The Annals of Probability
1982, Vol. 10, No. 1, 124–131

ASYMPTOTIC BEHAVIOUR OF SYMMETRIC POLYNOMIAL STATISTICS

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A general theorem of Hoeffding (Hoeffding (1948)) implies that the limit distribution of symmetric statistics under natural conditions is normal, but rather frequently the variance of this normal distribution vanishes. Even in the very simple case when the statistics are elementary symmetric polynomials of independent random variables $\{X_i: P(X_i = \pm 1) = \frac{1}{2}, i = 1, 2, \cdots, n\}$, the limit distributions (in general) are not normal. These distributions will be determined below.

1. Limit distributions. Let $X_1, X_2, \cdots$ be a sequence of independent random variables, $P(X_i = \pm 1) = \frac{1}{2}$ $(i = 1, 2, \cdots)$ and

$$S_n(k) = \sum_{1 \le j_1 < j_2 < \cdots < j_k \le n} x_{j_1} x_{j_2} \cdots x_{j_k} \qquad (n = 1, 2, \cdots)$$

$$S_n(0) = 1$$

the kth elementary symmetric polynomial of $X_1, X_2, \cdots, X_n$.

THEOREM 1. *Denote by N a standard normal variable and by $\{H_k : k = 0, 1, \cdots\}$ the orthonormal system of polynomials with respect to the standard normal distribution (i.e., let $H_k = G_k/\sqrt{k!}$ where $\{G_k: k = 0, 1, \cdots\}$ is the system of Hermite polynomials). If k is a fixed nonnegative number and $n \to \infty$ then*

$$\text{(i)} \qquad S_n(k)\Big/\sqrt{\binom{n}{k}} \to_d H_k(N)$$

and

$$\text{(ii)} \qquad S_n(n-k)\Big/\sqrt{\binom{n}{n-k}} \to_d XH_k(N),$$

where X has a uniform distribution on the set $\{-1, +1\}$ and it is independent of N. (If k is an odd number then $XH_k(N) =_d H_k(N)$.)

REMARK. The referee called our attention to the paper Rubin and Vitale (1980) which overlaps Theorem 1, however for the sake of completeness we have not deleted this part of our paper (which had been completed before the appearance of the work of Rubin and Vitale).

THEOREM 2. *Denote $S_n(1)$ by S. If $k \to \infty$ and $n - k \to \infty$ then*

$$\left(\frac{k(n-k)}{n}\right)^{1/4} S_n(k)\Big/\sqrt{\binom{n}{k}} - (2/\pi)^{1/4} \exp\{S^2/4n\} \cdot \cos(-k\pi/2 + S \arcsin \sqrt{k/n}) \to_p 0.$$

THEOREM 3. *Denote by N a standard normal variable, let U have the uniform distribution on the interval $[0, 1]$ and U_q the uniform distribution on the finite set $\{0, 1/q, 2/q, \cdots, (q-1)/q\}$, where q is a natural number. Let N be independent of U and U_q. Suppose that $k \to \infty$, $n - k \to \infty$ and $k/n \to c$ $(0 \le c \le 1)$. Then*

Received July 2, 1979; revised December 1980.
AMS 1970 *subject classification.* Primary 60F05.
Key words and phrases. Symmetric polynomial, saddle-point method.

(i) $k^{1/4}S_n(k)/\sqrt{\binom{n}{k}} \to_d (2/\pi)^{1/4} \exp(N^2/4) \cos(2\pi U)$ *if* $c = 0$,

(ii) $(n-k)^{1/4}S_n(k)/\sqrt{\binom{n}{k}} \to_d (2/\pi)^{1/4} \exp(N^2/4) \cos(2\pi U)$ *if* $c = 1$,

(iii) $n^{1/4}S_n(k)/\sqrt{\binom{n}{k}} \to_d \left(\frac{2}{\pi c(1-c)}\right)^{1/4} \exp(N^2/4) \cos(2\pi U)$ *if* $(2\pi)^{-1} \arcsin \sqrt{c}$ *is an irrational number, or if it is rational,* $\sqrt{n}\,|k/n - c| \to \infty$ *and* $0 < c < 1$.

(iv) $n^{1/4}S_n(k)/\sqrt{\binom{n}{k}} \to_d \left(\frac{2}{\pi c(1-c)}\right)^{1/4} \exp(N^2/4) \cos\left(2\pi U_q + \frac{b}{2\sqrt{c(1-c)}}N\right)$ *if* $(2\pi)^{-1} \arcsin \sqrt{c}$ *is a rational number of the form* p/q *where* p *and* q *are relative prime numbers,* q *is divisible by* 4, $\sqrt{n}\,|k/n - c| \to b$ *and* $0 < c < 1$.

THEOREM 4. *If* $n \to \infty$ *and* $k/n \to c$ $(0 \le c \le 1)$ *then*

$$\limsup_{n\to\infty} |S_n(k)|^{1/n} = \exp\{H(c, 1-c)/2\}$$

with probability one, where $H(c, 1-c) = -c \log c - (1-c)\log(1-c)$.

REMARKS. 1. The method of proofs below can easily be applied for more general random variables too, e.g., Theorem 1 (i) remains valid for an arbitrary sequence of independent, identically distributed random variables with expectation 0 and variance 1, but in this paper we do not want to go into such details.

2. As it will be seen from the proofs below, our Theorem 1 can be generalized to obtain the following functional limit theorem:

Let $X_1, X_2, \cdots$ *be* i.i.d. *rv's with expectation* 0 *and variance* 1. *Define a random element of the Skorohod space* $D(0, 1)$ *by*

$$S_{nk}(t) = \begin{cases} 0 & \text{if } 0 \le t \le (k-1)/n \\ S_j(k)/\sqrt{\binom{n}{k}} & \text{if } (j-1)/n < t \le j/n \end{cases}$$

$$(j = k, k+1, \cdots, n).$$

If k *is fixed and* $n \to \infty$ *then*

$$S_{nk}(t) \to_d t^{k/2}H_k(t^{-1/2}W(t)) \quad \text{in} \quad D(0, 1),$$

where $W(t)$ $(0 \le t \le 1)$ *is a standard Wiener process.*

3. The probability density function of $\exp(N^2/4)\cos(2\pi U)$ (which appears in Theorem 3) is

$$2^{1/2}\pi^{-3/2}\int_1^\infty t^{-2}((t^2 - x^2)\log t)^{-1/2}\,dt \qquad \text{if } |x| < 1,$$

and

$$2^{1/2}\pi^{-3/2}\int_1^\infty (tx)^{-2}((t^2 - 1)\log|tx|)^{-1/2}\,dt \qquad \text{if } |x| > 1.$$

The only local minimum of this function is taken at the point $x = 0$ (the value of the local minimum is $1/\pi$). If $x \to \pm 1$ then the limit of the density function is $+\infty$.

4. In Theorem 3 (iv) if the denominator q does not divide by 4, then the asymptotic

distribution of $n^{1/4}S_n(k)/\sqrt{\binom{n}{k}}$ depends on the mod 4 behaviour of k while $n \to \infty$.

2. Proof of the theorems.

PROOF OF THEOREM 1. (i): Denote $Q_n(k) = \sum_{j=1}^n X_j^k$.
If $n \geq k$ then by Newton's identities

$$Q_n(k)S_n(0) - Q_n(k-1)S_n(1) + \cdots + (-1)^{k-1}Q_n(1)S_n(k-1) + (-1)^k kS_n(k) = 0.$$

Thus $S_n(k) = P_k(Q_n(1), Q_n(2), \cdots, Q_n(k))$, where P_k is a polynomial of degree k in k variables. Let $G_k(x) = k!P_k(x, 1, 0, \cdots, 0)$. It is easy to see that $G_0 = 1$, $G_1 = x$ and by Newton's identities

$$G_k = xG_{k-1} - (k-1)G_{k-2}, \qquad k = 2, 3, \cdots,$$

therefore G_k is the Hermite polynomial of degree k. Thus

$$S_n(k)/\sqrt{\binom{n}{k}} \sim \sqrt{k!}\,n^{-k/2}S_n(k) = P_k(n^{-1/2}Q_n(1), n^{-1}Q_n(2), \cdots, n^{-k/2}Q_n(k))\sqrt{k!}$$

$$\sim_d P_k(N, 1, 0, \cdots, 0)\sqrt{k!} = H_k(N) \qquad \text{as} \quad n \to \infty.$$

(ii): It is enough to observe that $S_n(n-k) = S_n(n)S_n(k)$, $P(S_n(n) = \pm 1) = 1/2$ and the asymptotic normality of $n^{-1/2}Q_n(1)$ is independent of $S_n(n)$.

PROOF OF THEOREM 2. Because of the symmetry we may suppose that

$$k/n \leq 1/2.$$

For any complex number z,

$$\sum_{k=0}^n S_n(k)z^k = S_n(n)\sum_{k=0}^n S_n(n-k)z^k = S_n(n)\prod_{j=1}^n (X_j + z)$$
$$= \prod_{j=1}^n (1 + X_j z) = (1-z)^{(n-S)/2}(1+z)^{(n+S)/2};$$

thus by Cauchy's formula

$$S_n(k) = \frac{1}{2\pi i}\int_{|z|=\rho} (1-z)^{(n-S)/2}(1+z)^{(n+S)/2}z^{-(k+1)}\,dz$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} (1-\rho e^{i\vartheta})^{(n-S)/2}(1+e^{i\vartheta})^{(n+S)/2}\rho^{-k}e^{-ik\vartheta}\,d\vartheta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\{u_n(\vartheta)\}\,d\vartheta = Re\left(\frac{1}{\pi}\int_0^{\pi} \exp\{u_n(\vartheta)\}\,d\vartheta\right)$$

for any $\rho = \rho_n > 0$, where

$$u_n(\vartheta) = \frac{n-S}{2}\log(1-\rho e^{i\vartheta}) + \frac{n+S}{2}\log(1+\rho e^{i\vartheta}) - k\log\rho - ik\vartheta.$$

This integral can be approximately evaluated in terms of the maximum value of $|\exp\{u_n(\vartheta)\}|$ taking account of the speed of its decrease on the contour of integration. If the path of integration is such that on a small section of it the absolute value of the integrand reaches its maximum and then rapidly decreases, it is natural to suppose that the quantity thus found yields a good approximation. If a function is harmonic in a domain then it cannot attain its maximum at interior points of this domain, therefore we have to analyse the behaviour of $u_n(\vartheta) = u_n(\rho, \vartheta)$ at the neighbourhood of its saddle-point. This is the essence of the saddle-point method of complex analysis.

One can easily compute that the only saddle-point (ρ_n, ϑ_n) of u_n is determined by the equations

$$\rho_n = \sqrt{\frac{k}{n-k}}, \qquad \cos\vartheta_n = \frac{S}{2\sqrt{k(n-k)}}$$

if $|S|/2\sqrt{k(n-k)} \leq 1$ is supposed, but the probability of the latter tends to 1 as $n \to \infty$ and $k \to \infty$. It is obvious that $\rho_n \leq 1$ and ϑ_n tends to $\pi/2$ in probability as $n \to \infty$.

For the Taylor expansion of $u_n(\vartheta)$ we need the following derivatives:

$u'_n(\vartheta_n) = \frac{\partial u_n}{\partial \vartheta}(\rho_n, \vartheta_n) = 0$ because (ρ_n, ϑ_n) is a saddle-point. For the sake of brevity in the following we shall always write ρ instead of ρ_n.

$$u''_n(\vartheta_n) = \frac{2n\rho^2 \sin \vartheta_n}{(1+\rho^2)^2(1-S^2/n^2)}\left(-\sin \vartheta_n + i\left(1 - 2\frac{k}{n}\right)\cos \vartheta_n\right)$$

hence $u''_n(\vartheta_n) \sim_p -\frac{2k(n-k)}{n}$. Finally

$$u'''_n(\vartheta) = \frac{i4\rho^2 e^{2i\vartheta}(1+\rho^2 e^{2i\vartheta})n - i\rho e^{i\vartheta}(1+6\rho^2 e^{2i\vartheta}+\rho^4 e^{4i\vartheta})S}{(1-\rho^2 e^{2i\vartheta})^3}$$

hence if $\pi/4 < \vartheta < 3\pi/4$ then

$$|u'''_n(\vartheta_n)| \leq 8\rho^2 n + 8\rho|S| \leq 16k(1+|\cos \vartheta_n|) \leq 32k.$$

Now let us compute the integral $(1/\pi)\int_0^\pi \exp\{u_n(\vartheta)\}\, d\vartheta$.

Let $\delta > 0$ be a sufficiently small fixed number. Since the function $|\exp\{u_n(\vartheta)\}|$ increases if $0 < \vartheta < \vartheta_n$ and decreases if $\vartheta_n < \vartheta < \pi$, hence by the Taylor expansion of $u_n(\vartheta)$

$$\left|\frac{1}{\pi}\int_0^{\vartheta_n-\delta} \exp\{u_n(\vartheta)\}\, d\vartheta\right| \leq |\exp\{u_n(\vartheta_n-\delta)\}|$$

$$= |\exp\{u_n(\vartheta_n)\}| \exp\{-C_1\delta^2 k\}$$

where $C_1 > 0$. A similar estimation holds for

$$\left|\frac{1}{\pi}\int_{\vartheta_n+\delta}^{\pi} \exp\{u_n(\vartheta)\}\, d\vartheta\right|.$$

Making use of the equality

$$e^{a+b} = e^a + O(be^{a+|b|})$$

we obtain

$$\int_{\frac{1}{\pi}|\vartheta-\vartheta_n|<\delta} \exp\{u_n(\vartheta)\}\, d\vartheta = exp\{u_n(\vartheta_n)\}\frac{1}{\pi}\int_{-\delta}^{\delta} \exp\left\{\frac{1}{2}u''_n(\vartheta_n)h^2 + O(k)h^3\right\} dh$$

$$= \exp\{u_n(\vartheta_n)\}\left[\frac{1}{\pi}\int_{-\delta}^{\delta}\exp\left\{\frac{1}{2}u''_n(\vartheta_n)h^2\right\} dh \right.$$

$$\left. +O(k)\int_{-\delta}^{\delta} |h^3| \exp\{-C_2kh^2\}\, dh\right]$$

$$\sim_p \exp\{u_n(\vartheta_n)\}\frac{1}{\pi}\int_{-\infty}^{\infty}\exp\left\{\frac{1}{2}u''_n(\vartheta_n)h^2\right\} dh$$

$$= \exp\{u_n(\vartheta_n)\}\left(-\frac{\pi}{2}u''_n(\vartheta_n)\right)^{-1/2}$$

where the argument of the value of square root is taken in the interval $(-\pi/2, \pi/2)$. All these imply that

$$S_n(k) \sim_p \mathrm{Re}\{(1-\rho e^{i\vartheta_n})^{(n-S)/2}(1+\rho e^{i\vartheta_n})^{(n+S)/2}\rho^{-k}e^{-ik\vartheta_n}\left(-\frac{\pi}{2}u''_n(\vartheta_n)\right)^{-1/2p}.$$

Here

$$\left| (1 - \rho e^{i\vartheta_n})^{(n-S)/2}(1 + \rho e^{i\vartheta_n})^{(n+S)/2} \rho^{-k}\left(-\frac{\pi}{2} u_n''(\vartheta_n)\right)^{-1/2} \right|$$

$$\cdot \sim_p (1 - S/n)^{(n-S)/4}(1 + S/n)^{(n+S)/4}(1 - k/n)^{(n-k)/2}(k/n)^{-k/2} \left(\frac{n}{\pi k(n-k)}\right)^{1/2}$$

$$\cdot \sim_p \left(\frac{2n}{\pi k(n-k)}\right)^{1/4} \sqrt{\binom{n}{k}} \exp\{S^2/4n\}.$$

In the sequel the argument of the approximate formula obtained for $S_n(k)$ will be discussed.

$$\begin{aligned}\arg\{(1 - \rho e^{i\vartheta_n})^{(n-S)/2}(1 + \rho e^{i\vartheta_n})^{(n+S)/2}\} &= \frac{n+S}{2} \arcsin\{(k/n)^{1/2}(1 + S/n)^{-1/2} \sin \vartheta_n\} \\ &\quad - \frac{n-S}{2} \arcsin\{(k/n)^{1/2}(1 - S/n)^{-1/2} \sin \vartheta_n\} \\ &= -\frac{S}{2}\sqrt{\frac{k}{n-k}} + S \arcsin \sqrt{k/n} + \omega_n\end{aligned}$$

where $\omega_n \to_p 0$ as $n \to \infty$.

Finally it suffices to notice that

$$\arg(-1/2\, u_n''(\vartheta_n))^{-1/2} \to_p 0$$

and

$$-k\vartheta_n = -k \arccos \frac{S}{2\sqrt{k(n-k)}} = -k\frac{\pi}{2} + \frac{S}{2}\sqrt{\frac{k}{n-k}} + \omega_n$$

where ω_n tends again to zero in probability, and the proof of Theorem 2 is complete.

REMARK. If $X_1, X_2, \cdots$ are positive random variables then the saddle-point of u_n is always on the real axis, i.e. $\vartheta_n = 0$. This fact simplifies the asymptotic expansions (see G. Halász and G. J. Székely (1976)).

PROOF OF THEOREM 3. We shall need the following

LEMMA. *Let (α_n) $n = 1, 2, \cdots$ be a sequence of real numbers and suppose that*

$$\lim_{n\to\infty} \sqrt{n}\, |\exp(2\pi i m \alpha_n) - 1| = +\infty$$

for every non-zero integer m. Then

$$(n^{-1/2}S, \{\alpha_n S\}) \to_d (N, U)$$

where $\{\cdot\}$ stands for fractional part.

PROOF. The central limit theorem assures the asymptotic normality of $n^{-1/2}S$, therefore it suffices to show that the conditional distribution of $\{\alpha_n S\}$ is asymptotically uniform, given the condition $n^{-1/2}S < x$. But this will be a consequence of Weyl's theorem if we prove that for every non-zero integer m and real x

$$\lim_{n\to\infty} E(\exp(2\pi i m \alpha_n S) \mid S < x\sqrt{n}) = 0.$$

To prove this equation let M be a large natural number and $x_0 < x_1 < \cdots < x_M = x$ a partition of the real line, for which the total variation of the standard normal density function φ in the interval $[x_{j-1}, x_j)$ is less than $1/M$ $(j = 1, 2, \cdots, M)$, in addition $x_0 < 0$ and $\varphi(x_0) = 1/M$. Then

$$\begin{aligned} E &= E(\exp(2\pi i m\alpha_n S) I(S < x\sqrt{n})) \\ &= \textstyle\sum_{\nu < x_0\sqrt{n}} P(S=\nu)\exp(2\pi i m\alpha_n \nu) + \sum_{j=1}^{M} \sum_{x_{j-1}\sqrt{n} \le \nu < x_j\sqrt{n}} P(S=\nu)\exp(2\pi i m\alpha_n\nu) \\ &= A_0 + \textstyle\sum_{j=1}^{M} A_j. \end{aligned}$$

Here $|A_0|$ is less than $P(S < x_0\sqrt{n}\,)$ which converges, as $n \to \infty$, to

$$\int_{-\infty}^{x_0} \varphi(t)\,dt \le -\frac{\varphi(x_0)}{x_0} = -\frac{1}{Mx_0}.$$

For $j = 1, 2, \cdots, M$

$$\begin{aligned} |A_j| \le \left| \frac{2}{\sqrt{n}} \varphi(x_{j-1}) \textstyle\sum_\nu \exp(2\pi i m\alpha_n\nu) \right| + \left| \textstyle\sum_\nu \exp(2\pi i m\alpha_n\nu) \dfrac{2}{\sqrt{n}} (\varphi(x_{j-1}) - \varphi(\nu/\sqrt{n})) \right| \\ + \left| \textstyle\sum_\nu \exp(2\pi i m\alpha_n\nu) \left(\dfrac{2}{\sqrt{n}} \varphi(\nu/\sqrt{n}) - P(S=\nu) \right) \right| \end{aligned}$$

where these summations run over the integers $x_{j-1}\sqrt{n} \le \nu < x_j\sqrt{n}$. In the right-hand side the first term is majorized by

$$\frac{2}{\sqrt{n}} \varphi(x_{j-1}) 2/|\exp(2\pi i m\alpha_n) - 1|$$

which tends to zero as $n \to \infty$ by the assumption. The second term is less than

$$\frac{2}{\sqrt{n}} \textstyle\sum_\nu |\varphi(x_{j-1}) - \varphi(\nu/\sqrt{n})| \le \dfrac{2}{M} (x_j - x_{j-1}),$$

and the last term is of order $O(1/\sqrt{n}\,)$, since

$$P(S=\nu) = \frac{2}{\sqrt{n}} \varphi(\nu/\sqrt{n}) + O(1/n)$$

uniformly for $x_{j-1}\sqrt{n} \le \nu < x_j\sqrt{n}$ (see e.g. Feller (1950) Ch. VII/2).

Thus

$$\limsup_{n\to\infty} |E| \le \frac{2}{M} (x - x_0 - 1/2x_0) = O\left(\frac{1}{M}\sqrt{\log M}\right)$$

which completes the proof of the Lemma.

Let us pass over to the proof of Theorem 3. The cases (i)–(iii) are simple consequences of Theorem 2 and the previous Lemma. Acutally it suffices to prove that for every nonzero integer m

$$\lim_{n\to\infty} \sqrt{n}\,|\exp(im \arcsin \sqrt{k/n}) - 1| = +\infty.$$

In the case (i) i.e. when $c = 0$

$$\begin{aligned} &\sqrt{n}\,|\exp(im \arcsin \sqrt{k/n}) - 1| \\ &= \sqrt{n}\,(|m| \arcsin \sqrt{k/n} + O(\arcsin^2 \sqrt{k/n})) \\ &= \sqrt{n}\,(|m| \sqrt{k/n} + O(k/n)) \sim |m| \sqrt{k}. \end{aligned}$$

If $c = 1$ then

$$\begin{aligned} |\exp(im \arcsin \sqrt{k/n}) - 1| &= |\exp(im \arccos \sqrt{(n-k)/n}) - 1| \\ &= |(e^{im\pi/2} - 1) - ime^{im\pi/2} \sqrt{(n-k)/n} + O((n-k)/n)|. \end{aligned}$$

If m does not divide by 4 then the first term is a nonzero constant while the other terms tend to zero. If m divides by 4 then the second term dominates, consequently

$$\sqrt{n}\,|\exp(im \arcsin\sqrt{k/n}) - 1| \sim m\sqrt{n-k}.$$

In the case (iii) if $(2\pi)^{-1}\arcsin\sqrt{c}$ is an irrational number then

$$\lim_{n\to\infty}|\exp(im \arcsin\sqrt{k/n}) - 1| = |\exp(im \arcsin\sqrt{c}) - 1| \neq 0.$$

If $(2\pi)^{-1}\arcsin\sqrt{c} = p/q$ and $\sqrt{n}\,|k/n - c| \to +\infty$, then

$$|\exp(im \arcsin\sqrt{k/n}) - 1|$$
$$= |(\exp(2\pi imp/q) - 1) + \frac{1}{2\sqrt{c(1-c)}}\exp(2\pi imp/q)(k/n - c) + O((k/n-c)^2)|.$$

Here if m does not divide by q then the first term is a nonzero constant while the other terms tend to zero. Otherwise the second term dominates and the condition imposed on the convergence rate of k/n is applicable.

Finally in the case (iv) let us start from the expansion

$$S\arcsin\sqrt{k/n} = S\arcsin\sqrt{c} + \frac{S}{2\sqrt{c(1-c)}}(k/n - c) + S\cdot O((k/n-c)^2).$$

On the right-hand side the second term is asymptotically equal to $\pm\frac{b}{2\sqrt{c(1-c)}}\frac{S}{\sqrt{n}}$ where the sign $\pm$ depends on the sign of $((k/n) - c)$. Since the limit distribution of $n^{-1/2}S$ is symmetrical, this sign does not play any role. The third term equals to $O(S/n)$. Concerning the first term it is easy to see that

$$\left(S/\sqrt{n}, \left\{\frac{p}{q}S\right\}\right) \to_d (N, U_q).$$

Finally if q divides by 4 then

$$\cos(-k\pi/2 + S\arcsin\sqrt{k/n}) \to_d \cos\left(2\pi U_q + \frac{b}{2\sqrt{c(1-c)}}N\right).$$

The proof of Theorem 3 is complete.

PROOF OF THEOREM 4. By the Chebyshev inequality

$$P\left(|S_n(k)| \geq n\sqrt{\binom{n}{k}}\right) \leq n^{-2}.$$

Thus the Borel-Cantelli lemma implies that a.s. $|S_n(k)| < n\sqrt{\binom{n}{k}}$ if n is large enough; therefore,

$$\limsup_{n\to\infty}|S_n(k)|^{1/n} \leq \lim_{n\to\infty}\binom{n}{k}^{1/2n} = \exp\left\{\frac{1}{2}H(c, 1-c)\right\}$$

with probability one.

One the other hand, the proof of Theorem 3 yields that

$$\liminf_{n\to\infty} P\left(|S_n(k)| > C\left(\frac{2n}{\pi k(n-k)}\right)^{1/4}\sqrt{\binom{n}{k}}\right) > 0$$

if $C > 0$ is sufficiently small. Thus

$$P(\limsup_{n\to\infty}|S_n(k)|^{1/n} \geq \exp\{\tfrac{1}{2}H(c, 1-c)\}) > 0;$$

therefore the zero or one law of Hewitt and Savage implies that this probability has to be equal to one.

REMARK. The determination of all a.s. accumulation points of $|S_n(k)|^{1/n}$ as $n \to \infty$ and $k/n \to c$ $(0 \le c \le 1)$ seems to be a more difficult problem.

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