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LOCAL LIMIT THEOREMS FOR SAMPLE EXTREMES

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A local limit theorem for maxima of i.i.d. random variables is proved. Also it is shown that under the so-called von Mises' conditions the density of the normalized maximum converges to the limit density in L_p $(0 < p \leq \infty)$ provided both the original density and the limit density are in L_p. Finally an occupation time result is proved. The methods of proof are different from those used for the corresponding results concerning partial sums.

1. Introduction. In extreme value theory not much is known about the quality of convergence in the case of weak convergence of the normalized maximum of a sample to one of the limit distributions. Attempts to establish uniform rates of convergence have been sporadic and not completely satisfactory. There is an unpublished manuscript by A. A. Balkema dealing with the general case and specific cases have been considered by P. Hall (1978) and W. J. Hall and J. Wellner (1978). Large deviation results can be found in de Haan and Hordijk (1972) and C. W. Anderson (1978). Our focus in this paper is on local limit results.

As usual, the results for maxima parallel those for sums but the methods are completely different. Also, in the theory of partial sums of i.i.d. random variables, local limit results are frequently related to occupation time theorems (Breiman, 1968, page 229; Darling and Kac, 1957). In extreme value theory this relation seems to be rather weak.

In Section 2 we give some preliminaries. Section 3 discusses the limit behavior of the probability the normalized sample maximum is in a certain interval. Section 4 treats density convergence in the uniform and L_p metrics. Finally in Section 5 we derive an occupation time result using the structure of extremal processes.

2. Preliminaries. We first recall some of the properties of distribution functions in the domain (D) of one of the extreme value distributions Φ_α, Ψ_α, and Λ (cf. Gnedenko, 1943). The properties are formulated in a way slightly different from de Haan, 1970.

If $F \in D(\Phi_\alpha)$, then the function $-\log F$ is regularly varying with exponent $-\alpha$ and

$$-\log F(x) = c(x)\exp\left\{-\int_1^x \frac{a(t)}{t}\,dt\right\}$$

with

$$a(x) = c(x)\left\{\int_1^\infty s^{-1}(-\log F(s))\,ds\right\}^{-1}$$

$$= \left[\int_x^\infty t^{-1}\{-\log F(t)\}\,dt\right]^{-1}\{-\log F(x)\} \quad \text{and} \quad \lim_{x\to\infty} a(x) = \alpha.$$

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If the von Mises condition $\lim_{x\to\infty}\{-\log F(x)\}^{-1}xF'(x) = \alpha$ holds, then $F \in D(\Phi_\alpha)$ and

$$-\log F(x) = \{-\log F(1)\}\exp\left\{-\int_1^x \frac{a(t)}{t}\,dt\right\}$$

with $a(x) = \{-F(x)\log F(x)\}^{-1}xF'(x)$.

We will further use the property that the positive norming constants $\{a_n\}$ in the limit relation $\lim_{n\to\infty}F^n(a_nx) = \Phi_\alpha(x)$ are regularly varying with exponent α^{-1} (being defined as the inverse function of $-1/\log F$ at the point n).

If $F \in D(\Lambda)$ and $F(x) < 1$ for all x then

$$-\log F(x) = c(x)\exp\left\{-\int_1^x \frac{a(t)}{f_1(t)}\,dt\right\}$$

with

$$a(x) + 1 = c.2c(x) = 2\left\{\int_x^\infty (-\log F(t)\,dt)\right\}^{-2}(-\log F(x))\left\{\int_x^\infty\int_y^\infty (-\log F(t))\,dt\,dy\right\},$$

$$f_1(x) = \left\{\int_x^\infty\int_y^\infty (-\log F(t))\,dt\,dy\right\}\left\{\int_x^\infty (-\log F(t))\,dt\right\}^{-1}$$

and

$$\lim_{x\to\infty}a(x) = 1, \qquad \lim_{x\to\infty}f_1'(x) = 0.$$

We also need $f_2(x) = \int_x^\infty (-\log F(t))\,dt/(-\log F(x))$, which is asymptotic to f_1. If the von Mises type condition

$$\lim_{x\to\infty}\frac{F'(x)\left(\int_x^\infty (-\log F(t))\,dt\right)}{\{-\log F(x)\}^2} = 1$$

holds, then $F \in D(\Lambda)$ and

$$-\log F(x) = (-\log F(1))\exp\left\{-\int_1^x \frac{a(t)}{f_0(t)}\,dt\right\}$$

with $a(x) = 1/F(x)$ and $f_0(x) = (-\log F(x))/F'(x)$. (The von Mises type condition given here is implied by the well-known von Mises condition given on page 112 of de Haan, 1970; see also von Mises, 1936). Any of the functions $f(f_0, f_1$, or $f_2)$ is called an auxiliary function and satisfies $\lim_{t\to\infty}f(t + xf(t))/f(t) = 1$ uniformly on finite x-intervals. In the limit relation $\lim_{n\to\infty}F^n(a_nx + b_n) = \Lambda(x)$ we may then take b_n as the inverse function of $1/(-\log F)$ at the point n and $a_n = f(b_n)$. We will also need the property that $\{a_n\}$ is slowly varying.

The following lemma on regularly varying functions is well-known.

LEMMA 1. *Suppose U is regularly varying with exponent α. Take $\varepsilon > 0$. Then there exists t_0 such that for $x \geq 1$, $t \geq t_0$*

$$(1-\varepsilon)x^{\alpha-\varepsilon} < \frac{U(tx)}{U(t)} < (1+\varepsilon)x^{\alpha+\varepsilon}.$$

PROOF. The inequalities follow easily from the representation for regularly varying functions (de Haan, 1970. Theorem 1.2.2; Feller, 1971, VIII. 9).

REMARK. Obviously, one can also prove that for any $\varepsilon_1, \varepsilon_2 > 0$ eventually

$$(1-\varepsilon_1)x^{\alpha-\varepsilon_2} < \frac{U(tx)}{U(t)} < (1+\varepsilon_1)x^{\alpha+\varepsilon_2}.$$

Next we prove an inequality which is crucial in our attack.

LEMMA 2. *Suppose $F \in D(\Lambda)$ with auxiliary function f and $\varepsilon > 0$. There exists a t_0 such that for $x \geq 0$, $t \geq t_0$*

$$(1-\varepsilon)\left\{\frac{-\log F(t)}{-\log F(t+xf(t))}\right\}^{-\varepsilon} \leq \frac{f(t+xf(t))}{f(t)} \leq (1+\varepsilon)\left\{\frac{-\log F(t)}{-\log F(t+xf(t))}\right\}^{\varepsilon}$$

and for $x < 0$, $t + xf(t) \geq t_0$

$$(1-\varepsilon)\left\{\frac{-\log F(t+xf(t))}{-\log F(t)}\right\}^{-\varepsilon} < \frac{f(t+xf(t))}{f(t)} < (1+\varepsilon)\left\{\frac{-\log F(t+xf(t))}{-\log F(t)}\right\}^{\varepsilon}.$$

PROOF. It is known (Balkema and de Haan, 1972) that there exists F_1 such that $-\log F_1$ has an increasing (negative) density and $-\log F(t) \sim -\log F_1(t)$ $(t \to \infty)$. Since F and F_1 have the same auxiliary function it is sufficient to prove the result for F_1. Let U be the inverse function of $1/\{-\log F_1(x)\}$, then $f \circ U$ is slowly varying (de Haan, 1974). We apply Lemma 1 for this function $f \circ U$. For $z > 0$ replace tx by $1/\{-\log F_1(U(t)) + zf(U(t))\}$ in the statement of Lemma 1. Then for $U(t) \geq t_0$, $z > 0$

$$(1-\varepsilon)\{-t\log F_1(U(t)+zf(U(t)))\}^{+\varepsilon} \leq \frac{f(U(t)+zf(U(t)))}{f(U(t))}$$
$$\leq (1+\varepsilon)\{-t\log F_1(U(t)+zf(U(t)))\}^{-\varepsilon}.$$

Introduce a new variable $s = U(t)$ to get the first statement of the lemma. For the second statement replace xt by t and t by $1/\{-\log F_1(u(t) + zf(U(t)))\}$.

REMARK. Here again we may take the epsilons in the exponents different from the other ones.

We now use Lemma 1 to prove a needed variant. This lemma can be found with a different proof in Pickands (1968).

LEMMA 3. *Suppose $V: \mathbb{R}^+ \to \mathbb{R}$ is measurable and satisfies*

$$\lim_{t\to\infty} \frac{V(tx) - V(t)}{L(t)} = \log x$$

for some positive (necessarily slowly varying) function L and all $x > 0$. Take $\varepsilon > 0$. Then for $x \geq 1$, $t \geq t_0$

$$(1-\varepsilon)^2 \frac{1 - x^{-\varepsilon}}{\varepsilon} - \varepsilon < \frac{V(tx) - V(t)}{L(t)} < (1+\varepsilon)^2 \frac{x^{\varepsilon} - 1}{\varepsilon} + \varepsilon.$$

PROOF. We may assume L is such that $V(x) = \int_1^x (L(s))/s\, ds + L(x)$ (de Haan, 1970, page 34, de Haan and Resnick, 1979, Proposition 2). Then

$$\frac{V(tx) - V(t)}{L(t)} = \int_1^x \frac{L(ts)}{L(t)} \frac{ds}{s} + \frac{L(tx)}{L(t)} - 1.$$

Apply Lemma 1 for $U = L$; then for $x \geq 1$, $t \geq t_0$

$$(1-\varepsilon)\frac{1 - x^{-\varepsilon}}{\varepsilon} + (1-\varepsilon)x^{-\varepsilon} - 1 < \frac{V(tx) - V(t)}{L(t)} < (1+\varepsilon)\frac{x^{\varepsilon} - 1}{\varepsilon} + (1+\varepsilon)x^{\varepsilon} - 1$$

and this is the statement of Lemma 3.

REMARK. Here also one can prove the refinement

$$(1-\varepsilon_1)\frac{1-x^{-\varepsilon_2}}{\varepsilon_2}-\varepsilon_1<\frac{V(tx)-V(t)}{L(t)}<(1+\varepsilon_1)\frac{x^{\varepsilon_2}-1}{\varepsilon_2}+\varepsilon_1.$$

3. Local limit theorem. We prove statements of the following form. Suppose $X_1, X_2, \cdots$ are i.i.d. random variables with common df F. Set $M_n = \bigvee_{i=1}^n X_i$ for $n = 1, 2, \cdots$. If $\lim_{n\to\infty} F^n(a_n x + b_n) = G(x)$ for all x then under an additional smoothness condition $\lim_{n\to\infty} a_n P\left\{a_n\left|\frac{M_n - b_n}{a_n} - x\right| \le h\right\} = 2hG'(x)$ uniformly for all x.

THEOREM 1. a) *If* $\lim_{n\to\infty} F^n(a_n x + b_n) = \Lambda(x) := \exp\{-e^{-x}\}$ *for all* x, $a_n \to \infty$ *and for all* x

$$\lim_{t\to\infty} f_2(t+x) - f_2(t) = \lim_{t\to\infty}\left\{\frac{\int_{t+x}^{\infty}(-\log F(s))\,ds}{-\log F(t+x)} - \frac{\int_{t}^{\infty}(-\log F(s))\,ds}{-\log F(t)}\right\} = 0,$$

then for all $h > 0$

$$\lim_{n\to\infty} a_n P\{a_n^{-1}(M_n - b_n) \in (x, x + a_n^{-1}h]\}$$
$$= \lim_{n\to\infty} a_n\{F^n(a_n x + b_n + h) - F^n(a_n x + b_n)\}$$
$$= h\Lambda'(x) \textit{ uniformly for all } x.$$

b) *If* $\lim_{n\to\infty} F^n(a_n x) = \Phi_\alpha(x)$ *and for all* x

$$\lim_{t\to\infty} t\left\{\frac{-\log F(t+x)}{\int_{t+x}^{\infty} s^{-1}(-\log F(s))\,ds} - \frac{-\log F(t)}{\int_{t}^{\infty} s^{-1}(-\log F(s))\,ds}\right\} = 0,$$

then for all $h > 0$

$$\lim_{n\to\infty} a_n P\{a_n^{-1} M_n \in (x, x + a_n^{-1}h]\} = \lim_{n\to\infty} a_n\{F^n(a_n x + h) - F^n(a_n x)\}$$
$$= h\Phi_\alpha'(x) \textit{ uniformly for all } x.$$

c) *If* $\lim_{n\to\infty} F^n(a_n x + b_n) = \Lambda(x)$ *for all* x, $a_n \to 0$, $F(x) < 1$ *for all* x *and for all* x

$$\lim_{t\to\infty} \frac{1}{f_1(t)}\left\{\frac{f_1(t+x)}{f_2(t+x)} - \frac{f_1(t)}{f_2(t)}\right\} = 0$$

where f_1 *and* f_2 *are defined in Section* 2, *then for all* $h > 0$

$$\lim_{n\to\infty} a_n^{-1} P\{a_n^{-1}(M_n - b_n) \in (x, x + a_n h]\}$$
$$= \lim_{n\to\infty} a_n^{-1}\{F^n(a_n x + b_n + a_n^2 h) - F^n(a_n x + b_n)\} = h\Lambda'(x)$$

uniformly for all x.

REMARK. For convenience we only treat the cases where $F(x) < 1$ for all x. The case where $F^n(a_n x + b_n) \to \Lambda(x)$ and a_n tends to a finite positive constant which we suppose without loss of generality to be one (e.g. F is exponential) can be discussed as follows: If $a_n \to 1$ then $F^n(x + b_n) \to \Lambda(x)$. Set $F^* = F \circ \log$ and $\alpha_n = \log b_n$. Then for $y > 0$ $(F^*(\alpha_n y))^n \to \Phi_1(y)$ and Part (b) of Theorem 1 is applicable: we obtain

$$\lim_{h\to\infty} e^{b_n} P[M_n - b_n \in (y, \log(e^y + he^{-b_n}))] = he^{-y}\Lambda'(y)$$

uniformly in y provided that

$$\lim_{y\to\infty} e^y\{f_2(y + xe^{-}y) - f_2(y)\} = 0$$

locally uniformly in x. (The von Mises condition (4.2) below implies this last condition.) If we replace $\log(e^y + he^{-b_n})$ by $y + he^{-b_n}$, the uniformity in y is no longer guaranteed.

PROOF. a) We have to prove

$$a_n[F^n(a_nx_n + b_n + h) - F^n(a_nx_n + b_n)] - h\Lambda'(x_n) \to 0$$

for any sequence $\{x_n\}(n \to \infty)$. Considering subsequences we may assume that both $\{x_n\}$ and $\{a_nx_n + b_n\}$ converge where $\{n\}$ now represents a subsequence of the integers. First consider the case $a_nx_n + b_n \to c < \infty$ (hence $x_n \to -\infty$). Then both $F^n(a_nx_n + b_n + h)$ and $F^n(a_nx_n + b_n)$ converge geometrically fast to zero; since a_n is slowly varying, the result follows.

Next consider the case $x_n \to -\infty$ and $a_nx_n + b_n \to\infty$ $(n \to \infty)$.

Using the simple inequality $u < -\log(1 - u)$ for $0 < u < 1$ twice we get

$$\begin{aligned} 0 &\le F^n(a_nx_n + b_n + h) - F^n(a_nx_n + b_n) \\ &\le F^n(a_nx_n + b_n + h)\{\log F^n(a_nx_n + b_n + h) - \log F^n(a_nx_n + b_n)\} \\ &= F^n(a_nx_n + b_n + h)(-\log F^n(a_nx_n + b_n))\left\{1 - \frac{-\log F(a_nx_n + b_n + h)}{-\log F(a_nx_n + b_n)}\right\} \\ &\le F^n(a_nx_n + b_n + h)\{-\log F^n(a_nx_n + b_n)\}\{-\log(-\log F(a_nx_n + b_n + h)) \\ &\quad + \log(-\log F(a_nx_n + b_n))\}. \end{aligned}$$

First we want to change the second factor. According to the representation of Section 2

$$\frac{-\log F^n(a_nx_n + b_n)}{-\log F^n(a_nx_n + b_n + h)} = \frac{c(a_nx_n + b_n)}{c(a_nx_n + b_n + h)} \exp\left\{\int_0^h \left(\frac{a(a_nx_n + b_n + s)}{f_1(a_nx_n + b_n + s)}\right) ds\right\},$$

which tends to 1 as $n \to \infty$ since $f_1 \to \infty$. So $-\log F^n(a_nx_n + b_n) < -(1 + \varepsilon) \log F^n(a_nx_n + b_n + h)$ for sufficiently large n.

The third factor multiplied by a_n can be written as

$$(*) \quad a_n[\log c(b_n + a_nx_n + h) - \log c(b_n + a_nx_n)] + a_n \int_{x_n}^{x_n + \frac{h}{a_n}} \left(\frac{a(b_n + sa_n)f(b_n)}{f_1(b_n + sa_n)}\right) ds.$$

The first term of (*) is asymptotic to

$$cf(b_n)\{c(b_n + a_nx_n + h) - c(b_n + a_nx_n)\}$$

which by Lemma 2 is at most

$$f(b_n + a_nx_n)\{c(b_n + a_nx_n + h) - c(b_n + a_nx_n)\}\left\{\frac{-\log F(b_n + a_nx_n)}{-\log F(b_n)}\right\}^{\varepsilon} (1 + \varepsilon)$$

for sufficiently large n. Now

$$\begin{aligned} f(t)(c(t + h) - c(t)) &\sim \frac{f_2(t)}{c}\left(\frac{f_1(t + h)}{f_2(t + h)} - \frac{f_1(t)}{f_2(t)}\right) \\ &= c^{-1}\frac{f_1(t + h)}{f_2(t + h)}(f_2(t) - f_2(t + h)) + c^{-1}(f_1(t + h) - f_1(t)). \end{aligned}$$

The first term of this expression tends to zero as $t \to \infty$ by assumption, the second one by the conditions for the domain of attraction ($\lim_{t\to\infty} f_1'(t) = 0$).

For the second term of (*) we use Lemma 2 again and get

$$\begin{aligned} a_n \int_{x_n}^{x_n+h/a_n} \left(\frac{a(b_n + sa_n)f(b_n)}{f(b_n + sa_n)}\right) ds &\le \frac{1 + \varepsilon}{1 - \varepsilon} a_n \int_{x_n}^{x_n+h/a_n} \left(\frac{-\log F(b_n + sa_n)}{-\log F(b_n)}\right)^{\varepsilon} ds \\ &\le \left(\frac{1 + \varepsilon}{1 - \varepsilon}\right) h \left(\frac{-\log F(b_n + x_na_n + h)}{-\log F(b_n)}\right)^{\varepsilon}. \end{aligned}$$

Collecting the inequalities, we get

$$0 \le a_n[F_n(a_nx_n + b_n + h) - F^n(a_nx_n + b_n)]$$
$$\le \frac{(1+\varepsilon)^2}{1-\varepsilon}(h+\varepsilon)\left\{\frac{-\log F(a_nx_n + b_n + h)}{-\log F(b_n)}\right\}^{1+\varepsilon} \exp\left[-\{-n \log F(a_nx_n + b_n + h)\}\right]$$

which tends to zero as $n \to \infty$ since $n(-\log F(a_nx_n + b_n + h_n)) \sim \{-\log F(b_n)\}^{-1}$ $\{-\log F(a_nx_n + b_n)\} \to \infty$.

For the case $x_n \to c > -\infty$ (hence $a_n x_n + b_n \to \infty$) one quickly sees, since $a_n \to \infty$ implies $F^n(a_nx_n + b_n + h) = F^n(a_n(x_n + ha_n^{-1}) + b_n) \to \Lambda(c)$, that in fact

$$a_n[F^n(a_nx_n + b_n + h) - F^n(a_nx_n + b_n)]$$
$$\sim F^n(a_nx_n + b_n + h)\{-\log F^n(a_nx_n + b_n)\}$$
$$\cdot a_n\{-\log(-\log F(a_nx_n + b_n + h)) + \log(-\log F(a_nx_n + b_n))\} \quad \text{as} \quad n \to \infty.$$

Now the product $F^n(a_nx_n + b_n + h)\{-\log F^n(a_nx_n + b_n)\}$ tends to the density $\Lambda'(c)$. For the third factor proceed as before, using the representation of Section 2 and the fact that $f(t + xf(t))/f(t) \to 1$ uniformly on finite intervals. It follows that the third factor tends to h.

b) First consider the case $a_nx_n \to c < \infty$ (hence $x_n \to$ const. ≤ 0); then from the convergence of $F^n(a_nx_n + h)$ and $F^n(a_nx_n)$ to zero at a geometric rate and regular variation of $\{a_n\}$, the result follows. Next consider the case $x_n \downarrow 0$ and $a_nx_n \to \infty$ $(n \to \infty)$. As under a), we get

$$0 \le a_n[F^n(a_nx_n + h) - F^n(a_nx_n)]$$
$$\le (1+\varepsilon)F^n(a_nx_n + h)\{-\log F^n(a_nx_n + h)\}$$
$$\cdot\{-a_n\log(-\log F(a_nx_n + h)) + a_n\log(-\log F(a_nx_n))\}.$$

The last factor multiplied by x_n can be written as

$$a_nx_n[\log c(a_nx_n + h) - \log c(a_nx_n)] + a_nx_n \int_{anxn}^{a_nx_n+h} \frac{a(s)}{s}\, ds.$$

Since $\lim_{s\to\infty}a(s) = \alpha$, the last term is easily seen to converge to αh. The first term is asymptotic to (const.) $a_nx_n\{c(a_nx_n + h) - c(a_nx_n)\}$ and hence converges to zero by the condition of the theorem. Finally we use Lemma 1 and the fact that $a_nx_n > 1$ for sufficiently large n to get

$$x_n^{-1}F^n(a_nx_n + h)\{-\log F^n(a_nx_n + h)\}$$
$$= x_n^{-1}\left(\frac{-\log F(a_nx_n + h)}{-\log F(a_n)}\right)\exp\left\{-\left(\frac{-\log F(a_nx_n + h)}{-\log F(a_n)}\right)\right\}$$
$$< (1+\varepsilon)x_n^{-1}\left(x_n + \frac{h}{a_n}\right)^{-\alpha-\varepsilon}\exp\left\{-(1-\varepsilon)\left(x_n + \frac{h}{a_n}\right)^{-\alpha+\varepsilon}\right\}$$
$$< (1+\varepsilon)x_n^{-1-\alpha-\varepsilon}\exp\{-(1-\varepsilon)x_n^{-\alpha+\varepsilon}(1+h)^{-\alpha+\varepsilon}\} \to 0$$

as $n \to \infty$ since $x_n \downarrow 0$.

Finally if $x_n \to c > 0$ we get as under a)

$$a_n\{F^n(a_nx_n + h) - F^n(a_nx_n)\} \sim F^n(a_nx_n + h)\{-\log F^n(a_nx_n)\}$$
$$\cdot[a_n\{-\log(-\log F(a_nx_n + h)) + \log(-\log F(a_nx_n))\}].$$

The first two factors tend to $c\alpha^{-1}\Phi_\alpha'(c)$ and for the third factor use the representation of Section 2 to see that it tends to αhc^{-1}.

c) This part of the proof is completely analogous to part a).

COROLLARY. *Under the conditions of part* a *one has*

$$\lim_{n\to\infty} a_n P\{x < M_n - b_n \le x + h\} = he^{-1}$$

and under the conditions of Part b

$$\lim_{n\to\infty} a_n P\{x < M_n - a_n \le x + h\} = \alpha h e^{-1}.$$

We conclude this section by showing that Theorem 1 is not true for every F in the domain of attraction of Λ with $|\log a_n| \to \infty$.

Take any positive differentiable function f with $\lim_{t\to\infty} f'(t) = 0$, $\lim_{t\to\infty} f(t) = \infty$. Define b_n by $n = \exp\left\{\int_0^{b_n} \frac{ds}{f(s)}\right\}$ for $n \ge 1$. We may construct a continuously differentiable function c_0 which is constructed as follows:

$$c_0(b_1) = 0, \qquad c_0(b_1 + 1) = \{f(b_1)\}^{-1}$$

and increasing in between. Furthermore for $b_1 + 1 < t < b'$ let $c_0'(t) = -\{f(t)\}^{-1}$, where b' is the value for which the function c_0 vanishes. Such b' exists since $\int_0^\infty \frac{dt}{f(t)} = \infty$. Take n such that $b_{n-1} < b' \le b_n$ and define $c_0(t) = 0$ for $b' \le t < b_n$ (with a small adaptation to keep c_0 continuously differentiable). Again we take $c_0(b_n + 1) = \{f(b_n)\}^{-1}$ and increasing in between, etc. This way we get:

1) $c_0(t) + \int_0^t \frac{ds}{f(s)}$ is non-decreasing since $f(t)c_0'(t) \ge -1$,

2) $\lim_{t\to\infty} c_0(t) = 0$ since $\lim_{t\to\infty} f(t) = \infty$ and

3) $f(b_n)\{c_0(b_n + 1) - c_0(b_n)\} = 1$ for infinitely many n.

Define now $F(x) = \exp\left\{-\exp\left[-c_0(x) - \int_0^x \frac{ds}{f(s)}\right]\right\}$; then by the representation in de Haan, 1970, page 92, $\lim_{n\to\infty} F^n(a_n x + b_n) = \Lambda(x)$ for all x, where $a_n \to \infty$. But (choose $x = 0$ and $h = 1$ in the corollary)

$$a_n[F^n(b_n + 1) - F^n(b_n)] \sim e^{-1} f(b_n)[c_0(b_n + 1) - c_0(b_n)] + e^{-1} a_n \int_0^{1/a_n} \frac{f(b_n)\, ds}{f(b_n + sf(b_n))}$$

and this converges to $2e^{-1}$ if n goes through the subsequence obtained above. If the corollary would be true, the limit should be just e^{-1}.

4. Density convergence. If we assume von Mises type conditions, we can prove the density of the normalized maximum converges to the density of the appropriate extreme value distribution in the L_p metric, $p \le \infty$ provided both F' and the limit extreme value density are in the space L_p.

THEOREM 2. *Let* $\{X_n, n \ge 1\}$ *be* i.i.d. *with df* $F(x)$ *which is absolutely continuous with bounded density* F'. *Let* $M_n = \vee_1^n X_i$ *and let* g_n *be the density of* M_n *normalized as described below.*

a) *If* $F'(x) > 0$ *for all* x *in a neighborhood of* ∞ *and*

$$\lim_{x\to\infty} \frac{xF'(x)}{-\log F(x)} = \alpha \tag{4.1}$$

then $g_n(x) = nF^{n-1}(a_n x)F'(a_n x)a_n \to \Phi_\alpha'(x)$ *uniformly in* x.

b) *If* $F'(x) > 0$ *for all* x *in a neighborhood of* ∞ *and*

$$\lim_{x\to\infty} \frac{F'(x)\int_x^\infty (-\log F(t))\,dt}{(\log F(x))^2} = 1 \tag{4.2}$$

then

$$g_n(x) = nF^{n-1}(a_n x + b_n)F'(a_n x + b_n)a_n \to \Lambda'(x)$$

uniformly in x.

PROOF.
a) We need show $g_n(x_n) \to \Phi_\alpha'(x_0)$ when $x_n \to x_0$ and $n \to \infty$ through some subsequence of the integers for the cases: (i) $x_n \to x_0 \in (0, \infty)$, (ii) $x_n \to \infty$, (iii) $x_n \to 0$, $a_n x_n \to \infty$, (iv) $x_n \to x_0 \le 0$, $a_n x_n \le K < \infty$.

Case (i). Write $g_n(x_n) = F^{n-1}(a_n x_n)\left(\dfrac{a_n x_n F'(a_n x_n)}{-\log F(a_n x_n)}\right)\cdot n(-\log F(a_n x_n))/x_n$. Now use the fact that both $F^n(a_n x) \to \Phi_\alpha(x)$ (which follows from the von Mises condition (4.1)—see de Haan, 1970, page 109) and $n(-\log F(a_n x)) \to x^{-\alpha}$ uniformly for $x \ge x_0$ (since we have monotone functions converging to continuous limits). So $g_n(x_n) \to \Phi_\alpha(x_0)\alpha x_0^{-\alpha}/x_0 = \Phi_\alpha'(x_0)$ as $n \to \infty$.

Case (ii). Since $n(-\log F(a_n x_n)) \to 0$, we see by writing g_n as in (i) that $g_n(x_n) \to 0$.

Case (iii). As above $g_n(x_n) \sim \alpha e^{-(n-1)(-\log F(a_n x_n))} n(-\log F(a_n x_n))/x_n$. Since $n(-\log F(a_n x_n)) \sim -\log F(a_n x_n)/(-\log F(a_n))$ we have by Lemma 1 for sufficiently large n: $(1-\varepsilon)x_n^{-(\alpha-\varepsilon)} \le n(-\log F(a_n x_n)) \le (1+\varepsilon)x_n^{-(\alpha+\varepsilon)}$. Therefore

$$\limsup_{n\to\infty} g_n(x_n) \le \limsup_{n\to\infty} \alpha e^{-\left(\frac{n-1}{n}\right)(1-\varepsilon)x_n^{-(\alpha-\varepsilon)}}(1+\varepsilon)x_n^{-(\alpha+\varepsilon+1)} = 0.$$

Case (iv). Since F' is assumed bounded, $g_n(x_n) = O(nF^{n-1}(K)a_n)$. Since na_n is regularly varying and $F^{n-1}(K) \to 0$ geometrically fast we have $nF^{n-1}(a_n x_n)a_n \to 0$.

b) Again we show $g_n(x_n) \to \Lambda'(x_0)$ when $x_n \to x_0$ and consider cases (i) $x_0 \in (-\infty, \infty)$, (ii) $x_0 = +\infty$, (iii) $x_0 = -\infty$, $a_n x_n + b_n \to +\infty$, (iv) $x_0 = -\infty$, $a_n x_n + b_n \le K$. Write

$$g_n(x_n) \sim F^{n-1}(a_n x_n + b_n)\left(\frac{f_0(b_n)}{f_0(a_n x_n + b_n)}\right) n(-\log F(a_n x_n + b_n))$$

where as in Section 2 $a_n = f_0(b_n)$, $f_0(x) = (-\log F(x))/F'(x)$.

Case (i). The result follows immediately since $F^{n-1}(a_n x + b_n) \to \Lambda(x)$ uniformly on $(-\infty, \infty)$, $f_0(b_n)/f_0(b_n + xf(b_n)) \to 1$ uniformly on finite intervals and $n(-\log F(a_n x + b_n)) \to e^{-x}$ uniformly on intervals bounded away from $-\infty$.

Case (ii). $F^n(a_n x_n + b_n) \to 1$, $n(-\log F(a_n x_n + b_n)) \to 0$ and from Lemma 2 for n large

$$\frac{f_0(b_n)}{f_0(b_n + x_n f(b_n))} \le \left(\frac{1}{1-\varepsilon}\right)\left(\frac{-\log F(b_n + x_n f_0(b_n))}{-\log F(b_n)}\right)^{-\varepsilon} \tag{4.3}$$

so $g_n(x_n) \le \left(\dfrac{1}{1-\varepsilon}\right)(n(-\log F(a_n x_n + b_n)))^{1-\varepsilon} \to 0$.

Case (iii). In this case $n(-\log F(a_n x_n + b_n)) \to \infty$. Using (4.3) we have for large n

$$g_n(x_n) \le e^{-\left(\frac{n-1}{n}\right)n(-\log F(a_n x_n + b_n))}\left(\frac{1}{1-\varepsilon}\right)(n(-\log F(a_n x_n + b_n)))^{1+\varepsilon} \to 0 \quad \text{as} \quad n \to \infty.$$

Case (iv). As in part (a) since F' is bounded

$$g_n(x_n) = O(na_n F^{n-1}(K)).$$

Since na_n is regularly varying and $F^{n-1}(K) \to 0$ at a geometric rate, the result follows.

REMARK. Local uniformity in case a) of our theorem has been proved by C. W. Anderson (1971) and pointwise convergence in case b) by Pickands (1967).

REMARK. If we suppose F' is ultimately non-increasing and $g_n(x)$ converges pointwise to the appropriate extreme value density, then (4.1) or (4.2) is true (recall the pointwise density convergence implies weak convergence). (Cf. de Haan, 1970, Theorem 2.7.1b and Theorem 2.7.3b). The same is true if we replace the requirement of monotonicity for F' by the requirement that F' is ultimately continuous; this follows from a so-called Croftian theorem (cf. Kendall 1968). It is easy to verify that locally uniform convergence of $g_n(x)$ entails (4.1) without any extra condition on F'. In case of just pointwise convergence, some extra condition on F' is necessary to get (4.1) or (4.2); this can be seen from Pompeiu's well known example of a strictly increasing differentiable function whose derivative vanishes on a dense subset of its interval of definition (cf. Bruckner, 1978). We thank Dr. A. A. Balkema for pointing out this example to us.

We now show that under the conditions of Theorem 2, a more general version of Theorem 1 holds.

COROLLARY.

a) *Suppose* (4.1). *For any sequence* $d_n \to \infty$ $(n \to \infty)$

$$\lim_{n\to\infty} d_n P\left\{x < \frac{M_n}{a_n} \le x + d_n^{-1}h\right\} = h\Phi_\alpha'(x)$$

uniformly for all x.

b) *Suppose* (4.2). *For any sequence* $d_n \to \infty$ $(n \to \infty)$

$$\lim_{n\to\infty} d_n P\left\{x < \frac{M_n - b_n}{a_n} \le x + d_n^{-1}h\right\} = h\,\Lambda'(x)$$

uniformly for all x.

PROOF.

a)

$$\lim_{n\to\infty} h^{-1}d_n \int_x^{x+d_n^{-1}h} g_n(t)\,dt = \Phi_\alpha'(x)$$

uniformly in x.

b)

$$\lim_{n\to\infty} h^{-1}d_n \int_x^{x+d_n^{-1}h} g_n(t)\,dt = \Lambda'(x)$$

uniformly in x.

We next consider density convergence in the L_p metric.

THEOREM 3. *Let* $\{X_n, n \ge 1\}$ *be* i.i.d. *with* df F *which is absolutely continuous with density* F'. *Set* $M_n = \bigvee_{i=1}^n X_i$ *and let* g_n *be the density of the normalized* M_n.

a) *Suppose* (4.1) *holds and* $g_n(x) = nF^{n-1}(a_n x)F'(a_n x)a_n$. *If* $\int_{-\infty}^{\infty} |F'(x)|^p\,dx < \infty$ *and* $p > (1+\alpha)^{-1}$ *then*

$$\int_{-\infty}^{\infty} |g_n(x) - \Phi_\alpha'(x)|^p\,dx \to 0$$

b) *Suppose* (4.2) *holds and* $g_n(x) = nF^{n-1}(a_n x + b_n)F'(a_n x + b_n)a_n$. *If* $\int_{-\infty}^{\infty} |F'(x)|^p\,dx < \infty$ *then*

$$\int_{-\infty}^{\infty} |g_n(x) - \Lambda'(x)|^p\,dx \to 0.$$

Remark. Under the von Mises type conditions (4.1) or (4.2) we thus get L_p convergence whenever F' and the limit density are in L_p.

Proof. (a) Since $(\Phi'_\alpha(x))^p = \alpha^p x^{-p(\alpha+1)}\exp\{-px^{-\alpha}\}$ we have $\int_0^\infty (\Phi'_\alpha(x))^p\,dx < \infty$ if and only if $p > (1+\alpha)^{-1}$. Hence

$$\lim_{M\to\infty} \int_{[0,M^{-1}]\cup[M,\infty)} (\Phi'_\alpha(x))^p\,dx = 0. \tag{4.4}$$

Next we prove that the right tail of $\int g_n^p$ is eventually small. For $M > 1$

$$\int_M^\infty (nF^{n-1}(a_n x)F'(a_n x)a_n)^p\,dx = \int_M^\infty (g_n(x))^p\,dx \le \int_M^\infty (na_n F'(a_n x))^p\,dx$$
$$= n^p a_n^{p-1}\int_{a_n M}^\infty (F'(y))^p\,dy.$$

Now using (4.1) and Karamata's Theorem (eg. de Haan, 1970, page 15, Theorem 1.2.9) we get since $-\log F$ is regularly varying

$$n^p a_n^{p-1}\int_{a_n M}^\infty (F'(y))^p\,dy \sim n^p a_n^{p-1}\alpha^p \int_{a_n M}^\infty (-\log F(y))^p y^{-p}\,dy$$
$$\sim n^p a_n^{p-1}\alpha^p (p(1+\alpha)-1)^{-1}(-\log F(a_n M))^p (a_n M)^{-p+1}$$
$$\sim n^p a_n^{p-1}\alpha^p (p(1+\alpha)-1)^{-1}(-\log F(a_n))^p M^{-p\alpha}(a_n M)^{-p+1}$$
$$= \alpha^p (p(1+\alpha)-1)^{-1} M^{1-p(\alpha+1)}$$

(as $n \to \infty$) where we used $-\log F(a_n) = n^{-1}$. We conclude

$$\lim_{M\to\infty}\lim_{n\to\infty}\sup \int_M^\infty (g_n(x))^p\,dx = 0. \tag{4.5}$$

We now consider the region $(-\infty, M^{-1}]$. It is convenient to define δ_n by $-\log F(\delta_n) \sim n^{-1/2}$ so that $\delta_n \to \infty$ but $\delta_n/a_n \to 0$ (since otherwise if along a subsequence $\delta_n/a_n \to c > 0$ then $n^{1/2} \sim n(-\log F(\delta_n)) \sim n(-\log F(a_n\cdot(\delta_n/a_n))) \to c^{-\alpha}$.)
We have

$$\int_{-\infty}^{M^{-1}} (g_n(x))^p\,dx = \int_{-\infty}^{\delta_n/a_n} (g_n(x))^p\,dx + \int_{\delta_n/a_n}^{M^{-1}} (g_n(x))^p\,dx = \text{I and II}.$$

Now

$$\text{I} = n^p a_n^{p-1}\int_{-\infty}^{\delta} (n(F^{n-1}(t))F'(t))^p\,dt \le n^p a_n^{p-1}F^{p(n-1)}(\delta_n)\int_{-\infty}^{\infty} (F'(t))^p\,dt$$

and since $n^p a_n^{p-1}$ is a regularly varying function of n and $F^{p(n-1)}(\delta_n) \le \exp\{-\tfrac{1}{2}pn^{1/2}\}$ the above goes to zero.

Now for given ε we have $F'(t) \le (\alpha+\varepsilon)t^{-1}(-\log F(t))$ ultimately. Noting that on the region of integration of II $x > \delta_n/a_n$ so that $a_n x > \delta_n \to \infty$, we have for sufficiently large n that

$$\text{II} \le \int_{\delta_n/a_n}^{M^{-1}} \left(\exp\left\{-\frac{n-1}{n}\cdot n(-\log F(a_n x))\right\}(\alpha+\varepsilon)n(-\log F(a_n x))\cdot x^{-1}\right)^p dx.$$

Applying Lemma 1 we get

$$\mathrm{II} \le \int_{\delta_n/a_n}^{M^{-1}} \left(\exp\left\{-\left(\frac{n-1}{n}\right)(1-\varepsilon)x^{-\alpha+\varepsilon}\right\}(\alpha+\varepsilon)(1+\varepsilon)x^{-\alpha-\varepsilon}x^{-1}\right)^p dx.$$

and for appropriate positive constants c_1, c_2 this is

$$\le c_1 \int_0^{c_2 M^{-1}} \exp\{-s^{-\alpha-\varepsilon}\} s^{-p(\alpha+\varepsilon)-p}\, ds < \infty.$$

We conclude $\lim_{M\to\infty}\lim_{n\to\infty}\sup \mathrm{II} = 0$. Hence also

$$\lim_{M\to\infty}\lim_{n\to\infty}\sup \int_{-\infty}^{M^{-1}} (g_n(x))^p\, dx = 0. \tag{4.6}$$

Next fix $M > 1$ consider $\int_{M^{-1}}^{M} |g_n(x) - \Phi_\alpha'(x)|^p\, dx$. Write

$$g_n(x) = F^{n-1}(a_n x)\frac{a_n x F'(a_n x)}{-\log F(a_n x)}\; \frac{n(-\log F(a_n x))}{x}.$$

Observe on $[M^{-1}, M]$

$$F^{n-1}(a_n x)/\Phi_\alpha(x) \to 1 \quad \text{uniformly,}$$

$$a_n x F'(a_n x)/(-\log F(a_n x)) \to \alpha \quad \text{uniformly,}$$

and

$$n(-\log F(a_n x))x^\alpha \to 1 \quad \text{uniformly.}$$

So on $[M^{-1}, M]$

$$g_n(x) = \Phi_\alpha(x)\alpha x^{-\alpha-1}(1+\zeta_n(x)) = (1+\zeta_n(x))\Phi_\alpha'(x)$$

where $\zeta_n(x) \to 0$ uniformly in $x \in [M^{-1}, M]$. Therefore

$$\int_{M^{-1}}^{M} |g_n(x) - \Phi_\alpha'(x)|^p\, dx = \int_{M^{-1}}^{M} |\Phi_\alpha'(x)|^p (\zeta_n(x))^p\, dx$$

and since $\Phi_\alpha' \in L_p$ and $\zeta_n \to 0$ uniformly on $[M^{-1}, M]$, we get

$$\lim_{n\to\infty} \int_{M^{-1}}^{M} |g_n(x) - \Phi_\alpha'(x)|^p\, dx = 0. \tag{4.7}$$

Finally write

$$\int_{-\infty}^{\infty} |g_n(x) - \Phi_\alpha'(x)|^p\, dx = \int_{-\infty}^{M^{-1}} + \int_{M^{-1}}^{M} + \int_{M}^{\infty}$$

$$\le 2^p\left\{\int_{-\infty}^{M^{-1}} |g_n(x)|^p\, dx + \int_{-\infty}^{M^{-1}} |\Phi_\alpha'(x)|^p\, dx + \int_M^\infty (g_n(x))^p\, dx \right.$$

$$\left. + \int_M^\infty (\Phi_\alpha'(x))^p\, dx\right\} + \int_{M^{-1}}^{M} |g_n(x) - \phi_\alpha'(x)|^p\, dx.$$

Now let $n \to \infty$ and then $M \to \infty$ and use (4.4), (4.5), (4.6), (4.7) to get the desired L_p convergence.

(b) As in part (a), we begin by noting

$$\lim_{M\to\infty} \int_{(-\infty,-M]\cup[M,\infty)} |\Lambda'(x)|^p\, dx = 0. \tag{4.8}$$

This is because $\Lambda' \in L_p$ for $p > 0$. Now we replace Λ' by g_n. We have for $M > 0$

$$\int_M^\infty (g_n(x))^p\,dx = \int_M^\infty (nF^{n-1}(a_n x + b_n)F'(a_n x + b_n)a_n)^p\,dx$$

$$= \int_M^\infty \left(F^{n-1}(a_n x + b_n)\frac{f_0(b_n)}{f_0(a_n x + b_n)}n(-\log F(a_n x + b_n))\right)^p dx.$$

Note $F^{N-1}(a_n x + b_n) \le 1$. Also, given $\varepsilon > 0$, for n large we have by Lemma 2

$$\frac{f_0(b_n)}{f_0(a_n x + b_n)} \le \left(\frac{1}{1-\varepsilon}\right)(n(-\log F(a_n x + b_n)))^{-\varepsilon}.$$

Therefore,

$$\int_M^\infty |g_n(x)|^p\,dx \le \left(\frac{1}{1-\varepsilon}\right)^p \int_M^\infty (n(-\log F(a_n x + b_n)))^{(1-\varepsilon)p}\,dx$$

$$= n^{p(1-\varepsilon)}a_n^{-1}\left(\frac{1}{1-\varepsilon}\right)^p \int_{a_n M + b_n}^\infty (-\log F(s))^{p(1-\varepsilon)}\,ds$$

and by Theorem 2.8.1, page 113 of de Haan, 1970, this is asymptotic to

$$n^{p(1-\varepsilon)}a_n^{-1}\left(\frac{1}{1-\varepsilon}\right)^p \frac{1}{p(1-\varepsilon)}(-\log F(a_n M + b_n))^{p(1-\varepsilon)-1}\int_{a_n M + b_n}^\infty (-\log F(s))\,ds$$

$$= (\text{const})(n(-\log F(a_n M + b_n)))^{p(1-\varepsilon)-1}\frac{\int_{a_n M + b_n}^\infty (-\log F(s))\,ds}{\int_{b_n}^\infty (-\log F(s))\,ds}$$

$$\cdot\frac{a_n^{-1}\int_{b_n}^\infty (-\log F(s))\,ds}{-\log F(b_n)}\cdot n(-\log F(b_n)).$$

Now since $F \in D(\Lambda)$ (from 4.2) we have $n(-\log F(a_n M + b_n)) \to e^{-M}$. Also by de Haan, 1970, page 90, Lemma 2.5.1

$$\int_{a_n M + b_n}^\infty (-\log F(s))\,ds \Big/ \int_{b_n}^\infty (-\log F(s))\,ds \to e^{-M}.$$

Recall

$$n(-\log F(b_n)) = 1, \qquad \int_{b_n}^\infty (-\log F(s))\,ds \Big/ (-\log F(b_n)) \sim f_0(b_n) \sim a_n$$

and we see

$$\limsup_{n\to\infty} \int_M^\infty |g_n(x)|^p\,dx \le (\text{const})e^{-Mp(1-\varepsilon)}.$$

Therefore,

$$\lim_{M\to\infty}\limsup_{n\to\infty} \int_M^\infty |g_n(x)|^p\,dx = 0. \tag{4.9}$$

For the region $(-\infty, -M]$, it is convenient to choose δ_n satisfying $-\log F(\delta_n) \sim n^{-1/2}$ so that $\delta_n \to \infty$ and $(\delta_n - b_n)/a_n \to -\infty$, since otherwise if along a subsequence $\delta_n - b_n/a_n \to c > -\infty$ we would have

$$\infty \leftarrow n^{1/2} \sim n(-\log F(\delta_n)) = n\left(-\log F\left(a_n\left(\frac{\delta_n - b_n}{a_n}\right) + b_n\right)\right) \to e^{-c}.$$

We now decompose the integral

$$\int_{-\infty}^{-M} |g_n(x)|^p\, dx = \int_{-\infty}^{(\delta_n-b_n)/a_n} + \int_{(\delta_n-b_n)/a_n}^{-M} = A + B.$$

For A we have

$$A = \int_{-\infty}^{(\delta_n-b_n)a_n} | nF^{n-1}(a_nx + b_n)F'(a_nx + b_n)a_n |^p\, dx$$

$$= n^p a_n^{p-1} \int_{-\infty}^{\delta_n} |F^{n-1}(y)F'(y)|^p\, dy$$

$$\leq n^p a_n^{p-1} F^{n-1}(\delta_n) \int_{-\infty}^{\infty} | F'(y)|^p\, dy$$

and since a_n is slowly varying and

$$F^{n-1}(\delta_n) = \exp\{-(-\log F(\delta_n))(n-1)\} \leq \exp\{-\tfrac{1}{2}n^{1/2}\}$$

for n large, we have $A \to 0$ as $n \to \infty$. For B we write

$$B = \int_{(\delta_n-b_n)/a_n}^{-M} \left| F^{n-1}(a_nx + b_n)\frac{f_0(b_n)}{f_0(a_nx + b_n)} n(-\log F(a_nx + b_n))\right|^p dx$$

which for given $\varepsilon > 0$ and n sufficiently large is bounded according to Lemma 2 by

$$\leq \int_{(\delta_n-b_n)/a_n}^{-M} | F^{n-1}(a_nx + b_n)\left(\frac{1}{1-\varepsilon}\right)(n(-\log\ F(a_nx + b_n)))^{1+\varepsilon} |^p\, dx.$$

Note Lemma 2 is applicable since $M > 0$ and if $x > (\delta_n - b_n)/a_n$ then $b_n + a_nx \geq b_n + a_n((\delta_n - b_n)/a_n) = \delta_n \to \infty$. In the above integral, make the change of variable $y = (n(-\log F(a_nx + b_n)))^{-1}$ and the integral becomes

$$\int_{(n(-\log F(\delta_n)))^{-1}}^{(n(-\log F(a_n(-M)+b_n)))^{-1}} e^{-\left(\frac{n-1}{n}\right)py^{-1}} y^{-(1-\varepsilon)p} H_n(dy)$$

where $H_n(y) = \left[\left(\frac{1}{-\log F}\right)^{\leftarrow}(ny) - b_n\right]\Big/ a_n$ where the arrow denotes the inverse function. Since $F \in D(\Lambda)$, $(1/(-\log F))^{\leftarrow}$ is in the class Π (de Haan, 1970, 1974) and satisfies $H_n(y) \to \log y$, $y > 0$, $n \to \infty$. The endpoints of the interval of integration both converge:

$$\theta_1(n) := (n(-\log F(\delta_n)))^{-1} \to 0$$

$$\theta_2(n) := (n(-\log F(a_n(-M) + b_n)))^{-1} \to e^{-M}.$$

The above integral is bounded by

$$\int_{\theta_1(n)}^{\theta_2(n)} g(y)H_n(dy)$$

(where $g(y) = e^{-(p/2)y^{-1}}y^{-(1-\varepsilon)p}$) and using partial integration this becomes

$$g(\theta_2(n))H_n(\theta_2(n)) - g(\theta_1(n))H_n(\theta_1(n)) - \int_{\theta_1(n)}^{\theta_2(n)} H_n(y)g'(y)\,dy = (i) + (ij) + (iij).$$

Now $(i) \to g(e^{-M}) \log e^{-M} = g(e^{-M})/M$ as $n \to \infty$ and note $\lim_{M\to\infty} g(e^{-M})/M = \lim_{s\downarrow 0} g(s)/(-\log s) = 0$. For (ij) and (iij) we need to use Lemma 3. For (iij), on the region of integration $y > \theta_1(n) = 1/(n(-\log F(\delta_n)))$ so $ny > n^{1/2}$. Therefore, for given ε and n sufficiently large (remember $b_n = (1/(-\log F))^{\leftarrow}(n)$ and suppose $\theta_2(n) < 1$)

$$\int_{\theta_1(n)}^{\theta_2(n)} g'(y)|\, H_n(y)|\, dy \le \int_{\theta_1(n)}^{\theta_2(n)} g'(y)\left[(1+\varepsilon)^2\frac{y^{\varepsilon}-1}{\varepsilon} + \varepsilon\right] dy.$$

It is readily seen $\lim_{M\to\infty}\lim_{n\to\infty}\sup$ of this bound is 0. We can handle (ij) similarly by using Lemma 3 and in conclusion we find

$$\lim_{M\to\infty}\lim_{n\to\infty}\sup \int_{-\infty}^{-M} |\, g_n(x)|^p\, dx = 0. \tag{4.10}$$

Finally on $[-M, M]$ we have

$$F^{n-1}(a_n x + b_n)\frac{f_0(b_n)}{f_0(a_n x + b_n)} n(-\log F(a_n x + b_n)) = (1 + \zeta_n(x))\Lambda'(x)$$

where $\zeta_n \to 0$ uniformly on $[-M, M]$ (recall $f_0(t + xf(t))/f_0(t) \to 1$ as $t \to \infty$ locally uniformly) so that

$$\int_{-M}^{M} |\, g_n(x) - \Lambda'(x)|^p\, dx = \int_{-M}^{M} \zeta_n(x)^p\, |\, \Lambda'(x)|^p\, dx \to 0 \text{ as } n \to \infty. \tag{4.11}$$

We finish by decomposing $\int_{-\infty}^{\infty} |g_n(x) - \Lambda'(x)|^p\, dx$ as done at the end of Part (a) and then using 4.8, 4.9, 4.10, 4.11. The proof is complete.

5. Occupation Times. Occupation time theorems for sums are frequently related to local limit theorems. See Breiman (1968), Darling and Kac (1957). For the case of maxima we have not found a direct connection but only one of analytic methodology which we discuss below.

In this section it is convenient to have a slightly different representation for $F \in D(\Lambda)$ and we proceed as in de Haan, 1970, Theorem 2.4.2. We restrict attention to the case where $F(x) < 1$ for all x. According to this result $F \in D(\Lambda)$, iff

$$\lim_{x\to\infty} \frac{(1 - F(x))\int_x^{\infty}\int_s^{\infty}(1 - F(u))\,du\,ds}{\left(\int_x^{\infty}(1 - F(s))\,ds\right)^2} = 1$$

and in this case

$$1 - F(x) = c(x)\exp\left\{-\int_1^x \frac{a(t)}{f^*(t)}\,dt\right\}$$

and we may set

$$f^*(x) = \int_x^\infty \int_s^\infty (1 - F(u))\,du\,ds \Big/ \int_x^\infty (1 - F(s))\,ds$$

and

$$a(x) = -1 + 2\frac{(1 - F(x))\int_x^\infty \int_x^\infty (1 - F(u))\,du\,ds}{\left(\int_x^\infty (1 - F(s))\,ds\right)^2}$$

$$c(x) = \frac{(1 - F(x))\int_x^\infty \int_s^\infty (1 - F(u))\,du\,ds}{\left(\int_x^\infty (1 - F(s))\,ds\right)^2} \Bigg/ \frac{\int_1^\infty \int_s^\infty (1 - F(u))\,du\,ds}{\left(\int_1^\infty (1 - F(s))\,ds\right)^2}$$

so that $a(x) \to 1$, $c(x) \to c$, $(f^*(x))' \to 0$ and hence $f^*(t + xf^*(t))/(f^*(t)) \to 1$ locally uniformly. In this case $F^n(a_n x + b_n) \to \Lambda(x)$ where $b_n = (1/(1 - F))^{\leftarrow}(n)$ and $a_n = f^*(b_n)$.

LEMMA 4. *Suppose $F \in D(\Lambda)$ and $a_n \to 0$ (i.e. $f^*(x) \to 0$, $x \to \infty$). Set $R(x) = -\log(1 - F(x))$ and suppose $\lim_{t\to\infty} f^*(t + x)/f^*(t) = 1$ for all x. Then for all x*

$$a_n(R(x + b_n) - \log n) \to x$$

or in terms of measures

$$a_n R(\cdot + b_n) \to_v m$$

where m is Lebesgue measure on $(-\infty, \infty)$ and $\to_v$ denotes vague convergence.

PROOF. Let $c_0(x) = -\log c(x)$ in the previous representation so that

$$R(x) = c_0(x) + \int_1^x \frac{a(t)}{f^*(t)}\,dt$$

and $c_0(x) \to c_0$. Now

$$a_n(R(x + b_n) - \log n) = a_n(c_0(x + b_n) - c_0(b_n)) + a_n \int_{b_n}^{x+b_n} \frac{a(t)}{f^*(t)}\,dt.$$

Since $a_n = f^*(b_n) \to 0$, the above is

$$o(1) + \int_0^x a(t + b_n)\frac{f^*(b_n)}{f^*(t + b_n)}\,dt \to \int_0^x 1\,dt = x$$

as $n \to \infty$ since $a(t + b_n) \to 1$ and $f^*(b_n)/f^*(t + b_n) \to 1$ locally uniformly.

REMARK. The condition $f^*(t + x)/f^*(t) \to 1$ is satisfied by $1 - F(x) = e^{-x^\alpha}$, $x \geq 0$ for

$\alpha > 1$ since in this case we may take

$$f^*(x) \sim (1 - F(x))/F'(x) = \frac{1}{\alpha} x^{-\alpha+1}.$$

Also in the case $F = N(0, 1)$ the condition is satisfied since

$$f^*(x) \sim (1 - F(x))/F'(x) = (1 - N(0, 1, x))/n(0, 1, x)$$
$$\sim \frac{n(0, 1, x)/x}{n(0, 1, x)} \sim x^{-1} \quad \text{as } x \to \infty$$

(by Mill's ratio). These two examples illustrate that a helpful sufficient condition is that $\lim_{t\to\infty} \frac{d}{dt} \log f^*(t) = 0$ (analogous to von Mises' conditions for the domain of attraction).

We now consider an occupation time theorem for maxima and phrase it in terms of random measures. For $x \in R$ define the point measure $\varepsilon_x(A) = 1$ if $x \in A$, 0 if $x \notin A$. The occupation time measure for maxima is then

$$0(\cdot) = \sum_{k=1}^{\infty} \varepsilon_{M_k}(\cdot).$$

It is more convenient to express 0 in terms of record values: Say X_j is a record of the sequence $\{X_n, n \geq 1\}$ if $M_j > M_{j-1}$ and let the record value indices be $L(j), j \geq 0$; i.e. $L(0) = 1$ and $L(j) = \inf\{k > L(j-1) : M_k > M_{k-1}\}$. Let $\Delta_j = L(j) - L(j-1), j \geq 0$. Then

$$0(\cdot) = \sum_{k=0}^{\infty} \Delta_{k+1} \varepsilon_{X_{L(k)}}(\cdot).$$

We first compute the Laplace functional (cf. Neveu, 1977, page 258) of 0.

PROPOSITION. *Suppose F is continuous. Then for g continuous with compact support we have*

$$E \exp\left\{-\int g(x) O(dx)\right\} = \exp\left\{-\int \frac{1 - e^{-g(x)}}{1 - e^{-g(x)}F(x)} R(dx)\right\}.$$

PROOF. We have

$$E \exp\left\{-\int g(x) O(dx)\right\} = E \exp\{-\textstyle\sum_{k=0}^{\infty} \Delta_{k+1} g(X_{L(k)})\}$$
$$= E(E(\exp\{-\textstyle\sum_{k=0}^{\infty} \Delta_{k+1} g(X_{L(k)})\} \mid X_{L(i)}, i \geq 0)).$$

Conditional on $\{X_{L(i)}, i \geq 0\}$, the interrecord times are independent geometrically distributed random variables (Shorrock, 1972) so the above is

$$= E(\textstyle\prod_{k=0}^{\infty} E(e^{-g(X_{L(k)})\Delta_{k+1}} \mid X_{L(i)}, i \geq 0))$$
$$= E \textstyle\prod_{k=0}^{\infty} \sum_{n=1}^{\infty} e^{-g(X_{L(k)})n} F(X_{L(k)})^{n-1}(1 - F(X_{L(k)}))$$
$$= E \textstyle\prod_{k=0}^{\infty} h(X_{L(k)})$$

where

$$h(x) = \frac{(1 - F(x))e^{-g(x)}}{1 - e^{-g(x)}F(x)}.$$

$$E \textstyle\prod_{k=0}^{\infty} h(X_{L(k)}) = E \exp\{-\sum_{k=0}^{\infty} (-\log h(X_{L(k)}))\}$$
$$= E \exp\left\{-\int (-\log h(x)) \textstyle\sum_{k=0}^{\infty} \varepsilon_{X_{L(k)}}(dx)\right\}$$

which is the Laplace functional of the point process with points $\{X_{L(k)}\}$ at the function

$-\log h$. We now make use of the well known fact that $\{X_{L(k)}\}$ is Poisson with mean measure R. The Laplace functional of such a process is well known to be

$$\exp\left\{-\int (1-e^{-(-\log h(x))})R(dx)\right\} = \exp\left\{-\int (1-h(x))R(dx)\right\}$$

and substituting the form of h gives the result.

THEOREM 4. *Suppose $F \in D(\Lambda)$ is continuous with auxiliary function f satisfying $\lim_{t\to\infty} f(t+x)/f(t) = 1$ and $\lim_{t\to\infty} f(t) = 0$. Set $\mu(A) = \int_A (1-F(y))^{-1}\,dy$ for A a Borel subset of R. Then*

$$a_n O(\cdot + b_n) = a_n \textstyle\sum_{k=1}^{\infty} \varepsilon_{M_k - b_n}(\cdot) \Rightarrow \mu$$

in the sense of weak convergence of stochastic point processes.

For a discussion of weak convergence of point processes see Neveu, 1977, page 282.

PROOF. It suffices to show the Laplace functional of $a_n O(\cdot + b_n)$ converges to that of μ. For g continuous with compact support we have

$$E\exp\left\{-\int g(x)a_n O(dx+b_n)\right\} = E\exp\left\{-\int a_n g(x-b_n)O(dx)\right\}$$

and from the proposition this equals

$$\exp\left\{-\int \frac{1-e^{-a_n g(x-b_n)}}{1-e^{-a_n g(x-b_n)}F(x-b_n)}\,R(dx)\right\} = \exp\left\{-\int \frac{a_n^{-1}(1-e^{-a_n g(y)})}{1-e^{-a_n g(y)}F(y)}\,a_n R(dy+b_n)\right\}.$$

We now use the fact that $a_n \to 0$, $a_n R(\cdot + b_n) \to_v m$ and obtain the convergence of the Laplace functionals to

$$\exp\left\{-\int g(y)\frac{dy}{1-F(y)}\right\}$$

so the proof is complete.

REMARK. It is easy to formulate a completely analagous version of this theorem in continuous time using the structure of extremal processes. Cf. Resnick (1974).

REMARK. An occupation time result for $a_n^{-1}(M_k - b_n)$, $k \geq 1$ is implied by the following:

$$E\exp\left\{-\int g(x)\textstyle\sum_{k=1}^{\infty} \varepsilon_{(M_k-b_n)/a_n}(dx)\right\} \to \exp\left\{-\int \frac{1-e^{-g(s)}}{1-e^{-g(s)}F(s)}\right\} ds.$$

This limit Laplace functional corresponds to the point process $\sum_{k=-\infty}^{\infty} \mathbf{n}_{k+1}\varepsilon_{\mathbf{t}_k}$ where $\{\mathbf{t}_k\}$ are the points of a homogeneous Poisson process on R and $(\mathbf{n}_k)$ are conditionally independent given $\{\mathbf{t}_k\}$ with

$$P[\mathbf{n}_{k+1} = n \mid \mathbf{t}_k] = F^{n-1}(\mathbf{t}_k)(1-F(\mathbf{t}_k))$$

for $n \geq 1$. So we have

$$\textstyle\sum_{k=1}^{\infty} \varepsilon_{(M_k-b_n)/a_n} \Rightarrow \sum_{k=-\infty}^{\infty} \mathbf{n}_{k+1}\varepsilon_{\mathbf{t}_k}.$$

An analogous result holds when $F \in D(\Phi_\alpha)$. Both results depend on the proposition and the fact that

$$F^n(a_n x + b_n) \to \Lambda(x) \quad (F^n(a_n x) \to \Phi_\alpha(x))$$

iff $$R(a_n x + b_n) - \log n \to x \, (R(a_n x) - \log n \to \alpha \log x).$$

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