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THE RATE OF CONVERGENCE OF A RANDOM WALK TO BROWNIAN MOTION

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This paper establishes a rate of convergence of a random walk to Brownian motion which is nearly best possible. The Skorokhod representation is employed in the proof.

1. Introduction and summary. Let x_i be a sequence of independent random variables with mean 0 and variance 1. Let $s_n(t) = \sum_{i=1}^{[nt]} x_i/n^{\frac{1}{2}}$, and $w(t)$ be the standard Brownian motion.

THEOREM. *Suppose* $E(\exp |x_i|^c) \leqq M < \infty$. *Then*

$$P(\max_{0 \leqq t \leqq 1} |s_n(t) - w(t)| > \alpha(\log n)^\beta n^{-\frac{1}{4}}) = O(n^{-q}) \tag{1.1}$$

for all q, *where* $\alpha = \alpha(c, q)$ *and* $\beta = c^{-1} + \frac{3}{2}$.

Rosencrantz (1967) proves that

$$|F_n(\lambda) - F(\lambda)| < A(\log n)^{\frac{1}{2}} n^{-(p-2)/2(p+1)}$$

where $F_n(\lambda) = P(\max_{1<k<n} |\sum_{i=1}^k x_i/n^{\frac{1}{2}}| < \lambda)$, and $F(\lambda) = P(\max_{0<t<1} |w(t)| < \lambda)$, if $E(|x_i|^p) < \infty, 2 < p < 4$, and gets a Lévy rate-of-convergence theorem. Heyde (1969) obtains a rate of convergence $A(\log n)^\lambda n^{-p/4(p+1)}$ for $p \geqq 4$, but his estimates are not sufficient for (1.1). By Theorem 2 of Sawyer (1972),

$$P(|s_n(1) - w(1)| \leqq \lambda c/n^{\frac{1}{4}}) = G(\lambda) ,$$

it is clear that the rate $O(n^{-\frac{1}{4}})$ cannot be improved, so a result like (1.1) is of value, since often the variables with which one is involved satisfy the hypothesis. From (1.1) one can draw the usual conclusions, namely

(i) If $\Phi(x)$ is any functional on $C[0, 1]$ such that $P(\Phi(w) \leqq \lambda)$ has a bounded density and $|\Phi(x) - \Phi(y)| \leqq C||x - y||$ then

$$\sup_\lambda |P(\Phi(s_n(1)) \leqq \lambda) - P(\Phi(w(1)) \leqq \lambda)| = O((\log n)^\beta n^{-\frac{1}{4}}) .$$

(ii) By Lemma 1.2 of Prokhorov (1956), if $P_n = P(s_n(\cdot) \in A)$, $L(P_n, W) = O((\log n)^\beta n^{-\frac{1}{4}})$, where $L(\ ,\ )$ is the Prohorov metric.

2. Establishing the result. By means of the Skorokhod (1965) representation, we can find independent random variables τ_i such that $w(\sum_{i=1}^k \tau_i)$ and $\sum_{i=1}^k x_i/n^{\frac{1}{2}}$

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have the same joint distribution. Then

$$\begin{aligned} P(\max_{0\leqq t\leqq 1} |s_n(t) - w(t)| > 2\varepsilon) &\leqq P(\max_{1\leqq l\leqq n} |w(\textstyle\sum_{i=1}^{l} \tau_i) - w(l/n)| > \varepsilon) \\ &\quad + P(\max_{0\leqq l<n} \max_{0\leqq t\leqq 1/n} |w(l/n) - w(l/n + t)| > \varepsilon) \\ &= A + B . \end{aligned}$$

Now

$$\begin{aligned} A &\leqq P(\max_{1\leqq l\leqq n} \max_{|s|\leqq\delta, l/n+s\geqq 0} |w(l/n + s) - w(l/n)| > \varepsilon) \\ &\quad + P(\max_{1\leqq l\leqq n} |\textstyle\sum_{i=1}^{l} \tau_i - l/n| > \delta) \\ &= C + D , \end{aligned}$$

where

$$C \leqq 4nP(\sup_{0\leqq s\leqq\delta} w(s) > \varepsilon) \leqq 4n \exp(-\varepsilon^2/2\delta)$$

and by a submartingale inequality

$$\begin{aligned} D &= P(\max_{1\leqq l\leqq n} |\textstyle\sum_{i=1}^{l} (n\tau_i - 1)| > n\delta) \\ &\leqq (n\delta)^{-2p} E((\textstyle\sum_{i-1}^{n} (n\tau_i - 1))^{2p}) \qquad \text{for all } p \geqq 1 . \end{aligned}$$

Let $y_i = n\tau_i - 1$. The y_i are independent with mean 0.

$$E((\textstyle\sum_{i=1}^{n} y_i)^{2p}) = \sum_{|\alpha|=2p} \frac{(2p)!}{\alpha_1!\, \alpha_2! \cdots \alpha_n!} E(y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_n^{\alpha_n}) .$$

If any $\alpha_i = 1$, $E(y^\alpha) = 0$, and so

$$E((\textstyle\sum_{i=1}^{n} y_i)^{2p}) = \sum_{|\alpha|=2p, \text{no}\,\alpha_i=1} \binom{2p}{\alpha} E(y^\alpha) ,$$

where

$$\begin{aligned} E(y^\alpha) &= E(y_1^{\alpha_1} \cdots y_n^{\alpha_n}) \\ &\leqq (E(y_1^{2p}))^{\alpha_1/2p} \cdots (E(y_n^{2p}))^{\alpha_n/2p} . \end{aligned}$$

Using the estimate of Sawyer (1967, (2.6)) we get

$$\begin{aligned} E(y_i^{2p}) &\leqq 2^{2p-1}(2(2p)!\, E(x_i^{4p}) + 1) \\ &\leqq 2^{2p}((2p)!\, 4pMc^{-1}\Gamma(4p/c) + 1) . \end{aligned}$$

Let $P(j, k)$ be the number of ways of putting j envelopes into k slots such that each slot gets at least two envelopes. From the trivial estimate $P(2p, k) < k^{2p}$ we estimate

$$\sum_{|\alpha|=2p, \text{no}\,\alpha_i=1} \binom{2p}{\alpha} \leqq pn^p p^{2p}/p!$$

Using the above estimate, we obtain

$$D \leqq D' = \delta^{-2p} n^{-p} M(2p)^{4p}(4p/ce)^{4p/c} .$$

We now set $D' = Mn^{-q}$ and solve for δ, with $q > 1$.

$$\delta = 4p^2(4p/ce)^{2/c} n^{-\frac{1}{2}+q/2p} .$$

Then set $C' = 4ne^{-\varepsilon^2/2\delta} = 4n^{1-q}$ and, with $p = \log n$, solve for ε

$$\begin{aligned}\varepsilon &= (2q\delta \log n)^{\frac{1}{2}} \\ &= 2^{\frac{3}{2}} q^{\frac{1}{2}} e^{q/4} (4/ce)^{1/c} n^{-\frac{1}{4}} (\log n)^{c^{-1}+\frac{3}{2}} .\end{aligned}$$

Term C dominates term B since $\delta > 1/n$. The proof of the result is now concluded. (For an alternate approach to this see Dudley (1972).)

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