu(\beta^2\gamma^{-1})\beta\gamma^{-1}e^{-m|\mu|(\beta-\gamma)}.$$

REMARK. The case $c = b$ is included in Corollary 1 but not Corollary 2. This seemingly innocuous distinction provides considerable insight into the asymptotic relation between the conditional probabilities (2) and the unconditional probabilities $P\{T \leq m\}$. An informal attempt to elucidate this relation is given at the end of the paper.

INFORMAL PROOF OF THEOREM 2. Consider first the case of a fixed $\mu_0 > 0$. The question of uniformity will be considered later. For $\mu_0 > 0$ it is easy to see that

$$P_{\mu_0}^{(m)}\{T < m, s_T < 0\} = o(P_{\mu_0}^{(m)}\{T < m, s_T > 0\}),$$

and hence without loss of generality one may assume that T is defined without the absolute value—as it is in Section 1. The main idea of the proof is to approximate the curve $\beta t^{1/2}$ by its tangent $\zeta + \eta t$ at a suitable value $t_0 \in (0, 1)$. For the line $\zeta + \eta t$ the appropriate value of μ_1 is $\mu_1 = 2(\zeta + \eta) - \mu_0$, and the identity (13) becomes

$$P_{\mu_0}^{(m)}\{T < m\}\exp[2m\zeta(\zeta + \eta - \mu_0)]$$

$$(20) \quad = \int_{\{T<m\}} (1 - T/m)^{-1/2}\exp\left\{-2(\zeta + \eta - \mu_0)\left[\frac{s_T - \zeta m - \eta T}{1 - T/m}\right] - \frac{1}{2}\frac{(s_T - \mu_1 T)^2}{m - T}\right\} dP_{\mu_1},$$

where $\mu_1 = 2(\zeta + \eta) - \mu_0$. The point of tangency t_0 is chosen so that $P_{\mu_1}\{T/m \to t_0\} = 1$. (Simple algebra gives $t_0 = (\mu_0/\beta)^2$, $\zeta = \mu_0/2$, $\mu_1 = \beta^2/\mu_0$, and $\eta = \beta^2/2\mu_0$.) A Taylor expansion yields

$$\begin{aligned} s_T - \zeta m - \eta T &= s_T - bT^{1/2} + m\{\beta(T/m)^{1/2} - \zeta - \eta T/m\} \\ &= s_T - bT^{1/2} - \beta^4(T - m_0)^2/8\mu_0^3 m + o_p(1), \end{aligned}$$

where $m_0 = mt_0$. Similarly

$$m^{-1}(s_T - \mu_1 T)^2 = \beta^4(T - m_0)^2/4\mu_0^2 m + O_p[(s_T - bT^{1/2})/m^{1/2}] + o_p(1).$$

Hence the integrand in (20) becomes

$$(21) \quad \begin{aligned} (1 - T/m)^{-1/2}\exp\left\{-\mu_0^{-1}(\beta^2 - \mu_0^2)\left[\frac{s_T - bT^{1/2}}{1 - T/m}\right] - \frac{1}{2}\frac{(1 - 2\mu_0^2\beta^{-2})\beta^6(T - m_0)^2}{4(1 - T/m)\mu_0^4 m}\right. \\ \left. + o_p(1) + O_p[(s_T - bT^{1/2})/m^{1/2}]\right\}. \end{aligned}$$

The limiting joint distribution of $s_T - bT^{1/2}$ and $(T - m_0)/m^{1/2}$ may be obtained as an application of Theorem 2 of Lai and Siegmund (1977) or Theorem 4.3 of Woodroofe (1976a). Substituting (21) into (20) and integrating with respect to the limiting distribution of $(s_T - bT^{1/2})$, $(T - m_0)/m^{1/2}$, and T/m produces the expression given in Theorem 2.

When $\mu_0 > 2^{-1/2}\beta$, the function of $(T - m_0)/m^{1/2}$ in (21) which must be integrated is unbounded, and hence some care is required to justify taking the limit inside the integral in the preceding paragraph. However, straightforward estimates show that

$$P_{\mu_0}^{(m)}\{T < m\} \sim P_{\mu_0}^{(m)}\{m_0 - \lambda m^{1/2} < T < m_0 + \lambda m^{1/2}\},$$

where λ tends to $+\infty$ arbitrarily slowly with m. By replacing the event $\{T < m\}$ in (20) by the smaller event

$$\{m_0 - \lambda m^{1/2} < T < m_0 + \lambda m^{1/2},\ s_T - bT^{1/2} < (\log m)^2\},$$

one may make the preceding argument precise. The details have been omitted. See Siegmund (1978) for a similar argument spelled out in detail.

To see that the preceding argument holds uniformly in μ_0 provided μ_0 is bounded away from 0 and β (hence t_0 is bounded away from 0 and 1) requires a tedious review of the various steps of the proof to see that each holds uniformly in μ_0. Perhaps the only non-obvious step is the classical renewal theorem, which must be applied to the ladder height renewal process determined by the distribution of $s_{\tau_+}(\tau_+ = \inf\{n : s_n > 0\})$. The Fourier analytic proof, given for example by Breiman (1968, page 218) seems well adapted to using boundedness of higher moments of s_{τ_+} to prove the required uniformity. The details are omitted.

To prove Corollary 1, note that the theorem immediately implies for $0 < \varepsilon < \gamma < \beta$

$$(22) \qquad P_0\{T < m, \varepsilon m < |s_m| < \gamma m\} \sim \beta m^{1/2} e^{-\beta^2 m/2}(2/\pi)^{1/2} \int_{\beta^2\gamma^{-1}}^{\beta^2\varepsilon^{-1}} x^{-1}\nu(x)\, dx.$$

Obviously, for $0 < \delta < 1$

$$P_0\{T < m, |s_m| < \varepsilon m\} \le P_0\{T \le \delta m\} + P_0\{\delta m < T < m, |s_m| < \varepsilon m\}.$$

It is easy to see that for small δ,

$$P_0\{T \le \delta m\} \le \textstyle\sum_{n \le \delta m} P_0\{|s_n| \ge b n^{1/2}\}$$

is small compared to the right hand side of (22); taking ε so small that $\beta\delta^{1/2} - \varepsilon > 0$ and using standard arguments one sees that $P_0\{\delta m < T < m, |s_m| < \varepsilon m\}$ is also small compared to the right hand side of (22). This proves the corollary when $\gamma < \beta$, and a similar argument to estimate $P_0\{T < m, (1 - \varepsilon)\beta m < |s_m| < \beta m\}$ handles the case $\gamma = \beta$.

The proof of Corollary 2 is almost immediate, for when $\mu \neq 0$, the entire contribution to the integral in (19) comes from the immediate neighborhood of $\mu_0 = \mathrm{c}m^{-1/2}$.

4. Discussion. It should be apparent from the preceding examples that the method given here is valid for fairly general random walks and curved boundaries. For example, it seems reasonably straightforward to consider curves $c(t) = \beta t^\gamma$ for $0 \le \gamma \le ½$ and exponential families as in Theorem 1 to obtain a simultaneous generalization of Theorems 1 and 2. Since the calculations are messy and the author is unaware of interesting applications, this generalization has not been pursued.

In considering extensions to more complicated curves, the important requirement is that the appropriate approximating tangent line have its point of tangency t_0 at neither 0 nor 1. Unfortunately these boundary cases can arise for a variety of reasons. The most obvious is that μ_0 may be a boundary case, e.g. $\mu_0 = c(1)$. However, for normal random walk and $c(t) = \beta t^\gamma$ with $½ < \gamma < 1$, the "appropriate tangent" is at $t_0 = 0$ for all $\mu_0 < c(1)$, so the method breaks down completely.

Corollary 1 to Theorem 2 with $\gamma = \beta$ yields the known asymptotic expression for $P_0\{T \le m\}$ by "unconditioning" $P_{\mu_0}^{(m)}\{T < m\}$. In general this is not an effective method for obtaining approximations to the corresponding unconditional probabilities, because the important values of μ_0 in the unconditioning integral may be boundary cases, for which the methods of this paper fail or are not particularly appropriate. For example, for $c(t) = \beta t^{1/2}$ and normal random walk with positive mean μ, the important values of μ_0 in the unconditioning integral

$$P_\mu\{T < m, s_m < m\beta\} = \int_{\mu_0 < \beta} P_{\mu_0}^{(m)}\{T < m\}\varphi[m^{1/2}(\mu_0 - \mu)]m^{1/2}\, d\mu_0$$

are values $\mu_0 = \beta - \xi/m$, for which the appropriate tangents are at $t_0 = 1$. Hence the methods of this paper do not apply directly, although a modification can be made to work.

The essential ingredient is to consider the process to be running backward in time from time m to time 0. For the reversed process the role of $m\zeta$ is played by ξ, which does not tend to $+\infty$, and hence the subsequent calculation is almost trivial. And in fact for these values of μ_0 the limiting behavior of $P_{\mu_0}^{(m)}\{T < m\}$ can be inferred by almost trivial arguments (e.g. Siegmund, 1978). Thus, although the methods of this paper give heuristic insight into the behavior of the unconditional probability $P_\mu\{T \leq m\}$, they do not appear likely to replace previously developed methods for obtaining mathematically rigorous results.

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