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FUNCTIONAL LIMIT THEOREMS FOR EXTREME VALUES OF ARRAYS OF INDEPENDENT RANDOM VARIABLES

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For an array $\{X_{ni}\}$ of independent, uniformly null random variables, several necessary and sufficient conditions are given for the convergence in distribution of its extremal process $\mathbf{M}_n = (M_n^1, M_n^2, \cdots)$ as $n \to \infty$, where $M_n^k(t) = k$th largest $\{X_{ni}: i/n \le t\}$, $t > 0$. It is shown that if $\mathbf{M}_n$ converges, then its limit is an extremal process of a Poisson process on the plane. The limit cannot be an extremal process of a non-Poisson, infinitely divisible point process, which is possible for certain stationary variables. A characterization of the convergence of $\mathbf{M}_n$, without the uniformly null assumption, is also given.

1. Introduction. Let $X_{n1}, X_{n2}, \cdots$ be a sequence of independent random variables $(n \ge 1)$ that take values in the interval (x_0, ∞) for some $x_0 \ge -\infty$. The kth extremal process of this sequence is defined by

$$M_n^k(t) := \begin{cases} k\text{th largest } \{X_{ni}: i/n \le t\} & \text{for} \quad t \ge k/n \\ M_n^k(k/n) & \text{for} \quad 0 < t < k/n. \end{cases} \tag{1.1}$$

Denote by ξ_n the point process on the Euclidean space $S := (0, \infty) \times (x_0, \infty)$ with points at locations $(i/n, X_{ni})$, $i \ge 1$. The M_n^k can also be viewed as the kth extremal process of ξ_n. The kth extremal process of a point process ξ on S with points at (T_i, X_i), $i \ge 1$, is defined by $\phi_k\xi(t) = k$th largest X_i with $T_i \le t$; see (2.1) below. Denote by D the set of nondecreasing right-continuous, piecewise constant functions from $R^+ := (0, \infty)$ to (x_0, ∞) that take a finite number of jumps in any compact subinterval of R^+. Endow D with the topology generated by the Levy metric: $y_n \to y$ in D if and only if $y_n(t) \to y(t)$ for all y-continuity points t; see Wichura (1974) and the M_1 topology in Skorohod (1956). Denote by D_1 the set D endowed with Skorohod's J_1 topology [3, 16, 22, 29]. In this paper, I give several necessary and sufficient conditions for the convergence in distribution of the random elements $\mathbf{M}_n := (M_n^1, M_n^2, \cdots)$, $n \ge 1$, in the product spaces D^∞ and D_1^∞. I also identify the family of possible limits of $\mathbf{M}_n$. In particular, the results yield limits for the joint distribution of the k-largest, $k \ge 1$, of independent, arbitrarily distributed random variables, and for the distributions of certain functionals of these maxima, such as their record levels and record times.

For the case $X_{ni} := (X_i - a_n)/b_n$ in which $X_1, X_2, \cdots$ are independent and identically distributed and a_n and b_n $(b_n \to \infty)$ are constants such that $\lim_{n\to\infty} P(X_1 \le b_n x + a_n)^n =: G(x)$, $x > x_0$, and G is continuous, it is known that $\xi_n \to_{\mathscr{D}} \xi$ and

$$\mathbf{M}_n = (\phi_1\xi_n, \phi_2\xi_n, \cdots) \to_{\mathscr{D}} (\phi_1\xi, \phi_2\xi, \cdots) \quad \text{in } D_1^\infty, \tag{1.2}$$

where ξ is a Poisson process on S with intensity $E\xi((0, t] \times (x, \infty)) = -\log G(x)^t$, and G is necessarily one of the three classical extreme value distributions; see Lamperti (1964), Resnick (1975), Mori and Oodaira (1976), Weissman (1975c) and Galambos (1978). Weissman actually proves that $\xi_n \to_{\mathscr{D}} \xi$ and (1.2) holds, if the X_i are independent with arbitrary distributions such that $M_n^1 \to_{\mathscr{D}} M$ in D, and $G(x) := P(M(1) \le x)$ is continuous; here ξ is necessarily a Poisson process with $-\log E\xi((0, t] \times (x, \infty)) = G(t^\theta x)$ $(\theta \ne 0)$ or =

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$G(x - c\log t)$ $(c \geq 0)$. Similar convergence theorems for the extremal processes of stationary and martingale-difference sequences appear in Leadbetter (1974), Mori (1977), Adler (1978), and Durrett and Resnick (1978). Other related references are [2, 4–12, 15, 17–20, 23–28, 30].

2. Main Results. Consider the process $\mathbf{M}_n = (M_n^1, M_n^2, \cdots)$ as defined in (1.1). In addition to the notation above, let $\mathcal{M}$ denote the set of nonnegative measures μ on S (with its Borel σ-field) such that $\mu(t, x) < \infty$, $(t, x) \in S$, where

$$\mu(t, x) := \mu((0, t] \times (x, \infty)), \quad t > 0, x \geq x_0,$$

(in particular, μ is finite on compacts). Endow $\mathcal{M}$ with the vague topology: the smallest topology that makes the mapping $\mu \to \int_S f d\mu$ continuous for any continuous real-valued function f on S with compact support; see Kallenberg (1975). Let $\mathcal{M}_\infty = \{\mu \in \mathcal{M} : \mu(t, x_0) = \infty, t > 0\}$, and denote by $\mathcal{M}_1$ the set of all $\mu \in \mathcal{M}_\infty$ such that for each line $L_t = \{t\} \times (x_0, \infty)$, $t > 0$, either $\mu\{L_t\} = 0$ or there is exactly one $x > x_0$ such that $\mu\{(t, x)\} > 0$ and $\mu\{L_t \backslash (t, x)\} = 0$. Denote by $\mathcal{N}$ the set of all integer-valued measures $\mu \in \mathcal{M}$ such that $\mu \in \mathcal{M}_\infty$, or

$$\mu(S) = \infty \quad \text{and} \quad \mu(t, x_0) < \infty, \quad t > 0.$$

Endow $\mathcal{N}$ with the relativized vague topology. Let $\mathcal{N}_\infty = \mathcal{M}_\infty \cap \mathcal{N}$ and $\mathcal{N}_1 = \mathcal{M}_1 \cap \mathcal{N}$. Finally, let $\phi_k (k \geq 1)$ denote the mapping from $\mathcal{N}$ to $\mathcal{D}$ defined by

$$\phi_k\mu(t) := \begin{cases} \inf\{x > x_0 : \mu(t, x) < k\} & \text{for} \quad t > \tau_k \\ \phi_k\mu(\tau_k +) & \text{for} \quad 0 < t \leq \tau_k, \end{cases} \tag{2.1}$$

where $\tau_k := \inf\{t > 0 : \mu(t, x_0) \geq k\}$. For any $\mu \in \mathcal{N}$ with atom locations (t_i, x_i), $i \geq 1$,

$$\phi_k\mu(t) = k\text{th largest } \{x_i : t_i \leq t\} \quad \text{for} \quad t \geq \tau_k.$$

In particular, note that $\mathbf{M}_n = (M_n^1, M_n^2, \cdots) = (\phi_1\xi_n, \phi_2\xi_n, \cdots)$.

This representation of $\mathbf{M}_n$ was first discussed in Pickands (1971) and later used in [1, 7, 18, 21, 24–26] to prove the convergence of $\mathbf{M}_n$ in (1.2) as a consequence of the convergence of ξ_n to a limit that is, with probability one, in the continuity set of each ϕ_k, $k \geq 1$. Mori and Oodaira (1976) partially describe the continuity set of these mappings. A more complete description is as follows.

LEMMA 2.1. *Suppose $\mu_n \to \mu$ in $\mathcal{N}$ and $\mu \in \mathcal{N}_\infty$. Then $\phi_k\mu_n \to \phi_k\mu$ in D for each $k \geq 1$. This convergence also holds in D_1 when $\mu \in \mathcal{N}_1$.*

PROOF. To prove $\phi_k\mu_n \to \phi_k\mu$ in D or D_1, it suffices to prove this convergence on any interval $[a, b]$ where a and b are $\phi_k\mu$-continuity points. Fix k and let $a < b$ be $\phi_k\mu$-continuity points. For the sake of brevity, choose a small enough so that $\phi_k\mu(a) < \phi_k\mu(b)$ (here $\phi_k\mu(0+) = x_0 < \phi_k\mu(b)$). Pick $x^\# < \phi_k\mu(a)$ and $t^\# > b$ such that $\phi_k\mu(t^\#) = \phi_k\mu(b)$ and

$$\mu(\partial B) = 0 \quad \text{where} \quad B = (0, t^\#) \times (x^\#, \infty).$$

Let $(t_1, x_1), \cdots, (t_m, x_m)$, with $t_1 \leq \cdots \leq t_m$ denote the locations of the atoms of μ in B, and let $B_1, \cdots, B_m$ denote open spherical neighborhoods of these points with diameters ε. Choose ε small enough so that the B_i's are disjoint and in B. Since $\mu_n \to \mu$ in $\mathcal{N}$, there is an integer N such that for all $n \geq N$

$$\mu_n(B_i) = \mu(B_i), \quad 1 \leq i \leq m, \quad \text{and} \quad \mu_n(B \backslash \cup_{i=1}^m B_i) = 0.$$

Then
$$|\phi_k\mu_n(b) - \phi_k\mu(b)| < \varepsilon, \quad n \geq N,$$

and hence $\phi_k\mu_n \to \phi_k\mu$ in D.

Now suppose $\mu \in \mathcal{N}_1$. Then, in the setting above, $t_1 < \cdots < t_m$. Choose the $\varepsilon < 1/2 \min\{t_i - t_{i-1}\}$ so that the B_i's are disjoint and in B. For each $n \geq N$, let (t_{nj}, x_{nj}), $1 \leq j \leq \mu_n(B_i)$, denote the atom locations of μ_n in B_i. Let λ_n be a mapping from $[a, t^\#]$ to itself such that $\lambda_n(a) = a$, $\lambda_n(t^\#) = t^\#$, $\lambda_n(t_i) = \min\{t_{nj} : 1 \leq j \leq \mu_n(B_i)\}$, $1 \leq i \leq m$, and λ_n is linear between these points. Then clearly

$$|\lambda_n(t) - t| \le \varepsilon \quad \text{and} \quad |\phi_k\mu_n(\lambda_n(t)) - \phi_k\mu(t)| \le \varepsilon, \quad a \le t \le b,$$

since the discontinuity points of $\phi_k\mu$ in $(0, b]$ are contained in $\{t_1, \cdots, t_m\}$. This proves $\phi_k\mu_n \to \phi_k\mu$ as functions on $[a, b]$ in Skorohod's J_1 topology, and hence this convergence holds in D_1.

Lemma 2.1 describes a subtle difference in the continuity of ϕ_k on the two spaces D and D_1. It is easy to construct μ_n that converge to some $\mu \notin \mathscr{N}_1$, and that for some k the $\phi_k\mu_n$ converge to $\phi_k\mu$ in D but not in D_1.

The following result characterizes the convergence of $\mathbf{M}_n$ in D^∞; the stronger convergence in D_1^∞ is characterized in Corollary 2.4. Here and below $X_{n1}, X_{n2}, \cdots$ are called uniformly null if

$$\lim_{n\to\infty} \max_{i\le mn} P(X_{ni} > x) = 0, \quad x > x_0, \quad \text{and} \quad m \ge 1.$$

Also, $[t]$ denotes the integer part of t.

THEOREM 2.2. *For independent, uniformly null $X_{n1}, X_{n2}, \cdots$, the following statements are equivalent.*

(i) $\mathbf{M}_n \to_{\mathscr{D}} \mathbf{M}$ *in* D^∞ *for some* $\mathbf{M}$.

(ii) $M_n^1 \to_{\mathscr{D}} Y$ *in* D *for some* Y.

(iii) $\sum_{i=1}^{[nt]} P(X_{ni} > x) \to \lambda(t, x)$ *for each* $(t, x) \in S$ *with* $\lambda(\partial\{(0, t] \times (x, \infty)\}) = 0$, *for some* $\lambda \in \mathscr{M}_\infty$.

(iv) $\xi_n \to_{\mathscr{D}} \xi$ *in* $\mathscr{N}$ *for some* $\xi \in \mathscr{N}_\infty$ a.s.

If any one of these statements holds, then $\mathbf{M} =_{\mathscr{D}} (\phi_1\xi, \phi_2\xi, \cdots)$, $Y =_{\mathscr{D}} \phi_1\xi$, ξ is a Poisson process with intensity λ, and

$$P(Y(t) \le x) = \exp -\lambda(t, x), \quad (t, x) \in S. \tag{2.2}$$

Hence any one of the limits λ, ξ, $\mathbf{M}$ and Y determines the other three.

PROOF. Clearly (i) implies (ii), and (ii) implies (iii) since under the independence and uniformly null assumptions

$$P(M_n^1(t) \le x) = \prod_{i=1}^{[nt]} P(X_{ni} \le x) = \exp \sum_{i=1}^{[nt]} \log P(X_{ni} \le x)$$
$$= \exp - (1 + o(1)) \sum_{i=1}^{[nt]} P(X_{ni} > x).$$

Also (iv) implies (i) by Lemma 2.1. I will finish the proof by showing that (iii) is equivalent to (iv); the proof that (iv) implies (iii) is included in order to prove the assertions following statement (iv).

Suppose (iv) holds. Let ξ_{ni} be the random element of $\mathscr{N}$ that has a unit mass at $(i/n, X_{ni})$. Then $\xi_n = \sum_{i=1}^\infty \xi_{ni}$. Let $\mathscr{B}(S)$ denote the relatively compact Borel sets of S, and let $\mathscr{N}_0$ denote the set of all nonnegative integer-valued measures on S, excluding the zero measure, that are finite on compacts. The uniformly null assumption implies that for any $B \in \mathscr{B}(S)$,

$$\sup_i P(\xi_{ni}(B) > 0) \le \max_{i\le mn} P(X_{ni} > x) = o(1), \tag{2.3}$$

where m and x are chosen such that $B \subset (0, m] \times (x, \infty)$. Then by Theorem 6.1 in [13], the supposition $\xi_n \to_{\mathscr{D}} \xi$ in $\mathscr{N}$ implies that

$$\sum_i P(\xi_{ni}(B_1) = k_1, \cdots, \xi_{ni}(B_m) = k_m) \to \Lambda\{\mu \in \mathscr{N}_0 : \mu(B_1) = k_1, \cdots, \mu(B_m) = k_m\} \tag{2.4}$$

for any integers $k_1, \cdots, k_m$ and rectangles $B_1, \cdots, B_m$ in $\mathscr{B}_\Lambda = \{B \in \mathscr{B}(S) : \Lambda\{\mu(\partial B) > 0\} = 0\}$, where Λ is some measure on $\mathscr{N}_0$ which satisfies

$$\int_{\mathscr{N}_0} (1 - e^{-\mu(B)})\Lambda(d\mu) < \infty \quad \text{for} \quad B \in \mathscr{B}(S).$$

In this setting, Λ is concentrated on the Dirac measures in $\mathscr{N}_0$ (those with one atom of unit size). To see this, first note that from (2.4) and the structure of ξ_{ni}, it follows for any disjoint rectangles $B_1, \cdots, B_m$ in $\mathscr{B}_\Lambda$ that

$$\Lambda\{\mu:\mu(B_1) > 0, \cdots, \mu(B_m) > 0\}$$
$$= \lim_{n\to\infty} \sum_i P(\xi_{ni}(B_1) > 0, \cdots, \xi_{ni}(B_m) > 0) = 0 \quad \text{for} \quad m \geq 2,$$

(2.5) and $$\Lambda\{\mu:\mu(B_1) \geq 2\} = \lim_{n\to\infty} \sum_i P(\xi_{ni}(B_1) \geq 2) = 0.$$

Then an application of Lemma 2.2 in [13] yields

(2.6) $$\Lambda\{\mu:\mu \text{ restricted to } B \text{ is not a Dirac measure or the zero measure}\} = 0$$

for any rectangle B in $\mathscr{B}_\Lambda$. Since there exist rectangles $B_1, B_2, \cdots$ in $\mathscr{B}_\Lambda$ such that $B_n \uparrow S$ (A6.1 in [13]), then it follows that (2.6) holds for $B = S$, and so Λ is concentrated on the Dirac measures in $\mathscr{N}_0$ as asserted.

Now let λ be measured on S defined by

$$\lambda(B) = \Lambda\{\mu:\mu(B) = 1\} \quad \text{for} \quad B \in \mathscr{B}(S).$$

Then from (2.4),

(2.7) $$\sum_{i=[ns]+1}^{[nt]} P(X_{ni} \in (x, y]) = \sum_i P(\xi_{ni}((s, t] \times (x, y]) = 1) \to \lambda((s, t] \times (x, y])$$

for any $s < t$ and $x < y$ with $\lambda(\partial\{(s, t] \times (x, y]\}) = 0$. This proves the convergence statement in (iii). From (2.5) and (2.7) it follows that ξ (the limit of ξ_n) is necessarily a Poisson process with intensity λ; see Corollary 7.5 of [13]. Since ξ is Poisson, then $\lambda(A) < \infty$ if and only if $\xi(A) < \infty$ a.s. for any Borel set A in S. Consequently, $\xi \in \mathscr{N}_\infty$ a.s. is equivalent to $\lambda \in \mathscr{M}_\infty$. This completes the proof that (iv) implies (iii).

Now suppose (iii) holds. This implies (2.7), and so by Corollary 7.5 of [13], $\xi_n \to_{\mathscr{D}} \xi$ where ξ is a Poisson process with intensity λ. Moreover, $\xi \in \mathscr{N}_\infty$ a.s. since $\lambda \in \mathscr{M}_\infty$. Thus (iv) holds.

This completes the proof that (i)–(iv) are equivalent statements. The arguments above also justify the last two assertions in Theorem 2.2.

REMARK 2.3. Mori (1977) describes a stationary strongly mixing sequence X_i for which $\mathbf{M}_n$, with $X_{ni} = (X_i - a_n)/b_n$, converges to the extremal process of a certain non-Poisson, infinitely divisible point process. This suggests, analogous to the central limit theory for sums, that for independent uniformly null arrays, the set of possible limits of $\mathbf{M}_n$ is the set of extremal processes of infinitely divisible point processes. Theorem 2.2 shows, however, that this is false. The next result, along with Theorem 2.2, shows that the set of limits of $\mathbf{M}_n$ is the set of extremal processes of Poisson processes with intensities in M_∞.

PROPOSITION 2.4. *For any* $\lambda \in \mathscr{M}_\infty$, *there exist independent, uniformly null* $X_{n1}, X_{n2}, \cdots$ *that satisfy* (iii) *in Theorem* 2.2.

PROOF. Fix $\lambda \in \mathscr{M}_\infty$ and let $A := \cup_{x>x_0} \{t:\lambda(\{t\} \times (x, \infty)) > 0\}$. This set is countable, since the sets in the union are countable and nondecreasing as $x \downarrow x_0$. Define $P(X_{ni} > x) := \min\{1, H_{ni}(x)\}$, $x > x_0$, $i > 1$, where

(2.8) $$H_{ni}(x) = \lambda(\{((i-1)/n, i/n]\backslash A\} \times (x, \infty)) + N_n^{-1} \sum_{s\in A} \lambda(\{s\} \times (x, \infty))\delta_i([[ns], [ns] + N_n]),$$

the N_n are integers such that $N_n \to \infty$ and $n^{-1}N_n \to 0$ as $n \to \infty$, and δ_i is the Dirac measure on R^+ with unit mass at i. The distributions of the X_{ni}'s are chosen so that they partition the mass of λ, with a special accounting for the atoms of λ. Since λ is atomless on the first set on the right side of (2.8), one can show that the X_{ni} are uniformly null. Furthermore, (iii) in Theorem 2.2 holds because for n sufficiently large

$$\sum_{i=1}^{[nt]} P(X_{ni} > x) = \lambda((0, [nt]/n] \times (x, \infty)) - N_n^{-1} \sum_{s\in A} \lambda(\{s\} \times (x, \infty))\min\{0, N_n - [nt] + [ns]\},$$

and the last term converges to zero.

The next result vis-á-vis Theorem 2.2 shows the differences between the convergence of $\mathbf{M}_n$ in D^∞ versus D_1^∞. This result follows from Theorem 2.2; Lemma 2.1; and the facts that convergence in D_1 implies convergence in D, and for ξ Poisson, $\lambda \in \mathcal{M}_1$ if and only if $\xi \in \mathcal{N}_1$.

COROLLARY 2.5. *For independent, uniformly null $X_{n1}, X_{n2}, \cdots$, the following statements are equivalent.*

(i) $\mathbf{M}_n \to_{\mathcal{D}} \mathbf{M}$ *in* D_1^∞ *for some* $\mathbf{M}$ *such that* $\mu \in \mathcal{M}_1$, *where* $\mu(t, x) = -\log P(M^1(t) \le x)$, $(t, x) \in S$.

(ii) $M_n^1 \to_{\mathcal{D}} Y$ *in* D *for some* Y *such that* $\mu \in \mathcal{M}_1$, *where* $\mu(t, x) = -\log P(Y(t) \le x)$, $(t, x) \in S$.

(iii) $\sum_{i=1}^{[nt]} P(X_{ni} > x) \to \lambda(t, x)$ *for each* $(t, x) \in S$ *with* $\lambda(\partial\{(0, t] \times (x, \infty)\}) = 0$, *for some* $\lambda \in \mathcal{M}_1$.

(iv) $\xi_n \to_{\mathcal{D}} \xi$ *in* $\mathcal{N}$ *for some* $\xi \in \mathcal{N}_1$ *a.s.*

The last two assertions of Theorem 2.2 *and Proposition* 2.4, *with* $\lambda \in \mathcal{M}_1$, *also hold in this setting; and hence* $\mu = \lambda$.

3. Additional Results. The uniformly null assumption in the results above can be relaxed at the expense of specifying the limit of $\mathbf{M}_n$ as follows (this limit is initially unspecified in Theorem 2.2 and Corollary 2.5).

COROLLARY 3.1. *Let ξ be a Poisson process on S with intensity $\lambda \in \mathcal{M}_1$ that is atomless. For independent $X_{n1}, X_{n2}, \cdots$, the following statements are equivalent.*

(i) $\mathbf{M}_n \to_{\mathcal{D}} (\phi_1\xi, \phi_2\xi, \cdots)$ *in* D_1^∞.

(ii) $\mathbf{M}_n \to_{\mathcal{D}} (\phi_1\xi, \phi_2\xi, \cdots)$ *in* D^∞.

(iii) $\prod_{i=1}^{[nt]} P(X_{ni} \notin (x, y]) \to \exp - \lambda((0, t] \times (x, y])$, $t > 0$, $x_0 < x < y$.

(iv) $\xi_n \to_{\mathcal{D}} \xi$ *in* $\mathcal{N}$.

PROOF. By Theorem 2.2 and Corollary 2.5 the statements (i), (ii) and (iv) are equivalent. Now (iv) implies (iii) since for any $I = (0, t] \times (x, y]$, $x < y$,

$$\prod_{i=1}^{[nt]} P(X_{ni} \notin (x, y]) = P(\xi_n(I) = 0) \to P(\xi(I) = 0) = \exp -\lambda(I).$$

Finally if (iii) holds, then (iv) follows by Theorem 4.7 in [13], since for $U = \cup_{j=1}^m (s_j, t_j] \times (x_j, y_j]$ where $s_1 < t_1 < \cdots < s_m < t_m$ and $x_1 < y_1 < \cdots < x_m < y_m$,

$$P(\xi_n(U) = 0) = \prod_{j=1}^m \prod_{i=[ns_j]+1}^{[nt_j]} P(X_{ni} \notin (x_j, y_j])$$

$$\to \exp - \sum_{j=1}^m \lambda((s_j, t_j] \times (x_j, y_j]) = P(\xi(U) = 0),$$

and for any $I = (s, t] \times (x, y]$

$$E\xi_n(I) = \sum_{i=[ns]+1}^{[nt]} P(X_{ni} \in (x, y]) \le - \sum_{i=[ns]+1}^{[nt]} \log P(X_{ni} \notin (x, y]) \to \lambda(I) = E\xi(I).$$

REMARKS 3.2. The preceding results with appropriate notational changes also hold for vector-valued, multiparameter extremal processes of the form $\mathbf{M}_n^k(\mathbf{t}) := (M_{n1}^k(\mathbf{t}), \cdots, M_{nd}^k(\mathbf{t}))$ $\mathbf{t} := (t_1, \cdots, t_m) \in (0, \infty)^m$, where

$$M_{n\ell}^k(\mathbf{t}) = k\text{th largest } \{X_n\ell(i) : i_1 \le [nt_1], \cdots, i_m \le [nt_m]\},\ i \le \ell \le d,$$

and $(X_{n1}(\mathbf{i}), \cdots, X_{nd}(\mathbf{i}))$, $\mathbf{i} := (i_1, \cdots, i_m) \in \{1, 2, \cdots\}^m$, are independent vectors for each $n \ge 1$. Here ξ_n is a point process on the $m + 2$-dimensional Euclidean space with points at $(i_1/n, \cdots, i_m/n, X_{n\ell}(\mathbf{i}), \ell)\mathbf{i} \in \{1, 2, \cdots\}^m$, $\ell = 1, \cdots d$. An example of this is $X_{n1}(i) = (X_i - a_n)/b_n$ and $X_{n2}(i) = (-X_i - c_n)/d_n$, and so $M_{n1}^k(t)$ and $M_{n2}^k(t)$ record the kth largest and kth smallest of the sequence $\{X_i\}$. The results above, and their vector analogues, can easily be extended to include (1) randomized norming constants $\tau_n(t)$ instead of $[nt]$ as in Durrett and Resnick (1978) and Galambos (1975), and (2) random (or deterministic) locations T_{ni} instead of i/n associated with X_{ni} as in Westcott (1977). Limit theorems for records and record times can also be obtained by applying Theorem 2.2 and the continuous mapping theorem as in Resnick (1975).

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