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A CLASS OF MULTIVARIATE NEW BETTER THAN USED DISTRIBUTIONS

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A class of multivariate distributions with new better than used (NBU) marginals is introduced. A number of necessary and sufficient conditions for a distribution to be a member in the class are given. Closure results, which are useful for the identification or construction of members in the class, are derived. In particular, simple proofs of some well known preservation properties of the class of the univariate NBU distributions are obtained. Two examples of replacement models, that give rise to multivariate NBU distributions in the class, are discussed.

1. Introduction. The class of "new better than used" distributions was introduced in the context of reliability theory where it arises in the study of replacement policies. A univariate distribution F or corresponding random variable T is said to be *new better than used* (NBU) if $P\{T \geq 0\} = 1$ and if

$$P\{T > s + t \mid T > s\} \leq P\{T > t\} \quad \text{for all} \quad s, t \geq 0. \tag{1.1}$$

With the notation $\bar{F}(t) = 1 - F(t) = P\{T > t\}$, this condition can be written in the form $F(x) = 0$ for all $x < 0$ and

$$\bar{F}(s + t) \leq \bar{F}(s)\bar{F}(t) \quad \text{for all} \quad s, t \geq 0. \tag{1.2}$$

It is well known (see Barlow and Proschan, 1975, page 159) that if F has an increasing hazard rate average (IHRA), then F is NBU, but not conversely.

Multivariate extensions of the IHRA property have been considered by Buchanan and Singpurwalla (1977), by Esary and Marshall (1979), and by Block and Savits (1980). The purpose of this paper is to discuss a multivariate extension of the NBU property that is quite analogous to the multivariate IHRA extension of Block and Savits (1980). Some ideas developed here lead naturally to other multivariate NBU conditions we discuss in another paper (Marshall and Shaked, 1980).

In the following, "increasing" stands for "nondecreasing" and "decreasing" stands for "nonincreasing."

2. Definition and equivalent conditions. The condition (1.2) that F is NBU can be rewritten in the form

$$P\{T \in (\alpha + \beta)A\} \leq P\{\mathbf{T} \in \alpha A\} P\{\mathbf{T} \in \beta A\} \tag{2.1}$$

for every $\alpha \geq 0$, $\beta \geq 0$ and every set $A = (s, \infty)$ such that $s \geq 0$. Sets A of the form (s, ∞) are open and have increasing indicator functions. They have natural multidimensional analogs: A set $A \subset \mathscr{R}^n$ is said to be an *increasing set*, or to be an *upper set*, if $\mathbf{x} \in A$ and $x_i \leq y_i$, $i = 1, \cdots, n$ implies $\mathbf{y} \in A$. Sets of this kind are also called *upper layers*, e.g., by Robertson and Wright (1974) and Steele (1978).

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In the remainder of this paper, *sets and functions are assumed to be Borel measurable* whenever measurability considerations are relevant.

2.1. DEFINITION. A random vector $\mathbf{T} = (T_1, \cdots, T_n)$ is said to be *multivariate new better than used* (MNBU) if $P\{T_i \geq 0, i = 1, \cdots, n\} = 1$ and if

$$P\{\mathbf{T} \in (\alpha + \beta)A\} \leq P\{T \in \alpha A\} P\{T \in \beta A\} \tag{2.2}$$

for every $\alpha \geq 0$, $\beta \geq 0$ and for every open upper set $A \subset [0, \infty)^n$.

A number of conditions equivalent to MNBU can be given but some terminology is useful for their statement.

A real function g defined on $[0, \infty)^n$ is said to be *subhomogeneous* if

$$\alpha g(\mathbf{t}) \leq g(\alpha \mathbf{t}) \text{ for every } \alpha \in [0, 1] \text{ and every } \mathbf{t} \geq 0, \tag{2.3.i}$$

or equivalently, if

$$\alpha g(\mathbf{t}) \geq g(\alpha \mathbf{t}) \text{ for every } \alpha \geq 1 \text{ and every } \mathbf{t} \geq 0. \tag{2.3.ii}$$

If equality holds in (2.3.i) for every $\alpha \in [0, 1]$ and every $\mathbf{t} \geq 0$, or if equality similarly holds in (2.3.ii), then g is said to be homogeneous. Homogeneous functions of the form

$$g(\mathbf{t}) = \max_{1 \leq j \leq m} \min_{1 \leq i \leq n} a_{ij} t_i, \quad 0 \leq a_{ij} \leq \infty, \quad i = 1, \cdots, n, \quad j = 1, \cdots, m \tag{2.4}$$

are particularly important examples.

2.2 THEOREM. *For a random vector* $\mathbf{T} = (T_1, \cdots, T_n)$ *such that*

$$P\{T_i \geq 0, i = 1, \cdots, n\} = 1,$$

the following conditions are equivalent:

(i) $\mathbf{T}$ *is* MNBU;

(ii) *for every* $\alpha > 0$, $\beta > 0$ *and every increasing binary* (*i.e., indicator*) *function* ϕ,

$$E\phi\left(\frac{1}{\alpha + \beta}\mathbf{T}\right) \leq E\phi\left(\frac{1}{\alpha}\mathbf{T}\right) E\phi\left(\frac{1}{\beta}\mathbf{T}\right); \tag{2.5}$$

(iii) *for every* $\alpha > 0$, $\beta > 0$, $\gamma \in (0, 1)$ *and every nonnegative increasing function* h *defined on* $[0, \infty)^n$,

$$Eh\left(\frac{1}{\alpha + \beta}\mathbf{T}\right) \leq Eh^{\gamma}\left(\frac{1}{\alpha}\mathbf{T}\right) Eh^{1-\gamma}\left(\frac{1}{\beta}\mathbf{T}\right); \tag{2.6}$$

(iv) *for every nonnegative increasing subhomogeneous function* g, $g(\mathbf{T})$ *has an* NBU *distribution*;

(v) *for every nonnegative increasing homogeneous function* g, $g(\mathbf{T})$ *has an* NBU *distribution.*

PROOF. The equivalence of these conditions is established by showing that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). Let ϕ be the indicator function of the increasing set A (not necessarily open) and fix $\alpha > 0$, $\beta > 0$. Esary, Proschan and Walkup (1967), pages 1468–1469 show that for every $\varepsilon > 0$, there exists an open increasing set A_ε possibly depending on α and β such that $A \subset A_\varepsilon$ and such that $P\{\mathbf{T} \in \alpha A_\varepsilon\} \leq P\{\mathbf{T} \in \alpha A\} + \varepsilon$ and $P\{\mathbf{T} \in \beta A_\varepsilon\} \leq P\{\mathbf{T} \in \beta A\} + \varepsilon$. Since (2.2) holds for A_ε

$$\begin{aligned} E\phi\left(\frac{1}{\alpha + \beta}\mathbf{T}\right) &= P\{\mathbf{T} \in (\alpha + \beta)A\} \leq P\{\mathbf{T} \in (\alpha + \beta)A_\varepsilon\} \leq P\{\mathbf{T} \in \alpha A_\varepsilon\} P\{\mathbf{T} \in \beta A_\varepsilon\} \\ &\leq [P\{\mathbf{T} \in \alpha A\} + \varepsilon][P\{\mathbf{T} \in \beta A\} + \varepsilon] = \left[E\phi\left(\frac{1}{\alpha}\mathbf{T}\right) + \varepsilon\right]\left[E\phi\left(\frac{1}{\beta}\mathbf{T}\right) + \varepsilon\right]. \end{aligned}$$

Upon letting $\varepsilon \to 0$, it follows that (2.5) holds.

(ii) $\Rightarrow$ (iii). Let h_k, $k = 1, 2, \cdots$ be an increasing sequence of increasing step functions with the property that $\lim_{k\to\infty} h_k = h$ pointwise; to be specific take

$$h_k(\mathbf{t}) = \frac{i-1}{2^k} \quad \text{if } \frac{i-1}{2^k} \le h(\mathbf{t}) < \frac{i}{2^k}, \quad i = 1, \cdots, k2^k$$
$$= k \qquad \text{if } h(\mathbf{t}) \ge k.$$

Denote the indicator function of a set A by I_A. Then

$$h_k(\mathbf{t}) = \textstyle\sum_{i=1}^{k2^k} \frac{1}{2^k} I_{A_{i,k}}(\mathbf{t})$$

where $A_{i,k} = \left\{\mathbf{t}: h(\mathbf{t}) \ge \frac{i}{2^k}\right\}$, $i = 1, \cdots, k2^k$, $k = 1, 2, \cdots$ is an increasing set. Note that $A_{1,k} \supset \cdots \supset A_{k2^k,k}$. Because of the monotone convergence theorem, it is sufficient to show that (2.6) holds for function h of the form

$$h(\mathbf{t}) = \textstyle\sum_{i=1}^{m} a_i I_{A_i}(\mathbf{t})$$

where $a_i \ge 0$, $i = 1, \cdots, m$ and $A_1 \supset \cdots \supset A_m$ are upper sets. For notational convenience, let $A_{m+1} = \phi$.

In the following, the first inequality follows from (ii), and the second equality is obtained by two changes of order of summation:

$$Eh\left(\frac{1}{\alpha+\beta}\mathbf{T}\right) = \textstyle\sum_{i=1}^{m} a_i P\{\mathbf{T} \in (\alpha+\beta)A_i\} \le \sum_{i=1}^{m} a_i P\{\mathbf{T} \in \alpha A_i\}\, P\{\mathbf{T} \in \beta A_i\}$$
$$= \textstyle\sum_{i=1}^{m}\sum_{j=1}^{m} [a_1 + \cdots + a_{\min(i,j)}] P\{\mathbf{T} \in \alpha(A_i - A_{i+1})\}\, P\{\mathbf{T} \in \beta(A_j - A_{j+1})\}$$
$$\le \textstyle\sum_{i=1}^{m}\sum_{j=1}^{m} (a_1 + \cdots + a_i)^\gamma (a_1 + \cdots + a_j)^{1-\gamma} P\{\mathbf{T} \in \alpha(A_i - A_{i+1})\}$$
$$\cdot P\{\mathbf{T} \in \beta(A_j - A_{j-1})\}$$
$$= [\textstyle\sum_{i=1}^{m} (a_1 + \cdots + a_i)^\gamma P\{\mathbf{T} \in \alpha(A_i - A_{i+1})\}]$$
$$\cdot[\textstyle\sum_{j=1}^{m} (a_1 + \cdots + a_j)^{1-\gamma} P\{\mathbf{T} \in \beta(A_j - A_{j+1})\}]$$
$$= Eh^\gamma\left(\frac{1}{\alpha}\mathbf{T}\right) Eh^{1-\gamma}\left(\frac{1}{\beta}\mathbf{T}\right).$$

(iii) $\Rightarrow$ (ii). This is trivial.

(ii) $\Rightarrow$ (iv). Let g be a nonnegative subhomogeneous increasing function. Fix $a > 0$ and set

$$\phi(\mathbf{t}) = I_{\{\mathbf{s}: g(\mathbf{s}) > a\}}(\mathbf{t}).$$

Let $\alpha \in (0, 1)$. In the following, the first inequality is obtained from (2.3.ii) and the second inequality follows from (ii):

$$P\{g(\mathbf{T}) > \alpha\, a\}\, P\{g(\mathbf{T}) > (1-\alpha)a\} \ge P\left\{g\left(\frac{1}{\alpha}\mathbf{T}\right) > a\right\} P\left\{g\left(\frac{1}{1-\alpha}\mathbf{T}\right) > a\right\}$$
$$= E\phi\left(\frac{1}{\alpha}\mathbf{T}\right) E\phi\left(\frac{1}{1-\alpha}\mathbf{T}\right) \ge E\phi(\mathbf{T}) = P\{g(\mathbf{T}) > a\}.$$

Since $a > 0$ and $\alpha \in (0, 1)$ are arbitrary, it follows that $g(\mathbf{T})$ is NBU.

(iv) $\Rightarrow$ (v). This is trivial.

(v) $\Rightarrow$ (i). Let $A \subset [0, \infty)^n$ be an open upper set and define the function g on $[0, \infty)^n$ by

$$g(\mathbf{t}) = \sup\left\{\theta > 0: \frac{1}{\theta}\mathbf{t} \in A\right\}, \text{ if } \left\{\theta > 0: \frac{1}{\theta}\mathbf{t} \in A\right\} \ne \phi$$
$$= 0, \qquad \text{otherwise.}$$

Then g is a nonnegative homogeneous increasing function and for every $\delta \geq 0$, $P\{g(\mathbf{T}) > \delta\} = P\{\mathbf{T} \in \delta A\}$. Since $g(\mathbf{T})$ is NBU,

$$P\{\mathbf{T} \in (\alpha + \beta)A\} = P\{g(\mathbf{T}) > \alpha + \beta\} \leq P\{g(\mathbf{T}) > \alpha\} P\{g(\mathbf{T}) > \beta\}$$
$$= P\{\mathbf{T} \in \alpha A\} P\{\mathbf{T} \in \beta A\}. \qquad \square$$

2.3 Remark. Various modifications of the conditions given in Theorem 2.2 are possible. In (iii), the nonnegative increasing functions can be replaced by the nonnegative increasing continuous functions. Also in (iii), it is sufficient to require that (2.6) holds for some $\gamma \in (0, 1)$. Condition (v) can be replaced by the seemingly weaker condition that requires $g(\mathbf{T})$ to be NBU for each g nonnegative continuous increasing homogeneous function, or even by the condition that requires $\mathbf{g}(\mathbf{T})$ to be NBU only for g of the form (2.4).

The equivalent conditions of Theorem 2.2 is an analog of a set of conditions that Block and Savits (1980) have used to define a multivariate increasing failure rate average (MIFRA) property. Because

(2.7) $\mathbf{T}$ is MIFRA if and only if $g(\mathbf{T})$ is IFRA for every nonnegative increasing homogeneous function g,

it follows that MIFRA $\Rightarrow$ MNBU.

3. Closure properties. The following closure properties are important for the evaluation of the MNBU concept introduced in Section 2, and they are also useful for the identification or construction of examples.

3.1. Property. If $\mathbf{T} = (T_1, \cdots, T_n)$ is MNBU, then any joint marginal is MNBU.

3.2. Property. If $\mathbf{T} = (T_1, \cdots, T_n)$ is MNBU and τ is the life function of a coherent system, then $\tau(\mathbf{T})$ is NBU.

3.3. Property. If $\mathbf{T} = (T_1, \cdots, T_n)$ is MNBU and $a_i \geq 0$, $i = 1, \cdots, n$, then $\Sigma a_i T_i$ is NBU.

Properties 3.1—3.3 follow directly from (v) with obvious choices of particular functions g. These results are important special cases of the following:

3.4 Property. If $\mathbf{T}$ is MNBU and g_j is a nonnegative subhomogeneous increasing function defined on $[0, \infty)^n$, $j = 1, \cdots, m$, then $(g_1(\mathbf{T}), \cdots, g_m(\mathbf{T}))$ is MNBU.

Proof. Let g be a nonnegative subhomogeneous increasing function defined on $[0, \infty)^m$. Then the composition $g(g_1(\mathbf{t}), \cdots, g_m(\mathbf{t}))$ is a nonnegative subhomogeneous increasing function defined on $[0, \infty)^n$. Consequently, the result follows from (iv). $\square$

It follows from 3.3 that if $(T_1, \cdots, T_n)$ is MNBU and $a_i \geq 0$, $i = 1, \cdots n$, then $(a_1 T_1, \cdots, a_n T_n)$ is MNBU.

3.5 Property. If $\mathbf{S} = (S_1, \cdots, S_m)$ and $\mathbf{T} = (T_1, \cdots, T_n)$ are MNBU and if $\mathbf{S}$ and $\mathbf{T}$ are independent, then $(\mathbf{S}, \mathbf{T})$ is MNBU.

Proof. To show that $(\mathbf{S}, \mathbf{T})$ satisfies (ii), let $\alpha \geq 0$, $\beta \geq 0$ and let ϕ be an increasing binary function defined on $\mathscr{R}^{m+n}$. Denote the distribution function of $\mathbf{S}$ by F and the distribution function of $\mathbf{T}$ by G.

In the following sequence, the first inequality follows from the fact that $\mathbf{S}$ satisfies (ii); the second inequality follows from the fact that $\mathbf{T}$ satisfies (ii) and that a product of increasing binary functions is an increasing binary function. The last inequality follows from $\phi \leq 1$.

$$E\left[\phi\left(\frac{1}{\alpha+\beta}\mathbf{S},\frac{1}{\alpha+\beta}\mathbf{T}\right)\right]=\int_{\mathbf{t}}\int_{\mathbf{s}}\phi\left(\frac{1}{\alpha+\beta}\mathbf{s},\frac{1}{\alpha+\beta}\mathbf{t}\right)dF(\mathbf{s})\,dG(\mathbf{t})$$

$$\le\int_{\mathbf{t}}\left[\int_{\mathbf{s}}\phi\left(\frac{1}{\alpha}\mathbf{s},\frac{1}{\alpha+\beta}\mathbf{t}\right)dF(\mathbf{s})\right]\left[\int_{\mathbf{s}'}\phi\left(\frac{1}{\beta}\mathbf{s}',\frac{1}{\alpha+\beta}\mathbf{t}\right)dF(\mathbf{s}')\right]dG(\mathbf{t})$$

$$=\int_{\mathbf{s}}\int_{\mathbf{s}'}\left[\int_{\mathbf{t}}\phi\left(\frac{1}{\alpha}\mathbf{s},\frac{1}{\alpha+\beta}\mathbf{t}\right)\phi\left(\frac{1}{\beta}\mathbf{s}',\frac{1}{\alpha+\beta}\mathbf{t}\right)dG(\mathbf{t})\right]dF(\mathbf{s})\,dF(\mathbf{s}')$$

$$\le\int_{\mathbf{s}}\int_{\mathbf{s}'}\left[\int_{\mathbf{t}}\phi\left(\frac{1}{\alpha}\mathbf{s},\frac{1}{\alpha}\mathbf{t}\right)\phi\left(\frac{1}{\beta}\mathbf{s}',\frac{1}{\alpha}\mathbf{t}\right)dG(\mathbf{t})\right]$$

$$\times\left[\int_{\mathbf{t}'}\phi\left(\frac{1}{\alpha}\mathbf{s},\frac{1}{\beta}\mathbf{t}'\right)\phi\left(\frac{1}{\beta}\mathbf{s}',\frac{1}{\beta}\mathbf{t}'\right)dG(\mathbf{t}')\right]dF(\mathbf{s})\,dF(\mathbf{s}')$$

$$\le\int_{\mathbf{s}}\int_{\mathbf{s}'}\int_{\mathbf{t}}\int_{\mathbf{t}'}\phi\left(\frac{1}{\alpha}\mathbf{s},\frac{1}{\alpha}\mathbf{t}\right)\phi\left(\frac{1}{\beta}\mathbf{s}',\frac{1}{\beta}\mathbf{t}'\right)dG(\mathbf{t})\,dG(\mathbf{t}')\,dF(\mathbf{s})\,dF(\mathbf{s}')$$

$$=E\phi\left(\frac{1}{\alpha}\mathbf{S},\frac{1}{\alpha}\mathbf{T}\right)E\phi\left(\frac{1}{\beta}\mathbf{S},\frac{1}{\beta}\mathbf{T}\right).$$

□

3.6. COROLLARY. *If $T_1, \cdots, T_n$ are independent NBU random variables, then*
(a) $\mathbf{T}=(T_1,\cdots,T_n)$ *is MNBU,*
(b) $g(T_1,\cdots,T_n)$ *is NBU whenever g is a nonnegative subhomogeneous increasing function.*

PROOF. (a) is immediate from Property 3.5. (b) follows from (a) and (iv).
The class of univariate NBU distributions is closed under formation of coherent systems and under convolutions (see Barlow and Proschan, 1975, pages 182–184); according to Corollary 3.6, these well-known facts are special cases of Properties 3.2 and 3.3.

3.7 PROPERTY. If $\mathbf{T}^{(\ell)}$, $\ell=1,2,\cdots$ is a sequence of MNBU random vectors that converges in distribution to $\mathbf{T}$, then $\mathbf{T}$ is MNBU.

PROOF. This fact can be easily verified by first proving that (iii) holds for every nonnegative increasing continuous function h and then by using Remark 2.3. □

4. Examples.

4.1 *A replacement model.* Suppose that devices $d_1,\cdots,d_5$ are available to perform tasks t_1, t_2, t_3. Upon failure of d_1 (which performs all three tasks simultaneously), it is replaced by d_2 (which performs tasks t_1 and t_2) and by d_3 (which performs only task t_3). When device d_2 fails, it is replaced by d_4 (which performs only task t_1) and by d_5 (which performs task t_2). Let X_i be the lifelength of the ith device, $i=1,\cdots,5$, and let T_j be the time that t_j is performed using these devices, $j=1,2,3$. Then

$$T_1=X_1+X_2+X_4,\quad T_2=X_1+X_2+X_5,\quad T_3=X_1+X_3.$$

It follows from Property 3.4 that if $X_1,\cdots,X_5$ are independent NBU, then (T_1,T_2,T_3) is MNBU. Thus, for example, $\tau(T_1,T_2,T_3)$ is NBU when τ is the life function of a coherent system; this fact is not easy to verify directly.

4.2 *Freund's* (1961) *distribution.* Suppose that devices d_1 and d_2 are placed in service together, and are subjected to respective constant hazard rates λ_1 and λ_2 until one or the other fails. From the earliest failure time on, the remaining device d_i is subjected to a new constant hazard rate $\mu_i>\lambda_i$. If T_j is the life length of $d_j, j=1,2$ then it can be shown that

(T_1, T_2) has the same joint distribution as

$$(4.1) \qquad (\min(X_1, X_2 + X_3), \quad \min(X_2, X_1 + X_4))$$

where X_1, X_2, X_3 and X_4 are independent exponential random variables with rates λ_1, λ_2, $\mu_1 - \lambda_1$, $\mu_2 - \lambda_2$ respectively. Since exponential random variables are IFRA, it follows easily that (T_1, T_2) satisfy the "MIFRA condition" of Block and Savits (1980). Using quite involved analysis, these authors obtain the special case $\lambda_2 = \mu_2$ that (T_1, X_2) is MIFRA.

In the above model, (T_1, T_2) is MNBU whenever $X_1, \cdots, X_4$ are independent NBU random variables. Random variables of the form (4.1) arise from a model somewhat like that of Example 4.1.

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