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A LIMIT THEOREM FOR SLOWLY INCREASING OCCUPATION TIMES

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Let B_t be a two-dimensional Brownian motion and $f(x)$ be a bounded measurable function vanishing outside a compact set. Then $(1/\lambda)\int_0^{e^{\lambda t}} f(B_s)\,ds$ converges to const. $\ell(M^{-1}(t), 0)$ as $\lambda \to \infty$, where $\ell(t, x)$ and $M(t)$ are the local time and the maximum process of a one-dimensional Brownian motion, respectively. In the present article we generalize this theorem for more general Markov processes as follows: Let X_t be a Markov process and $f(x)$ be a nonnegative, bounded measurable function on the state space. If the expectation of $\int_0^t f(X_s)\,ds$ is asymptotically equal to a slowly varying function $L(t)$ as $t \to \infty$, then, $(1/\lambda)\int_0^{L^{-1}(\lambda t)} f(X_s)\,ds$ converges to $\ell(M^{-1}(t), 0)$ as $\lambda \to \infty$, in the sense of the convergence of all finite-dimensional marginal distributions.

1. Introduction. Let $\{X_t\}_{t\ge 0}$ be a temporally homogeneous Markov process with values in a measurable space $(S, \mathscr{B})$ and let $f(x) \ge 0$ be a bounded measurable function defined on S. Then the random variable $\int_0^t f(X_s)\,ds$ is called the *occupation time* and the following theorem due to Darling and Kac [2] is well known.

THEOREM A. *If there exist $\alpha \in [0, 1]$ and a slowly varying function $L(t)$ such that*

$$\text{(DK)} \qquad \lim_{t\to\infty} E_x\left[\int_0^t f(X_s)\,ds\right]/t^\alpha L(t) = C > 0, \quad x \in S$$

the convergence being uniform on $K = \{x; f(x) > 0\}$, then, the law of $\int_0^t f(X_s)\,ds/t^\alpha L(t)$ converges weakly to the Mittag-Leffler distribution with exponent α, as $\lambda \to \infty$.

(Darling and Kac [2] stated condition (DK) using Laplace transforms. So precisely speaking, their condition is a little weaker than (DK).)

Furthermore, if $\alpha \ne 0$, N. H. Bingham [1] proved a functional-type limit theorem:

THEOREM B. *If, in addition, $\alpha \ne 0$, the continuous processes $\int_0^{\lambda t} f(X_s)\,ds/\lambda^\alpha L(\lambda)$ converge in distribution to the inverse of one-sided stable process with exponent α.*

However, in case $\alpha = 0$, we do not have such a theorem because "one-sided stable process of exponent 0" does not make sense. As a matter of fact, one can easily see that if $\alpha = 0$ and if

$$\int_0^{\lambda t} f(X_s)\,ds/L(\lambda) \to_{\text{f.d.}} Y(t) \quad \text{as} \quad \lambda \to \infty$$

(which denotes the convergence of all finite-dimensional marginal distributions) for some right-continuous process $Y(t)$, then $Y(t) = Y(0+)$, $t > 0$ holds with probability one ($Y(0+)$ is exponentially distributed by Theorem A.). Nonetheless, the case $\alpha = 0$ is of interest because this class includes Cauchy processes and two-dimensional Brownian motions (and of course the Bessel process with exponent 2). The following problem was raised by D.

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Stroock: What is the limiting process of

$$A_\lambda(t) = \frac{1}{\lambda}\int_0^{n(\lambda t)} f(X_s)\,ds \left(\text{instead of } \frac{1}{L(\lambda)}\int_0^{\lambda t} f(X_s)\,ds\right)$$

where $n(.)$ is the inverse function of $L(.)$?

For this problem, S. Kotani, studying the behavior of stable processes with small index α, conjectured that the limiting process should equal $Z(t) = \ell(M^{-1}(t), 0)$ where $\ell(t, x)$ and $M(t)$ are respectively the local time and the maximum process of a (common) one-dimensional Brownian motion starting at 0 (cf. S. Watanabe [8] page 260). In the case of two-dimensional Brownian motion, his conjecture was proved by himself and the author [4]:

THEOREM C. *Let B_t be a two-dimensional Brownian motion and $f(x)$ be a measurable function vanishing outside a compact set. Then, putting $n(t) = e^{2t}$ (or, te^{2t}),*

$$A_\lambda(t) = \frac{1}{\lambda}\int_0^{n(\lambda t)} f(B_s)\,ds \to_{\text{f.d.}} \frac{1}{\pi}\int f(y)\,dy\, Z(t) \quad as \quad \lambda \to \infty$$

where $Z(t) = \ell(M^{-1}(t), 0)$ is the same as before.

(In [4] the right-hand side is multiplied by π, which is a mistake.)

Here, it should be remarked that Bingham proved not only convergence of the finite dimensional distributions, but also weak convergence of the process over function space. However, it is *impossible* to strengthen Theorem C in a similar way as long as we stick to the uniform topology or Skorokhod's J-topology because the set of all continuous functions is closed under J-topology but the limiting process $Z(t)$ is a jump process while $A_\lambda(t)$, $\lambda > 0$ are continuous ([4] discussed weak M_1-convergence). The reader should also notice that if we apply this theorem for $f(x) = 1_{\{x;|x|<1\}}(x)$ we have

$$\frac{1}{\varepsilon^2\log(1/\varepsilon)}\int_0^{(1/\varepsilon)^{2(t-1)}} 1_{\{x;|x|\le\varepsilon\}}(B_s)\,ds \to_{\text{f.d.}} Z(t)$$

as $\varepsilon \to 0$, because $B_{\varepsilon^2 t}$ and εB_t are identical in law ($\varepsilon = e^{-\lambda}$).

In the present paper, we will give an extension of Theorem C for more general Markov processes satisfying (DK) with $\alpha = 0$. As an example, the case of Cauchy process will be discussed. A similar result for Markov processes with discrete time parameter will be given in the last section.

2. Main theorem. Let X_t, $t \ge 0$ be a Markov process on $(S, \mathscr{B})$ with stationary transitions and $f(x)$ be a measurable non-negative function defined on S.

THEOREM 2.1. *Suppose X_t and $f(.) \ge 0$ satisfy*

$$\lim_{t\to\infty} E_x\left[\int_0^t f(X_s)\,ds\right] \Big/ L(t) = C > 0, \quad x \in S, \tag{2.1}$$

the convergence being uniform for $x \in \{x; f(x) \ne 0\}$, where $L(t)$ ($\uparrow \infty$, $t \uparrow \infty$) is a slowly varying function. Then for a non-negative continuous increasing function $n(t)$, $t \ge 0$, such that $L(n(t))/t \to 1$ as $t \to \infty$, we have

$$A_\lambda(t) = \frac{1}{\lambda}\int_0^{n(\lambda t)} f(X_s)\,ds \to_{\text{f.d.}} CZ(t) \quad as \quad \lambda \to \infty,$$

where $Z(t) = \ell(M^{-1}(t), 0)$ is the process stated in the previous section.

REMARK 2.2. The *rate* of $n(t)$ is not uniquely determined by the relation $L(n(t))/t \to 1$ as $t \to \infty$ (eg. for $L(t) = \log t$, not only e^t but also te^t is possible.). However, since $L(n(t)) \sim t$ implies $L(t) \sim n^{-1}(t)$, $L(t)$ in (2.1) can be replaced by $n^{-1}(t)$. Therefore, it suffices to prove Theorem 2.1 assuming $L(t) = n^{-1}(t)$.

As an example of Theorem 2.1, we next consider the case of Cauchy process. Let X_t be a Markov process with independent increments (in R^1) such that $E_x[e^{i\xi X(t)}] = \exp(-t|\xi| + i\xi x)$, ξ, $x \in R^1$, and $f(x)(x \in R^1)$ be a bounded measurable function vanishing outside a compact set. Then since

$$E_x[f(X_s)] = \frac{s}{\pi}\int_{-\infty}^{\infty} f(y)/\{(y-x)^2 + s^2\}\, dy \sim \frac{1}{\pi}\int f(y)\, dy \frac{1}{s}$$

as $s \uparrow \infty$, we see that

$$E_x\left[\int_0^t f(X_s)\, ds\right] \sim \frac{1}{\pi}\int f(y)\, dy \log t \quad \text{as} \quad t \to \infty.$$

Therefore, setting $L(t) = \log t$ and $n(t) = e^t$, we have

$$\frac{1}{\pi}\int_0^{e^{\lambda t}} f(X_s)\, ds \to_{\text{f.d.}} \frac{1}{\pi}\int f(y)\, dy\, Z(t) \quad \text{as} \quad \lambda \to \infty. \tag{2.2}$$

Furthermore, if we set $f(x) = 1_{[-1,1]}(x)$ and $\lambda = \log(1/\varepsilon)$, (2.2) implies

$$\frac{1}{\varepsilon \log(1/\varepsilon)}\int_0^{(1/\varepsilon)^{t-1}} 1_{[-\varepsilon,\varepsilon]}(X_s)\, ds \to_{\text{f.d.}} \frac{2}{\pi} Z(t) \quad \text{as} \quad \varepsilon \downarrow 0\ (P_0).$$

Here we have used the fact that $X_{\lambda t}$ is equivalent in law to λX_t (w.r.t. P_0).

Before we prove the theorem, we review some facts about the process $Z(t) = \ell(M^{-1}(t), 0)$ for the convenience of the reader although we will not use them in the rest of this article.

$Z(t)$ does not have the ordinary Markov property. However, the inverse process $Y(t) = Z^{-1}(t)$ is a process with independent increments and appears in some limit theorems for sums or maxima of i.i.d. random variables (cf. Lamperti [5] and S. Watanabe [8]). For the semi-group and the generator, see [8] and for the transition probabilities see [5]. The finite-dimensional joint distribution function is known to be

$$P[Y(t_1) \le x_1, \cdots, Y(t_j) \le x_j] = G(x_1)^{t_1} G(x_2)^{t_2 - t_1} \cdots G(x_j)^{t_j - t_{j-1}}, \quad G(x) = e^{-1/x}, \tag{2.3}$$

for $0 < t_1 < \cdots < t_j$, $0 < x_1 < \cdots < x_j$.
(Since $Y(t)$ is a non-decreasing function, (2.3) determines the distribution function.) Therefore, we have

$$\begin{aligned} &P[Z(t_1) > x_1, Z(t_2) > x_2, \cdots, Z(t_j) > x_j] \\ &\qquad = \exp\{-x_1/t_1 - (x_2 - x_1)/t_2 - \cdots - (x_j - x_{j-1})/t_j\} \end{aligned} \tag{2.4}$$

for $0 \le t_1 \le \cdots \le t_j$ and $0 \le x_1 \le \cdots \le x_j$.

We next explain the idea of the proof of Theorem 2.1. For $j = 1, 2, \cdots$, define $m_j(t_1, t_2, \cdots, t_j) = E[Z(t_1)Z(t_2) \cdots Z(t_j)]$, $t_i \ge 0$. Then by (2.4) we have

$$m_1(t_1) = t_1, \tag{2.5}$$

$$m_2(t_1, t_2) = (t_1 + t_2) \min(t_1, t_2), \tag{2.6}$$

$$m_j(t_1, \cdots, t_j) = \textstyle\sum_\pi t_{1(\pi)} t_{2(\pi)} \cdots t_{j(\pi)} \quad \text{if} \quad t_1 \le t_2 \le \cdots \le t_j \tag{2.7}$$

where $k(\pi) = \min\{\pi(k), \pi(k+1), \cdots, \pi(j)\}$. (Here π runs over all permutations of j objects.)

In order to prove Theorem 2.1, it is sufficient to show

$$\lim_{\lambda\to\infty} E_x[A_\lambda(t_1) \cdots A_\lambda(t_j)] = C^j m_j(t_1, \cdots, t_j) \tag{2.8}$$

for $j = 1, 2, \cdots$ and $t_i \geq 0$ (see Section 4). For the proof of (2.8), we will prove the convergence of the Laplace transforms. (This idea is due to [1] and [2].)

$$\lim_{\lambda\to\infty} \mathscr{L}E_x[A_\lambda(t_1) \cdots A_\lambda(t_j)](s_1, \cdots, s_j) = C^j \mathscr{L}m_j(s_1, \cdots, s_j) \tag{2.9}$$

where

$$\mathscr{L}f(t_1, \cdots, t_j)(s_1, \cdots, s_j) = s_1 \cdots s_j \int_0^\infty \cdots \int_0^\infty e^{-\Sigma s_i t_i} f(t_1, \cdots, t_j)\, dt_1 \cdots dt_j.$$

Since it does not seem easy to know the explicit form of the right-hand side of (2.9), we will show the following instead of (2.9):

$$\lim_{\lambda\to\infty} \mathscr{L}E_x[A_\lambda(t_1) \cdots A_\lambda(t_j)](s_1, \cdots, s_j) = C^j \Phi^{(j)}(s_1, \cdots, s_j) \tag{2.10}$$

where $\Phi^{(j)}$ does not depend on X_t or $f(x)$ as long as (2.1) holds.

To see that (2.10) implies (2.9), we need only to apply both (2.10) and Theorem C to two-dimensional Brownian motions. Then, by the uniqueness of the limits, we see that $\Phi^{(j)}$ should equal the Laplace transform of m_j.

REMARK 2.3. For $j = 1, 2$, it is easy to have the explicit form of $\mathscr{L}m_1$ and $\mathscr{L}m_2$ directly from (2.5) and (2.6);

$$\mathscr{L}m_1(s_1) = 1/s_1, \tag{2.11}$$

$$\mathscr{L}m_2(s_1, s_2) = \frac{1}{s_1 s_2} + \frac{2}{(s_1 + s_2)^2}. \tag{2.12}$$

These facts will also be proved by computing the left-hand side of (2.10) as we explained above (see next section).

REMARK 2.4. The assumption that $f(x)$ is non-negative is necessary to derive (2.8) from (2.9). However, in many examples we can drop this assumption using additional arguments. On the other hand, the condition that C (in (2.1)) does not vanish is essential. In case $C = 0$, we are very likely to have another kind of limit theorem (cf. [3]).

3. Preliminaries. In this section we will prove (2.10) and (2.9). Let $L(t)$, $t \geq 0$ be a continuous, increasing function varying slowly at ∞ such that $L(0+) = 0$, $L(\infty-) = \infty$ and $n(t)$ be its inverse (cf. Remark 2.2.). For $\lambda > 0$, $t \geq 0$, $x \in S$, $j = 1, 2, \cdots$, and $s_i > 0$ ($i = 1, 2, \cdots$), define

$$\phi_\lambda^{(1)}(s_1; x, t) = \int_t^\infty \exp(-s_1\xi)\, E_x[f(X_{n(\lambda\xi)-n(\lambda t)})]n'(\lambda\xi)\, d\xi \tag{3.1}$$

$$\phi_\lambda^{(j)}(s_1, \cdots, s_j; x, t) = \int_t^\infty \exp(-s_1\xi)\, E_x[f(X_{n(\lambda\xi)-n(\lambda t)})\phi_\lambda^{(j-1)}(s_2, \cdots, s_j; X_{n(\lambda\xi)-n(\lambda t)}, \xi)]n'(\lambda\xi)\, d\xi, \quad j = 2, 3, \cdots \tag{3.2}$$

$$\Phi_\lambda^{(j)}(s_1, \cdots, s_j; x) = \textstyle\sum_\pi \phi_\lambda^{(j)}(s_{\pi(1)}, s_{\pi(2)}, \cdots, s_{\pi(j)}; x, 0) \tag{3.3}$$

$$\phi^{(1)}(s_1; t) = \int_t^\infty \exp(-s_1\xi)\, s_1\xi\, d\xi, \tag{3.4}$$

$$\phi^{(j)}(s_1, \cdots, s_j; t) = \int_t^\infty \exp(-s_1\xi)\{s_1\phi^{(j-1)}(s_2, \cdots, s_j; \xi) - \frac{d}{d\xi}\phi^{(j-1)}(s_2, \cdots, s_j; \xi)\}\xi\, d\xi, \quad j \geq 2, \tag{3.5}$$

$$\Phi^{(j)}(s_1, \cdots, s_j) = \sum_\pi \phi^{(j)}(s_{\pi(1)}, \cdots, s_{\pi(j)}; 0), \quad j \geq 1. \tag{3.6}$$

Here, it should be remarked $\phi_\lambda^{(j)}$ and $\Phi_\lambda^{(j)}$ depends on X_t and $f(x)$ but $\phi^{(j)}$ or $\Phi^{(j)}$ does not. For $j = 1, 2$, we see

$$\Phi^{(1)}(s_1) = 1/s_1, \quad \Phi^{(2)}(s_1, s_2) = \frac{1}{s_1 s_2} + \frac{2}{(s_1 + s_2)^2}$$

(cf. Remark 2.3.) Of course, $\Phi^{(j)}$ is the function we needed in (2.10), which will be proved in Lemma 3.3. The following lemma explains the meaning of $\Phi_\lambda^{(j)}$.

LEMMA 3.1.

$$\mathscr{L}E_x[A_\lambda(t_1)\cdots A_\lambda(t_j)](s_1, \cdots, s_j) = \Phi_\lambda^{(j)}(s_1, \cdots, s_j; x) \quad s_i > 0. \tag{3.7}$$

PROOF. By integration by parts, the right-hand side of (3.7) can be written as

$$\int_0^\infty \cdots \int_0^\infty \exp(-\Sigma_i s_i t_i)\, E_x[f(X_{n(\lambda t_1)})\cdots f(X_{n(\lambda t_j)})]n'(\lambda t_1)\cdots n'(\lambda t_j)dt_1\cdots dt_j$$
$$= \sum_\pi \int\cdots\int_{0<t_1<\cdots<t_j} \exp(-\Sigma_i s_{\pi(i)} t_i)\, E_x[\textstyle\prod_i f(X_{n(\lambda t_i)})]n'(\lambda t_1)\cdots n'(\lambda t_j)dt_1\cdots dt_j.$$

Therefore, it suffices to show

$$\phi_\lambda^{(j)}(s_1, \cdots, s_j; x, t) = \int\cdots\int_{t<t_1<\cdots<t_j} \exp(-\Sigma\, s_i t_i)\, E_x[\textstyle\prod_i f(X_{n(\lambda t_i)-n(\lambda t)})]\prod_i n'(\lambda t_i)dt_1\cdots dt_j.$$

However, this follows from (3.2) and the Markov property by induction.

The following proposition, the proof of which will be postponed until Section 5, plays an essential role in the present article. In fact, all assumptions in Theorem 2.1 are used only to prove Proposition 3.2.

PROPOSITION 3.2. *For all $s_i > 0$ ($i = 1, 2, \cdots$) and $x \in S$,*

$$\lim_{\lambda\to\infty} \phi_\lambda^{(j)}(s_1, \cdots, s_j; x, t) = C^j\phi^{(j)}(s_1, \cdots, s_j; t), t \geq 0, \tag{3.8}$$

the convergence being uniform for $(t, x) \in [0, \infty) \times \{x; f(x) \neq 0\}$.

Clearly, Proposition 3.2 combined with definitions (3.3) and (3.6) implies

$$\lim_{\lambda\to\infty} \Phi_\lambda^{(j)}(s_1, \cdots, s_j; x) = C^j\Phi^{(j)}(s_1, \cdots, s_j), \quad x \in S.$$

Thus, by Lemma 3.1, we have the result mentioned in (2.10):

LEMMA 3.3.

$$\lim_{\lambda\to\infty} \mathscr{L}E_x[A_\lambda(t_1)\cdots A_\lambda(t_j)](s_1, \cdots, s_j) = C^j\Phi^{(j)}(s_1, \cdots, s_j), \quad x \in S.$$

As we explained in the previous section, we next prove that $\Phi^{(j)}$ is the Laplace transform of $E[Z(t_1)\cdots Z(t_j)]$ using the result for two-dimensional Brownian motions (Theorem C). Let B_t be a two-dimensional Brownian motion and $f(x)$ be the indicator function of $\{x; |x| < 1\}$. Then, like the case of Cauchy process,

$$E_x\left[\int_0^t f(B_s)ds\right] = \int_0^t ds \int_{|y|<1} \frac{1}{2\pi s} \exp(-|x-y|^2/2s)\, dy$$

is asymptotically equal to (½) log t as $t \to \infty$. Therefore, (2.1) is satisfied with $L(t) =$ (½) log t (or, the inverse of te^{2t}) and $C = 1$. Thus applying Lemma 3.3, we have

$$\lim_{\lambda\to\infty} \mathscr{L} E_x[A_\lambda(t_1) \cdots A_\lambda(t_j)] = \Phi^{(j)}(s_1, \cdots, s_j) \tag{3.9}$$

where

$$A_\lambda(t) = \frac{1}{\lambda}\int_0^{n(\lambda t)} f(B_s)\, ds,\ n(t) = e^{2t} \quad (\text{or} \quad te^{2t}).$$

On the other hand, by Theorem C,

$$\lim_{\lambda\to\infty} \mathscr{L} E_x[A_\lambda(t_1) \cdots A_\lambda(t_j)] = \mathscr{L} E[Z(t_1) \cdots Z(t_j)]. \tag{3.10}$$

From (3.9) and (3.10) we conclude the following.

LEMMA 3.4. $\Phi^{(j)}(s_1, \cdots, s_j) = \mathscr{L} E[Z(t_1) \cdots Z(t_j)](s_1, \cdots, s_j)$.

Combining Lemmas 3.3 and 3.4, we have (2.9):

PROPOSITION 3.5. *If X_t and $f(x)$ satisfy* (2.1),

$$\lim_{\lambda\to\infty} \mathscr{L} E_x[A_\lambda(t_1) \cdots A_\lambda(t_j)] = C^j \mathscr{L} E[Z(t_1) \cdots Z(t_j)].$$

4. Proof of Theorem 2.1. Observe that $E_x[A_\lambda(t_1) \cdots A_\lambda(t_j)]$ defines a non-negative Lebesgue-Stieltjes measure on $[0, \infty)^j$ for every x and $\lambda(>0)$, and that $m_j(t_1, \cdots, t_j) = E[Z(t_1) \cdots Z(t_j)]$ is continuous on $[0, \infty)^j$ (cf. (2.7)). Therefore, Proposition 3.5 implies

$$\lim_{\lambda\to\infty} E_x[A_\lambda(t_1) \cdots A_\lambda(t_j)] = C^j E[Z(t_1) \cdots Z(t_j)], \quad x \in S, \quad t_i \geq 0.$$

Since t_i, $i = 1, 2, \cdots$ are arbitrary and repetition is allowed, we have

$$\lim_{\lambda\to\infty} E_x[A_\lambda(t_1)^{k_1} \cdots A_\lambda(t_m)^{k_m}] = C^j E[Z(t_1)^{k_1} \cdots Z(t_m)^{k_m}] \tag{4.1}$$

where $j = k_1 + \cdots + k_m$.

To prove the theorem, it remains to show that the right-hand side of (4.1) determines the joint law of $(Z(t_1), \cdots, Z(t_m))$ for all $0 \leq t_1 < \cdots < t_m$, $m = 1, 2, \cdots$. To this end it suffices to prove that, for arbitrary $\xi_1, \cdots, \xi_m (\in R)$, the law of $Y = \Sigma\, \xi_i Z(t_i)$ is determined by its moments. However, since $Z(t)$ is non-decreasing in t(w.p.1), we have

$$\begin{aligned} E[Y^k] &\leq \{\textstyle\sum_i |\xi_i|\}^k E[Z(T)^k] \\ &= \{\textstyle\sum_i |\xi_i|\}^k k! T^k, \quad \text{where } T = \max_i t_i. \end{aligned} \tag{4.2}$$

Here we have used that $Z(T)$ is exponentially distributed (see Theorem A or (2.7)). By the well known Carleman test, (4.2) implies that the law of Y is determined by its moments, which completes the proof of Theorem 2.1.

5. Proof of Proposition 3.2. Throughout this section we assume all conditions in Theorem 2.1 as well as $L(0) = 0$, $n(t) = L^{-1}(t)$ (see Remark 2.2.). Define $T_\lambda(t; t_0)$ by

$$\begin{aligned} n(\lambda T_\lambda(t; t_0)) &= n(\lambda t) - n(\lambda t_0) && \text{if } t \geq t_0 \geq 0 \\ T_\lambda(t; t_0) &= 0 && \text{if } 0 \leq t \leq t_0. \end{aligned} \tag{5.1}$$

Or equivalently,

$$T_\lambda(t; t_0) = \begin{cases} \dfrac{1}{\lambda} L(n(\lambda t) - n(\lambda t_0)) & \text{if } t \geq t_0 \geq 0 \\ 0 & \text{if } 0 \leq t \leq t_0. \end{cases} \tag{5.2}$$

Then we clearly have

(5.3) $$0 = T_\lambda(t_0; t_0) \le T_\lambda(t; t_0) \le T_\lambda(t; 0) = t.$$

The following lemma is a simple modification of the assumption (2.1) and can be proved without assuming that L varies slowly.

LEMMA 5.1.

(5.4) $$\lim_{\lambda\to\infty} \sup_{t,t_0,t\ge t_0} |E_x[\tilde{A}_\lambda(n(\lambda t) - n(\lambda t_0))] - CT_\lambda(t; t_0)|/(t+1) = 0,$$

the convergence being uniform for $x \in \{x; f(x) \ne 0\}$, *where* $\tilde{A}_\lambda(t) = A_\lambda(L(t))$.

PROOF. By (5.1), we see that $\tilde{A}_\lambda(n(\lambda t) - n(\lambda t_0)) = A_\lambda(T_\lambda(t; t_0))$. On the other hand, by (2.1), we have $|E_x[A_\lambda(t)] - Ct|/(1+t)$ converges to 0. Thus keeping (5.3) in mind, we obtain the assertion.

Throughout this paper, the assumption that $L(t)$ is a slowly varying function is used only for the proof of the following key lemma.

LEMMA 5.2. *For all* $c > 1$,

$$\lim_{\lambda\to\infty}\sup_{t,t_0;t\ge ct_0} |T_\lambda(t; t_0) - t|/(1+t) = 0.$$

PROOF. Since $L(t)$ varies slowly, $n(t) = L^{-1}(t)$ increases rapidly:

$$\lim_{\lambda\to\infty} n(\alpha\lambda)/n(\lambda) = 0, \quad \text{if} \quad 0 < \alpha < 1.$$

Therefore, for all $\delta > 0$,

$$\limsup_{\lambda\to\infty}\sup_{t,t_0;t\ge ct_0,t>\delta} n(\lambda t_0)/n(\lambda t) \le \lim_{\lambda\to\infty}\sup_{t>\delta} n(\lambda t/c)/n(\lambda t) = 0.$$

Thus for all sufficiently large λ we can assume

$$n(\lambda t_0)/n(\lambda t) < \tfrac{1}{2} \quad \text{if} \quad t \ge ct_0, \quad t > \delta.$$

So we see

$$T_\lambda(t; t_0)/t = L(n(\lambda t) - n(\lambda t_0))/(\lambda t)$$
$$> L(\tfrac{1}{2}n(\lambda t))/L(n(\lambda t)) \quad \text{if} \quad t \ge ct_0, \quad t > \delta.$$

Since $\lim_{x\to\infty} L(\tfrac{1}{2}x)/L(x) = 1$, we have

$$\liminf_{\lambda\to\infty}\inf_{t,t_0;t\ge ct_0,t>\delta} T_\lambda(t; t_0)/t \ge \liminf_{\lambda\to\infty}\inf_{t>\delta} L(\tfrac{1}{2}n(\lambda t))/L(n(\lambda t)) = 1.$$

Taking into account that $T_\lambda(t; t_0) \le t$, we conclude

$$\lim_{\lambda\to\infty}\sup_{t,t_0;t\ge ct_0,\delta} |T_\lambda(t; t_0) - t|/(1+t) = 0.$$

It remains to show

(5.5) $$\lim_{\delta\downarrow 0}\limsup_{\lambda\to\infty}\sup_{t,t_0;t\ge ct_0,t\le\delta} |T_\lambda(t; t_0) - t|/(1+t) = 0$$

However, (5.5) is clear because $T_\lambda(t; t_0) \le t \le \delta$ if $t \le \delta$.

PROPOSITION 5.3. *Let* $p(t)$, $t > 0$ *be a bounded function with bounded first derivative. Then for every* $s > 0$,

(5.6) $$\lim_{\lambda\to\infty} E_x\left[\int_{t_0}^{\infty} e^{-st} p(t) f(X_{n(\lambda t)-n(\lambda t_0)}) n'(\lambda t)\, dt\right]$$
$$= C\int_{t_0}^{\infty} e^{-st}\{sp(t) - p'(t)\} t\, dt, \quad t_0 \ge 0, \quad x \in S,$$

the convergence being uniform for $(t_0, x) \in [0, \infty) \times \{x; f(x) \neq 0\}$.

PROOF. By integration by parts, the left-hand side of (5.6) equals

$$\lim_{\lambda\to\infty} \int_{t_0}^{\infty} e^{-st}\{sp(t) - p'(t)\} E_x[\tilde{A}_\lambda(n(\lambda t) - n(\lambda t_0))]\, dt.$$

Taking into account that $sp(t) - p'(t)$ is bounded, we see that this equals

$$C \lim_{\lambda\to\infty} \int_{t_0}^{\infty} e^{-st}\{sp(t) - p'(t)\} E_x[T_\lambda(t; t_0)]\, dt$$

thanks to Lemma 5.1. By Lemma 5.2, we have

$$\lim_{\lambda\to\infty} \int_{ct_0}^{\infty} \{sp(t) - p'(t)\} E_x[T_\lambda(t; t_0)\}\, dt$$

$$= \int_{ct_0}^{\infty} e^{-st}\{sp(t) - p'(t)\} t\, dt, \quad \text{uniformly in } t_0,$$

for every $c > 1$. Thus it remains to show that

$$\Delta(c) = \limsup_{\lambda\to\infty} \sup_{t_0 \geq 0} \left| \int_{t_0}^{ct_0} e^{-st}\{sp(t) - p'(t)\} E_x[T_\lambda(t; t_0)]\, dt \right|$$

tends to 0 as $c \downarrow 1$. Since $T_\lambda(t; t_0) \leq t$ by (5.3), we have

$$\begin{aligned} \Delta(c) &\leq \| sp(t) - p'(t) \|_\infty \sup_{t_0 \geq 0} \int_{t_0}^{ct_0} e^{-st} t\, dt \\ &\leq \| sp(t) - p'(t) \|_\infty e^{-st_0}(ct_0)(ct_0 - t_0) \\ &\leq \| sp(t) - p'(t) \|_\infty\, 2c(c-1)s^{-2}. \end{aligned}$$

Therefore, $\Delta(c) \to 0$ as $c \downarrow 1$, which completes the proof.

REMARK 5.4. In the proof of Lemma 3.1, we used integration by parts and we now reversed it in the above. The aim of this procedure is to handle with the appearance of Dirac's δ-measure in the limit. (This also explains why we have $p'(t)$ in the right-hand side of (5.6).)

We return to the proof of Proposition 3.2. For $j = 1$, the assertion follows immediately from Proposition 5.3 (set $p(t) = 1$). Assume that (3.8) holds for $j - 1$. Then, by definition (3.2),

$$\lim_{\lambda\to\infty} \phi_\lambda^{(j)}(s_1, \cdots, s; x, t)$$

$$= \lim_{\lambda\to\infty} \int_t^{\infty} \exp(-s_1\xi) E_x[f(X_{n(\lambda\xi)-n(\lambda t)}) \phi_\lambda^{(j-1)}(s_2, \cdots, s_j; X_{n(\lambda\xi)-n(\lambda t)}, \xi)] n'(\lambda\xi)\, d\xi$$

$$= \lim_{\lambda\to\infty} \int_t^{\infty} \exp(-s_1\xi) E_x[f(X_{n(\lambda\xi)-n(\lambda t)})] C^{j-1} \phi^{(j-1)}(s_2, \cdots, s_j; \xi) n'(\lambda\xi)\, d\xi.$$

Applying Proposition 5.3 again with $p(t) = \phi^{(j-1)}$, we see that this equals

$$C^j \int_t^\infty \exp(-s_1\xi)\left\{s_1\phi^{(j-1)}(s_2, \cdots, s_j; \xi) - \frac{d}{d\xi}\phi^{(j-1)}(s_2, \cdots, s_j; \xi)\right\}\xi \, d\xi$$

which equals $C^j\phi^{(j)}$ by the definition (3.5). Thus the proof is completed.

6. The case of Markov processes with discrete parameter. Let X_k, $k = 1, 2, \cdots$ be a temporary homogeneous Markov process with values in a measurable space $(S, \mathscr{B})$ and $f(x) \geq 0$ a bounded measurable function on S. Then, define $Y_t = X_{[t]}$ where $[t]$ denotes the maximum integer not exceeding t. Although Y_t is not temporally homogeneous, the method we used in the above can be applied with a little modification and we have the following.

THEOREM 6.1. *Let $L(t)$ be a slowly varying function tending to ∞ as t goes to ∞, and $n(t)$ $(t \geq 0)$ be a function such that $L(n(t)) \sim t$ as $t \to \infty$. If*

(6.1) $$\lim_{N\to\infty} E_x[\textstyle\sum_{k=0}^N f(X_k)]/L(N) = C > 0, \quad \textit{for} \quad x \in S,$$

the convergence being uniform for $x \in \{x; f(x) \neq 0\}$, then

$$\frac{1}{\lambda}\textstyle\sum_{k=0}^{[n(\lambda t)]} f(X_k) \to_{\text{f.d.}} C\ell(M^{-1}(t), 0), \quad as \quad \lambda \to \infty.$$

Examples for random walks satisfying (6.1) are easily obtained using local limit theorems such as Corollary 3 of C. Stone [6] (see also [7]).

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REFERENCES

[1] BINGHAM, N. H. (1971). Limit theorems for occupation times of Markov processes. *Z. Wahrsch. verw. Gebiete* **17** 1–22.
[2] DARLING, D. A. and KAC, M. (1957). Occupation times for Markov processes. *Trans. Amer. Math. Soc.* **84** 444–458.
[3] KASAHARA, Y. (1981). Two limit theorems for occupation times of Markov processes. *Japan. J. Math.* **7** 291–300.
[4] KASAHARA, Y. and KOTANI, S. (1979). On limit processes for a class of additive functional of recurrent diffusion processes. *Z. Wahrsch. verw. Gebiete* **49** 133–153.
[5] LAMPERTI, J. W. (1964). On extreme order statistics, *Ann. Math. Statist.* **35** 1726–1737.
[6] STONE, C. (1965). A local limit theorem for nonlattice multidimensional distribution functions. *Ann. Math. Statist.* **36** 546–551.
[7] STONE, C. (1967). On local and ratio limit theorems. *Proc. 5th Berkeley Sympos. Math. Statis. Probab.* **2** 217–224. Univ. Calif.
[8] WATANABE, S. (1980). A limit theorem for sums of non-negative i.i.d. random variables with slowly varying tail probabilities. *Proc. 5th International Conf. Multivariate Anal.* 249–261. North-Holland, Amsterdam.

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