The Annals of Probability
1973, Vol. 1, No. 4, 650-662

RECORD VALUES AND MAXIMA

BY SIDNEY I. RESNICK

Stanford University

$\{X_n, n \geqq 1\}$ are i.i.d. random variables with continuous df $F(x)$. X_j is a record value of this sequence if $X_j > \max\{X_1, \cdots, X_{j-1}\}$. We compare the behavior of the sequence of record values $\{X_{L_n}\}$ with that of the sample maxima $\{M_n\} = \{\max(X_1, \cdots, X_n)\}$. Conditions for the relative stability (a.s. and i.p.) of $\{X_{L_n}\}$ are given and in each case these conditions imply the relative stability of $\{M_n\}$. In particular regular variation of $R(x) \equiv -\log(1 - F(x))$ is an easily verified condition which insures a.s. stability of $\{X_{L_n}\}$, $\{M_n\}$ and $\{\sum_{j=1}^n M_j\}$. Concerning limit laws, X_{L_n} may converge in distribution without $\{M_n\}$ having a limit distribution and vice versa. Suitable differentiability conditions on $F(x)$ insure that both sequences have a limit distribution.

1. Introduction and preliminaries. Let $\{X_n, n \geqq 1\}$ be a sequence of independent, identically distributed (i.i.d.) random variables with common distribution $F(\cdot)$. X_j is a *record value* of this sequence iff $X_j > \max\{X_1, \cdots, X_{j-1}\}$. By convention X_1 is a record value. The indices at which record values occur are given by the random variables $\{L_n, n \geqq 0\}$ defined by $L_0 = 1$, $L_n = \min\{j \mid j > L_{n-1}, X_j > X_{L_{n-1}}\}$. We assume throughout that $F(\cdot)$ is continuous in order to be able to use nonparametric techniques in our analysis.

The emphasis of this paper is on the relationships of the $\{X_{L_n}\}$ sequence to the sequence of maxima $\{M_n\}$ where $M_n = \max\{X_1, \cdots, X_n\}$. These relationships are weaker than expected, even though $\{X_{L_n}\}$ is an embedded subsequence of $\{M_n\}$.

A sequence of random variables $\{Z_n\}$ is *relatively stable* (henceforth just *stable*) if there exist normalizing constants $\{B_n\}$ such that $Z_n/B_n \to 1$ as $n \to \infty$. If the convergence is with probability 1 then $\{Z_n\}$ is *almost surely* (a.s.) *stable* while if the convergence is in measure we say $\{Z_n\}$ is *stable in probability* (i.p.).

In the following sections we determine stability conditions for $\{X_{L_n}\}$, $\{M_n\}$ and $\{\sum_{j=1}^n M_j\}$ and show $\{X_{L_n}\}$ stable implies $\{M_n\}$ stable but not the reverse. The approach is to reduce questions about extremes to questions about sums of i.i.d. random variables. In the last section we discuss the relationship between the existence of limit laws for $\{M_n\}$ and existence of limit laws for $\{X_{L_n}\}$.

A central role will be played by $R(x) = -\log(1 - F(x))$ and its inverse $R^{-1}(x) = \inf\{y \mid R(y) > x\}$. Both R and R^{-1} are non-decreasing; R maps $[-\infty, \infty]$ onto $[0, \infty]$. If $y = R(x)$ then we have $R^{-1}(y-) \leqq x \leqq R^{-1}(y)$, but for convenience we shall simply write $x = R^{-1}(y)$. The asymptotics we deal with will be unaffected by this convention.

Received February 14, 1972; revised January 2, 1973.

AMS 1970 *subject classifications.* Primary 60F05.

Key words and phrases. Record values, maxima, extreme values, regular variation, limiting distributions, relative stability.

Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Probability*.
www.jstor.org

This significant property of $R(x)$ is that if a random variable X has distribution $F(x)$, then $R(X)$ is exponentially distributed: $P[R(X) \leqq x] = 1 - e^{-x}$ for $x \geqq 0$.

The following basic lemma was presented in [16]:

LEMMA 1. *$\{X_n, n \geqq 1\}$ is an* i.i.d. *sequence with $P[X_n \leqq x] = 1 - e^{-x}$, $x \geqq 0$. Then $X_{L_n} = X_{L_0} + \sum_{j=1}^{n} (X_{L_j} - X_{L_{j-1}}) \equiv \sum_{j=0}^{n} Y_j$ where $\{Y_j, j \geqq 0\}$ is an* i.i.d. *sequence distributed according to the same exponential distribution.*

This lemma and the fact that R is non-decreasing and continuous lead to the following representations:

$$R(X_{L_n}) = \sum_{j=0}^{n} Y_j \,, \qquad X_{L_n} = R^{-1}(\textstyle\sum_{j=0}^{n} Y_j) \tag{1}$$

where $\{Y_j, j \geqq 0\}$ are i.i.d. exponentially distributed random variables.

It is convenient to define the *right end* x_0 of a distribution function $F(x)$ as $x_0 = \sup\{x \mid F(x) < 1\}$.

To every distribution function $F(x)$ corresponds the *associated distribution* $F^{(a)}(x)$ defined to be that distribution whose R-function is $R^{\frac{1}{2}}(x)$. Formally: $1 - F^{(a)}(x) = e^{-R^{\frac{1}{2}}(x)}$ for $-\infty \leqq x \leqq \infty$.

The theory of limit laws for record values is closely related to extreme value theory (see [16]). The following is pertinent: Let $\{X_n, n \geqq 1\}$ be i.i.d. with common distribution $F(x)$. If there exist normalizing constants $\alpha_n > 0$, β_n, such that

$$P[\alpha_n^{-1}(M_n - \beta_n) \leqq x] = F^n(\alpha_n x + \beta_n) \to_c G(x) \tag{2}$$

where G is nondegenerate, then G belongs to the type of one of the three *extreme value distributions* [2], [10]:

$$\begin{aligned} \Lambda(x) &= \exp\{-e^{-x}\} && -\infty < x < \infty \\ \Phi_\alpha(x) &= 0 && \text{if } x < 0\,, \\ &= \exp\{-x^{-\alpha}\} && \text{if } x \geqq 0\,; \\ \Psi_\alpha(x) &= \exp\{-(-x)^\alpha\} && \text{if } x < 0\,, \\ &= 1 && \text{if } x \geqq 0 \end{aligned}$$

where α is a positive constant. Abbreviate (2) by $F \in DM(G)$ which indicates that normalized maxima from F converge in distribution to G.

Similarly $\{X_{L_n}\}$ has limiting record value distribution $H(\cdot)$ if normalizing constants $a_n > 0$, b_n exist such that

$$P[X_{L_n} \leqq a_n x + b_n] \to_c H(x) \tag{3}$$

where H is nondegenerate. Write $R \in DR(H)$ to indicate weak convergence of the record values to H. In [16] it was proved that the limiting record value distributions are of the form $N(-\log(-\log G(x)))$ where N is the standard normal distribution and G is an extreme value distribution. Further $R(x) \in DR(N(-\log(-\log G(x)))$ iff the associated distribution $F^{(a)}(x) \in DM(G(x))$.

The equivalence between record value behavior of a distribution and extreme value behavior of the associated distribution is valid also for stability i.p. but breaks down for a.s. stability.

2. Stability in probability. In this and the following section we assume F has infinite right end. In the contrary case when $x_0 < \infty$ we have $M_n \to x_0$ a.s. and $X_{L_n} \to x_0$ a.s.

It is well known ([2], [10]) that $\{M_n\}$ is stable i.p. iff $1 - F(x)$ is *rapidly varying*:

$$\lim_{x\to\infty} (1 - F(tx))/(1 - F(x)) = 0 \tag{4}$$

for all $t > 1$. In this case it is always true that $M_n/R^{-1}(\log n) \to_P 1$.

THEOREM 1. *The following are equivalent*:

(i) *There exist* $B_n > 0$ *such that* $X_{L_n}/B_n \to_P 1$.
(ii) $X_{L_n}/R^{-1}(n) \to_P 1$.
(iii) $\lim_{x\to\infty} \{R(tx) - R(x)\}/R^{\frac{1}{2}}(tx) = \infty$ *for all* $t > 1$.
(iv) $\lim_{x\to\infty} \{R(tx) - R(x)\}/R^{\frac{1}{2}}(x) = \infty$ *for all* $t > 1$.

PROOF. The equivalence of (i), (ii), (iii) was proven in [16], Theorem 2.1 and obviously (iii) implies (iv). Given (iv) suppose (iii) does not hold. Then there exist $t > 1$, $x_n \to \infty$ and $c < \infty$ such that $\lim_{n\to\infty} \{R(tx_n) - R(x_n)\}/R^{\frac{1}{2}}(tx_n) = c$. This with (iv) entails $\lim_{n\to\infty} R(x_n)/R(tx_n) = 0$ so that

$$\begin{aligned}\{R(tx_n) - R(x_n)\}/R^{\frac{1}{2}}(tx_n) &= \{R^{\frac{1}{2}}(tx_n) - R^{\frac{1}{2}}(x_n)\}\{R^{\frac{1}{2}}(tx_n) + R^{\frac{1}{2}}(x_n)\}/R^{\frac{1}{2}}(tx_n)\\ &= (1 - o(1))\{R^{\frac{1}{2}}(tx_n) + R^{\frac{1}{2}}(x_n)\} \to \infty\end{aligned}$$

as $n \to \infty$ which gives a contradiction. This completes the proof.

THEOREM 2. $\{X_{L_n}\}$ *is stable* i.p. iff *maxima of* i.i.d. *random variables distributed according to the associated distribution* $F^{(a)}(x)$ *are stable* i.p.; *i.e.*, iff $1 - F^{(a)}(x)$ *is rapidly varying.*

PROOF. From (iii) of Theorem 1 we have:

$$\infty \leftarrow \{R(tx) - R(x)\}/R^{\frac{1}{2}}(tx) = \{R^{\frac{1}{2}}(tx) - R^{\frac{1}{2}}(x)\}\{R^{\frac{1}{2}}(tx) + R^{\frac{1}{2}}(x)\}/R^{\frac{1}{2}}(tx) .$$

Since the second factor is bounded by 2, we must have $R^{\frac{1}{2}}(tx) - R^{\frac{1}{2}}(x) \to \infty$ as $x \to \infty$ for all $t > 1$. This is equivalent to $F^{(a)}$ satisfying (4). The converse is almost the same.

COROLLARY 1. $\{X_{L_n}\}$ *is stable* i.p. iff $R^{-1}((\log x)^2)$ *is slowly varying* iff $R^{-1}(x + cx^{\frac{1}{2}}) \sim R^{-1}(x)$ *as* $x \to \infty$ *for all real* c.

PROOF. A distribution H satisfies (5) iff $H^{-1}(1 - x^{-1})$ is slowly varying ([2]), so $\{X_{L_n}\}$ is stable iff $F^{(a)}(x)$ satisfies (4) iff $(R^{\frac{1}{2}})^{-1}(\log x) = R^{-1}((\log x)^2)$ is slowly varying. The second statement of the corollary follows by a change of variable.

THEOREM 3. $X_{L_n}/R^{-1}(n) \to_P 1$ *implies* $M_n/R^{-1}(\log n) \to_P 1$, *but the converse is not true.*

PROOF. If $\{X_{L_n}\}$ is stable i.p. then Theorem 1 (iv) holds, which entails $R(tx) - R(x) \to \infty$ as $x \to \infty$ for all $t > 1$. This is equivalent to the rapid variation of $1 - F(x)$.

As a counterexample to the converse consider $R(x) = (\log x)^2$, $x \geqq 1$. Then $R^{-1}(\log y) = \exp\{(\log y)^{\frac{1}{2}}\}$ is slowly varying so that $\{M_n\}$ is stable i.p., but $R^{-1}((\log y)^2) = y$ is not slowly varying. Hence $\{X_{Ln}\}$ fails to be stable i.p.

3. Almost sure stability. In discussing a.s. stability of $\{X_{L_n}\}$, it is clear from Theorem 1 that we need consider no other normalizing constants except $\{R^{-1}(n)\}$.

For what follows it is convenient to set $Z_n = (\sum_{j=0}^{n} Y_j - n)/(2n \log\log n)^{\frac{1}{2}}$ where $\{Y_j, j \geqq 0\}$ are the i.i.d. exponentially distributed random variables of representation (1). It follows by the Law of the Iterated Logarithm [12] that almost surely:

$$\text{(5)} \qquad \limsup_{n\to\infty} Z_n = 1 \,, \qquad \liminf_{n\to\infty} Z_n = -1 \,.$$

THEOREM 4. *A sufficient condition for* $\lim_{n\to\infty} X_{L_n}/R^{-1}(n) = 1$ a.s. *is:*

$$\text{(6)} \qquad \text{For all real } t: \quad \lim_{s\to\infty} R^{-1}(s + t(s \log\log s)^{\frac{1}{2}})/R^{-1}(s) = 1 \,.$$

REMARKS CONCERNING CONDITION (6). Note first that the convergence in (6) is uniform on finite t intervals. By the inversion technique described in [6] (see also [2], [4]), (6) is fully equivalent to the more easily verified condition:

$$\text{(7)} \qquad \forall x > 1: \quad \lim_{t\to\infty} \{R(tx) - R(t)\}/(2R(t) \log\log R(t))^{\frac{1}{2}} = \infty \,.$$

It is clear that (7) implies Theorem 1 (iii), so either (6) or (7) is sufficient for $X_{L_n}/R^{-1}(n) \to_P 1$. Note if R^{-1} is regularly varying with exponent α, $0 \leqq \alpha < \infty$ (equivalent to R regularly varying with exponent α^{-1} via [2]), then (6) or (7) is satisfied.

PROOF OF THEOREM 4. Keeping in mind the uniform convergence in (6), we have:

$$\begin{aligned} \lim_{n\to\infty} X_{L_n}/R^{-1}(n) &= \lim_{n\to\infty} R^{-1}(\textstyle\sum_{j=0}^{n} Y_j)/R^{-1}(n) \\ &= \lim_{n\to\infty} R^{-1}(n + Z_n(2n \log\log n)^{\frac{1}{2}})/R^{-1}(n) \\ &= 1 \quad \text{a.s.} \end{aligned}$$

where the first equality follows by (1), the second by the definition of Z_n and the last by (5).

REMARKS. (i) Condition (6) is sufficient but not necessary for a.s. stability. A counterexample has been constructed by A. A. Balkema and will appear elsewhere.

(ii) Almost sure stability will not hold if for all x:

$$\lim_{t\to\infty} R^{-1}(t + x(2t \log\log t)^{\frac{1}{2}})/R^{-1}(t) = e^{cx} \qquad 0 < c \leqq \infty \,.$$

See [6].

(iii) Added information about how close (6) is to a necessary and sufficient condition is provided by the following reasoning: For any continuous distribution $X_{L_n}/R^{-1}(n) = R^{-1}(n + Z_n(2n \log\log n)^{\frac{1}{2}})/R^{-1}(n)$ so that $\limsup_{n\to\infty} X_{L_n}/R^{-1}(n) \geqq 1$

a.s. and $\liminf_{n\to\infty} X_{L_n}/R^{-1}(n) \leqq 1$ a.s. Hence stability is equivalent to

$$0 = P[X_{L_n} \geqq (1+\varepsilon)R^{-1}(n) \text{ i.o.}] = P[X_{L_n} \leqq (1-\varepsilon)R^{-1}(n) \text{ i.o.}],$$

for all $\varepsilon > 0$, which from (1) is the same as:

$$(8) \qquad P[\textstyle\sum_{j=0}^{n} Y_j \geqq R((1+\varepsilon)R^{-1}(n+1)) \text{ i.o.}] = P[\textstyle\sum_{j=0}^{n} Y_j \leqq R((1-\varepsilon)R^{-1}(n)) \text{ i.o.}] = 0 .$$

We wish to apply Feller's general form of the law of the iterated logarithm [7], but to do this we must suppose the following monotone convergences:

$$(9) \qquad \lim_{x\to\infty} \uparrow \{R(tx) - R(x)\}/R^{\frac{1}{2}}(tx) = \infty$$

$$(10) \qquad \lim_{x\to\infty} \uparrow \{R(tx) - R(x)\}/R^{\frac{1}{2}}(x) = \infty$$

for all $t > 1$ (cf. Theorem 3). Supposing (9) and (10), (8) is equivalent to the convergence of two series which in turn can be shown equivalent to the convergence of two integrals. The convergence of the integral corresponding to $\liminf_{n\to\infty} X_{L_n}/R^{-1}(n) \geqq 1$ necessitates convergence of the integral corresponding to $\limsup_{n\to\infty} X_{L_n}/R^{-1}(n) \leqq 1$. The result is that under (9) and (10) a.s. stability of $\{X_{L_n}\}$ is equivalent to

$$(11) \qquad \forall \varepsilon > 0: \ \int_{-\infty}^{\infty} \frac{R(y) - R((1-\varepsilon)y)}{R^{\frac{1}{2}}(y)} \times \exp\left\{-\frac{1}{2}\left(\frac{R(y) - R((1-\varepsilon)y)}{R^{\frac{1}{2}}(y)}\right)^2\right\} d\log R(y) < \infty .$$

However, in the presence of (9) it can be proven that (11) and (6) are equivalent, which leads to:

PROPOSITION 1. *In the presence of* (9) *and* (10) *a necessary and sufficient condition for* $X_{L_n}/R^{-1}(n) \to 1$ a.s. *is* (6).

The possibility of other limit points besides 1 for $\{X_{L_n}/R^{-1}(n)\}$ is discussed in [6].

For what follows we need the following result [1], [17]: Suppose $F(x) < 1$ for all x. Then there exist normalizing constraints b_n, $n > 1$ such that $M_n/b_n \to 1$ a.s. iff

$$(12) \qquad \text{For all } \ 0 < \varepsilon < 1: \ \int_1^{\infty} (1 - F((1-\varepsilon)x))^{-1}\, dF(x) < \infty .$$

In this case $b_n \sim R^{-1}(\log n)$, $n \to \infty$.

THEOREM 5. *Condition* (6) *is also sufficient for* $\lim_{n\to\infty} M_n/R^{-1}(\log n) = 1$ a.s.

PROOF. Let $\mu(n)$ be the number of record values in $\{X_1, \cdots, X_n\}$ so that $M_n = X_{L_{\mu(n)}}$. An iterated logarithm theorem holds for $\mu(n)$ (see [14]) and this gives the representation $\mu(n) = \log n + Z_n^*(2\log n \log\log\log n)^{\frac{1}{2}}$ where $\limsup_{n\to\infty} Z_n^* = 1$, $\liminf_{n\to\infty} Z_n^* = -1$, a.s. Hence

$$M_n/R^{-1}(\log n) = \{X_{L_{\mu(n)}}/R^{-1}(\mu(n))\}\{R^{-1}(\mu(n))/R^{-1}(\log n)\} .$$

The first factor converges to 1 a.s. by Theorem 4, and the second factor equals $R^{-1}(\log n + Z_n^*(2 \log n \log\log\log n)^{\frac{1}{2}})/R^{-1}(\log n)$ which converges to 1 via (6).

REMARK. When (9) and (10) hold, we have by Proposition 1 that

$$\lim_{n\to\infty} X_{L_n}/R^{-1}(n) = 1 \quad \text{a.s.} \Rightarrow \lim_{n\to\infty} M_n/R^{-1}(\log n) = 1 \quad \text{a.s.}$$

EXAMPLES. (i) Let $R(x) = (\log x)^2$, $x \geqq e$. In Theorem 3 we showed that X_{L_n} coming from this R-function are not stable i.p. Hence $\{X_{L_n}\}$ is not a.s. stable. However, (12) reduces to a convergent gamma integral and hence $\{M_n\}$ is a.s. stable.

(ii) Let $N(x)$ be the standard normal distribution with density $n(x) = (2\pi)^{-\frac{1}{2}}e^{-x^2/2}$. Then as is well known: $1 - N(x) \sim n(x)/x$ as $x \to \infty$, which entails $R(x) = -\log(1 - F(x)) \sim -\log n(x) + \log x \sim \frac{1}{2}x^2$ as $x \to \infty$ so that R is regularly varying exponent 2. Hence (5) is satisfied, and since $R^{-1}(x) \sim (2x)^{\frac{1}{2}}$ as $x \to \infty$, we have a.s.:

$$\lim_{n\to\infty} X_{L_n}/(2n)^{\frac{1}{2}} = 1 , \qquad \lim_{n\to\infty} M_n/(2 \log n)^{\frac{1}{2}} = 1 .$$

The close relationship between record values from the distribution $F(x)$ and maxima from the associated distribution $F^{(a)}(x)$ which was shown to hold for limit laws and stability, now breaks down:

THEOREM 6. *Under condition* (9) *if maxima from* $F^{(a)}(x) = 1 - \exp\{-R^{\frac{1}{2}}(x)\}$ *are* a.s. *stable*, *then record values from* $F(x) = 1 - \exp\{-R(x)\}$ *are* a.s. *stable. The converse is false.*

PROOF. According to (12), a.s. stability of maxima from $F^{(a)}(x)$ is equivalent to

$$\forall \varepsilon , \quad 0 < \varepsilon < 1 : \quad \int_{-\infty}^{\infty} \exp\{-(R^{\frac{1}{2}}(y) - R^{\frac{1}{2}}((1-\varepsilon)y))\}\, dR^{\frac{1}{2}}(y) < \infty .$$

To show $\{X_{L_n}\}$ a.s. stable, we verify (7). Note, however, that a simple proof by contradiction argument (similar to the one used in Theorem 1) shows that (7) is equivalent to

$$(7') \qquad \forall x > 1 : \quad \lim_{t\to\infty} \{R(tx) - R(t)\}/(2R(tx) \log\log R(tx))^{\frac{1}{2}} = \infty .$$

Suppose for some $x > 1$, (7′) is false. Then there exist $\varepsilon > 0$, $y_n \uparrow \infty$ and α, $0 < \alpha < \infty$ such that:

$$(13) \qquad \lim_{n\to\infty} \{R(y_n) - R((1-\varepsilon)y_n)\}/(R(y_n) \log\log R(y_n))^{\frac{1}{2}} = \alpha .$$

We can and do suppose that the sequence $\{y_n\}$ is so thin that $R(y_{n-1})/R(y_n) \to 0$. Set $h(y)) = \{R(y) - R((1-\varepsilon)y\}/R^{\frac{1}{2}}(y)$ so that from (13) we have

$$h(y_n)/(\log\log R(y_n))^{\frac{1}{2}} \to \alpha$$

and hence

$$h(y_n)/\log R^{\frac{1}{2}}(y_n) \to 0 .$$

So for given ζ, $0 < \zeta < 1$, we have for sufficiently large n, $n \geqq n_0$ say, that $h(y_n) < \zeta \log R^{\frac{1}{2}}(y_n)$.

Note that $R^{\frac{1}{2}}(y) - R^{\frac{1}{2}}((1-\varepsilon)y) \leqq h(y)$. Hence, keeping in mind that (9) entails

$h(y)$ non-decreasing, we have:

$$\begin{aligned}\int_{-\infty}^{\infty} \exp\{-(R^{\frac12}(y) - R^{\frac12}((1-\varepsilon)y))\}\, dR^{\frac12}(y) &\geqq \int_{-\infty}^{\infty} \exp\{-h(y)\}\, dR^{\frac12}(y) \\ &\geqq \sum_{n=n_0}^{\infty} \int_{y_n}^{y_{n+1}} \\ &\geqq \sum_{n=n_0}^{\infty} e^{-h(y_{n+1})}(R^{\frac12}(y_{n+1}) - R^{\frac12}(y_n)) \\ &\geqq \sum_{n=n_0}^{\infty} e^{-\zeta \log R^{\frac12}(y_{n+1})}(R^{\frac12}(y_{n+1}) - R^{\frac12}(y_n)) \\ &\approx \sum_{n=n_0}^{\infty} (R^{\frac12}(y_{n+1}))^{1-\zeta} = \infty\,.\end{aligned}$$

This proves the first assertion.

REMARK. This assertion can be proved if instead of (9) one assumes that

$$\lim_{s\to\infty} R^{-1}(s + t(s \log\log t)^{\frac12})/R^{\frac12}(s)$$

exists for all real t. The significance of this type of assumption is explained in [6].

To show that the converse in Theorem 6 is false, consider the R-function $R(x) = (\log x \log\log x)^2$, $x \geqq e^e$. Letting $M_n^{(a)}$ be the maxima of i.i.d. random variables drawn from the associated distribution, we can show: (i) $\{M_n^{(a)}\}$ stable i.p.; (ii) $\{M_n^{(a)}\}$ *not* stable a.s.; (iii) $\{M_n\}$ stable a.s.; and (iv) $\{X_{L_n}\}$ stable a.s. The proof of (i) follows by noting that $\log(1+\varepsilon)x \log\log(1+\varepsilon)x - \log x \log\log x \to \infty$, which implies $1 - F^{(a)}(x)$ is rapidly varying. The verification of (ii) comes from showing the integral in (12) diverges. To prove (iv), show that (7) holds. This is done most easily by verifying the sufficient condition (see [6] Theorem 2):

$$\lim_{x\to\infty} xR'(x)/(R(x) \log\log R(x))^{\frac12} = \infty\,.$$

Finally (iv) $\Rightarrow$ (iii) by Theorem 5.

We next consider a.s. stability of $\{\sum_{j=1}^n M_j\}$. Our result is related to a theorem of Grenander [11] concerning stability i.p. of successive sums of minima from an i.i.d. sequence. An a.s. version of the Grenander result was later proved by O. Frank [9]. The author is grateful to Dr. H. Cohn for pointing out these references.

THEOREM 7. *If* $\{M_n\}$ *is* a.s. *stable, so is* $\{\sum_{j=1}^n M_j\}$:

$$\lim_{n\to\infty} \sum_{j=1}^n M_j / \sum_{j=1}^n R^{-1}(\log j) = 1 \quad \text{a.s.}$$

If in addition $R^{-1}(x)$ *is regularly varying exponent* α, $0 \leqq \alpha < \infty$ *(equivalently* $R(x)$ *is regularly varying exponent* α^{-1} *via* [2], *Corollary* 1.2.1), *then* a.s. $\lim_{n\to\infty} \sum_{j=1}^n M_j/nR^{-1}(\log n) = 1$.

PROOF. The first assertion follows from the following analytical lemma: If $a_n > 0$, $b_n > 0$ for sufficiently large n, $\sum a_n = \infty$ and $\lim_{n\to\infty} a_n/b_n = 1$, then $\lim_{n\to\infty} \sum_{j=1}^n a_j / \sum_{j=1}^n b_j = 1$. For the second result note that if R^{-1} is regularly varying, then $R^{-1}(\log x)$ is slowly varying since $\log x$ is slowly varying ([2] page 21). Hence

$$\sum_{j=1}^n R^{-1}(\log j) \sim \int_1^n R^{-1}(\log x)\, dx \sim nR^{-1}(\log n)$$

as $n \to \infty$ via [8] page 281 or [2] page 15.

The Grenander result does not seem to come from these methods, apparently due to the impossibility of finding constants b_n such that $M_n - b_n \to 0$ as $n \to \infty$ a.s. (or even i.p.—see [10]) for the case of maxima drawn from a negative exponential distribution. Instead we obtain the following disjoint result:

THEOREM 7′. $X_1, X_2, \cdots$ *are* i.i.d. *with common continuous distribution* $F(x)$. *Define* $Z_n = \min\{X_1, \cdots, X\}$ *and* $Q(x) = -\log F(x)$. *Suppose* $Q^{-1}(x) = x^{-\alpha}L(x)$ *as* $x \to \infty$ *where* $\infty > \alpha \geqq 0$ *and* L *is slowly varying. Then*

(i) $$\lim_{n\to\infty} Z_n/Q^{-1}(\log n) = 1 \quad \text{a.s.}, \qquad \textit{and}$$

(ii) $$\lim_{n\to\infty} \textstyle\sum_{j=1}^{n} Z_j/nQ^{-1}(\log n) = 1 \quad \text{a.s.}$$

4. Comparison of the domains of attraction of the limiting record value distributions and the extreme value distributions. When will both $\{X_{L_n}\}$ and $\{M_n\}$ have limiting distributions? That is, for a distribution $F(x)$ with R-function $R(x)$, when it is true that $F(x) \in DM(G(x))$, for some extreme value distribution $G(x)$, and $R(x) \in DR(H(x))$ for some limiting record value distribution $H(x)$.

In this section we will designate the three limiting record value distributions by $N(x)$, $N_{1\alpha}(x)$ and $N_{2\alpha}(x)$ where $N(x)$ is the standard normal distribution, $N_{1\alpha}(x) = N(-\log(-\log \Phi_\alpha(x)))$, $N_{2\alpha}(x) = N(-\log(-\log \Psi_\alpha(x)))$ where $\Phi_\alpha(x)$ and $\Psi_\alpha(x)$ are the extreme value distributions given after (2).

In the case that $\{M_n\}$ has limit distribution $G(x)$ and $\{X_{L_n}\}$ has limit law $H(x)$ we will say that the pair $(F(x), R(x))$ is *dually attracted* to $(G(x), H(x))$ and write

$$(F(x), R(x)) \in D(G(x), H(x)) .$$

THEOREM 8. *The only possibilities for dual attraction are*:

$$(F(x), R(x)) \in D(\Lambda(x), N(x))$$
$$(F(x), R(x)) \in D(\Lambda(x), N_{1\alpha}(x))$$
$$(F(x), R(x)) \in D(\Lambda(x), N_{2\alpha}(x)) .$$

Thus a necessary condition for dual attraction is that $F(x) \in DM(\Lambda(x))$.

PROOF. We use freely the facts obtained from the duality:

$$R(x) \in DR(N(-\log(-\log G(x)))) \qquad \text{iff} \quad 1 - e^{-R^{\frac{1}{2}}(x)} \in DM(G(x)) .$$

Suppose $R(x) \in DR(N_{1\alpha}(x))$. Then ([16] Theorem 4.2) for all $t \geqq 1$:

$$\lim_{x\to\infty} \{R(tx) - R(x)\}/R^{\frac{1}{2}}(x) = \alpha \log t$$

and hence

$$\lim_{x\to\infty} R(tx) - R(x) = \infty .$$

This precludes $1 - F(x)$ being regularly varying and thus $F(x) \notin DM(\Phi_\alpha(x))$ for all $\alpha > 0$. Also $R(x) \in DR(N_{1\alpha}(x))$ implies that $F(x) < 1$ for all x and hence $F(x) \notin DM(\Psi_\alpha(x))$ since distributions in the domain of $\Psi_\alpha(x)$ must have finite right end.

The demonstration that $R(x) \in DR(N_{2\alpha}(x))$ implies $F(x) \notin DM(\Phi_\alpha(x))$ and

$F(x) \notin DM(\Psi_\alpha(x))$ is almost the same as the previous case. Now assume $R(x) \in DR(N(x))$. Then $1 - e^{-R^{\frac{1}{2}}(x)} \in DM(\Lambda(x))$ which implies if the right end is infinite that $e^{-R^{\frac{1}{2}}(x)}$ is rapidly varying:

$$e^{-R^{\frac{1}{2}}(tx)}/e^{-R^{\frac{1}{2}}(x)} \to 0$$

for all $t > 1$ as $x \to \infty$. By Theorem 2 this means that

$$\lim_{x\to\infty} \{R(tx) - R(x)\}/R^{\frac{1}{2}}(x) = \infty$$

for all $t > 1$ and this precludes the regular variation of $1 - F(x)$ so that $F(x) \notin DM(\Phi_\alpha(x))$ for any $\alpha > 0$. The remaining cases are disposed of in a similar manner.

Before continuing we make the following conventions: Let $F(x)$ be a distribution with R-function $R(x)$ and right end x_0 ($x_0 \leqq \infty$). Suppose the first and second derivatives of $R(x)$ exist in some neighborhood of x_0 and are denoted by $r(x)$ and $r'(x)$ respectively. Then

(i) $F(x)$ is a *Von Mises function of type* $\Lambda(x)$ if ultimately $r(x) > 0$ and if $(1/r(x))' \to 0$ or equivalently if $r'(x)/r^2(x) \to 0$ as $x \to x_0$.

(ii) $F(x)$ is a *Von Mises function of type* $\Phi_\alpha(x)$ if $x_0 = \infty$ and there exists $\alpha > 0$ such that $xr(x) \to \alpha$ as $x \to \infty$.

(iii) $F(x)$ is a *Von Mises function of type* $\Psi_\alpha(x)$ if $x_0 < \infty$ and there exists $\alpha > 0$ such that $(x_0 - x)r(x) \to \alpha$ as $x \uparrow x_0$.

It is known (see [9] or [2] pages 109–112) that (i), (ii), (iii) are respectively sufficient for $F(x) \in DM(\Lambda(x))$, $F(x) \in DM(\Phi_\alpha(x))$, $F(x) \in DM(\Psi_\alpha(x))$.

Returning to the content of Theorem 8 let us show that the intersection of the domains of attraction of $\Lambda(x)$ and the limiting record value distributions is indeed nonempty. To do this, consider the class of R-functions given by

$$R_\alpha(x) = \left(\frac{\beta}{2} \log x\right)^\alpha$$

for $x \geqq e$ and $\alpha > 0$, $\beta > 0$. Call the corresponding distribution $F_\alpha(x)$. Then the following conditions quickly emerge:

(i) For $\alpha > 1$, $F_\alpha(x)$ is a Von Mises function of type $\Lambda(x)$.

(ii) For $\alpha = 1$, $F_1(x)$ is a Von Mises function of type $\Phi_{\beta/2}(x)$. This follows immediately from the fact that $xr_1(x) = \beta/2$.

(iii) For $\alpha < 1$, $F_\alpha(x) \notin DM(\Phi_\gamma(x))$ for any $\gamma > 0$ and $F_\alpha(x) \notin DM(\Lambda(x))$.

The first assertion follows quickly by showing $1 - F_\alpha(x)$ is rapidly varying and hence not regularly varying. The second assertion can be proven by using the criterion for attraction to $\Lambda(x)$ given in [2] page 76 or [13].

From (i), (ii) and (iii) and the Duality Theorem ([16] Theorem 4.1) we have the following:

(i) For $\alpha > 2$, $R_\alpha(x) \in DR(N(x))$, $F_\alpha(x) \in DM(\Lambda(x))$.

(ii) For $\alpha = 2$, $R_2(x) \in DR(N_{1\beta}(x))$, $F_2(x) \in DM(\Lambda(x))$.

(iii) For $1 < \alpha < 2$, $R_\alpha(x) \notin DR(N_{1\alpha}(x))$ for any γ, $F_\alpha(x) \in DM(\Lambda(x))$,
For $1 < \alpha < 2$, $R_\alpha(x) \notin DR(N(x))$.

(iv) For $\alpha = 1$, $R_\alpha(x) \notin DR(N_{1\gamma}(x))$ for any γ, $F_1(x) \in DM(\Phi_{\beta/2}(x))$
For $\alpha = 1$, $R_\alpha(x) \notin DR(N(x))$.

Thus we see that $DM(\Lambda(x)) \cap DR(N(x)) \neq \emptyset$, and $DM(\Lambda(x)) \cap DR(N_{1\beta}(x)) \neq \emptyset$ for any $\beta > 0$ and by symmetry considerations it is clear that $DM(\Lambda(x)) \cap DR(N_{2\beta}(x)) \neq \emptyset$. Further it is possible that $\{M_n\}$ has a limiting distribution but not $\{X_{L_n}\}$.

Our most precise results about the overlap of these domains of attraction are contained in the following:

THEOREM 9. *Let $F(x)$ be a distribution with R-function $R(x)$ and associated distribution $F^{(a)}(x) = 1 - e^{-R^{\frac{1}{2}}(x)}$.*

(i) *If $R(x) \in DR(N(x))$ and $F^{(a)}(x)$ is a Von Mises function of type $\Lambda(x)$, then $F(x)$ is a Von Mises function of type $\Lambda(x)$ and $F(x) \in DM(\Lambda(x))$.*

(ii) *If $R(x) \in DR(N_{1\alpha}(x))$ and $F^{(a)}(x)$ is a Von Mises function of type $\Phi_{\alpha/2}(x)$, then $F(x)$ is a Von Mises function of type $\Lambda(x)$ and $F(x) \in DM(\Lambda(x))$.*

(iii) *If $R(x) \in DR(N_{2\alpha}(x))$ and $F^{(a)}(x)$ is a Von Mises function of type $\Psi_{\alpha/2}(x)$, then $F(x)$ is a Von Mises function of type $\Lambda(x)$ and $F(x) \in DM(\Lambda(x))$.*

PROOF. Because of the Duality Theorem ([16] Theorem 4.1) it suffices to show: If $F(x)$ is a Von Mises function of type $\Lambda(x)$, $\Phi_\alpha(x)$ or $\Psi_\alpha(x)$ then $1 - e^{-R^2(x)}$ is Von Mises of type $\Lambda(x)$. To prove this when $F(x)$ is a Von Mises function of type $\Lambda(x)$ note $(R^2(x))' = 2R(x)r(x)$ and

$$\left(\frac{1}{(R^2(x))'}\right)' = \frac{r'(x)}{2R(x)r^2(x)} + \frac{1}{2R(x)} \to 0 \qquad \text{as } x \to \infty .$$

In the case that $F(x)$ is a $\Phi_\alpha(x)$ type Von Mises function we have $r(x) \sim \alpha x^{-1}$. Hence $r'(x) \sim -\alpha x^{-2}$ as $x \to \infty$ ([2] page 23). But:

$$\left(\frac{1}{(R^2(x))'}\right)' = \frac{r'(x)}{2Rr^2(x)} + \frac{1}{2R(x)}$$

The second term goes to zero and the first term is asymptotic to

$$\frac{-\alpha x^{-2}}{2R(x)\alpha^2 x^{-2}} = -\frac{1}{2\alpha R(x)} \to 0$$

as $x \to \infty$. The third case is handled in a similar manner and this completes the proof.

We have already seen that it is possible for $\{M_n\}$ to have a limiting distribution but not $\{X_{L_n}\}$. The converse situation can hold as well: It can happen that $R(x) \in DR(H(x))$, for some limiting record value distribution $H(x)$, but $F(x)$ is not attracted to any extreme value distribution. Thus Theorem 9 cannot be extended to all continuous distributions. This is surprising in view of the recent de Haan–Balkema result [5] which states that if $F(x) \in DM(\Lambda(x))$ then $F(x)$ is close to a $\Lambda(x)$-type Von Mises function $F_v(x)$ in the sense that $(1 - F(x))/(1 - F_v(x)) \to 1$ as $x \to x_0$ (x_0 is the right end of $F(x)$).

The following example by Laurens de Haan shows the possibility that $R(x) \in DR(N((x))$ but $F(x) \notin DM(\Lambda(x))$ (and hence by Theorem 8 $F(x)$ is not attracted to any extreme value distribution). By the Duality Theorem ([16] Theorem 4.1) it suffices to exhibit a distribution $F(x)$ such that $F^{(a)}(x) \in DM(\Lambda(x))$ but $F(x) \notin DM(\Lambda(x))$. Define $F^{(a)}(x)$ by $1 - F^{(a)}(x) = e^{-R^{\frac{1}{2}}(x)}$ where $R^{\frac{1}{2}}(x) = x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}} \sin x$, $x \geqq 1$. Then $R^{\frac{1}{2}}(x) - x^{\frac{1}{2}} \to 0$ as $x \to \infty$ so $1 - F^{(a)}(x) \sim e^{-x^{\frac{1}{2}}}$ as $x \to \infty$. Since the distribution $1 - e^{-x^{\frac{1}{2}}}$ is a Von Mises function of type $\Lambda(x)$ we have by tail equivalence [15] that $F^{(a)}(x) \in DM(\Lambda(x))$.

Since $R(x) = x + \sin x + (\sin^2 x)/4x$ and $R(x) - (x + \sin x) \to 0$ as $x \to \infty$ we have that $F(x)$ is tail equivalent to $F_1(x) = 1 - \exp\{-(x + \sin x)\}$ and $F(x) \in DM(\Lambda(x))$ iff $F_1(x) \in DM(\Lambda(x))$. Hence it suffices to show $F_1(x) \notin DM(\Lambda(x))$. Proceeding by contradiction we have that if $F_1(x) \in DM(\Lambda(x))$ then there exists an auxiliary function $f(t) \geqq 0$ such that

$$\lim_{t \to \infty} R_1(t + xf(t)) - R_1(t) = x$$

for all x where $R_1(t) = -\log(1 - F_1(t))$. The above relation must hold along the sequence $t_n = 2n\pi$ where by the periodicity of $\sin x$ we have that $R_1(t_n + xf(t_n)) - R_1(t_n) = xf(t_n) + \sin xf(t_n)$. Take a further subsequence t_n' such that $f(t_n') \to c \in [0, \infty]$. A contradiction is obtained by showing the incompatibility of the following relations:

$$R_1(t_n' + xf(t_n')) - R_1(t_n') \to xc + \sin xc$$
$$R_1(t_n' + xf(t_n')) - R_1(t_n') \to x\,.$$

If $c = 0$ or ∞ the contradiction is clear (interpreting $\lim_{n \to \infty} xf(t_n + \sin xf(t_n) = \infty$ for the case $c = \infty$). If $0 < c < \infty$ we have for all x that $x = xc + \sin xc$ or $\sin xc = x(1 - c)$ which is not true. This completes the counterexample.

To close this discussion of limit laws we present some results about the shape of $R(x)$ when $R(x)$ is attracted to some limiting record value distribution.

THEOREM 10. *Suppose $R(x) \in DR(H(x))$ where $H(x)$ is one of the three limiting record value distributions. Then there exists a type-$\Lambda(x)$ Von Mises function $F^*(x)$ with corresponding R-function $R^*(x)$ such that*

$$F^*(x) \in DM(\Lambda(x)) \qquad \textit{and} \qquad R(x) \sim R^*(x)$$

as $x \to x_0$ (x_0 is the right end of the distribution).

PROOF. If $R(x) \in DR(N_{1\alpha}(x))$ then by duality we have:

$$R(x) \sim R^*(x) = \left(\frac{\alpha}{2} \log x\right)^2$$

(see Remarks following Theorem 4.2, [16]). It is quickly checked that this $R^*(x)$ is a Von Mises function of type $\Lambda(x)$. Similar techniques prove the result when $R(x) \in DR(N_{2\alpha}(x))$ so now suppose $R(x) \in DR(N(x))$ and the right end of $F(x)$ is x_0.

Duality shows that $F^{(a)}(x) \in DM(\Lambda(x))$ and hence from [5] $F^{(a)}(x)$ is tail equivalent to a $\Lambda(x)$-type Von Mises function; i.e., there exist $c(x)$, $r(x)$ such that $c(x) \to 1$ and $r'(x)/r^2(x) \to 0$ as $x \to \infty$ and

$$1 - F^{(a)}(x) = c(x) \exp\{-\int_{-\infty}^{x} r(t)\, dt\} .$$

Set $R^*(x) = (\int_{-\infty}^{x} r(t)\, dt)^2$. Since

$$R^{\frac{1}{2}}(x) = -\log(1 - F^{(a)}(x)) = -\log c(x) + \int_{-\infty}^{x} r(t)\, dt \sim \int_{-\infty}^{x} r(t)\, dt$$

we must have

$$R(x) \sim (\int_{-\infty}^{x} r(t)\, dt)^2 = R^*(x) .$$

But $1 - \exp\{-\int_{-\infty}^{x} r(t)\, dt\}$ is a Von Mises function of type $\Lambda(x)$ and hence by Theorem 9 so is $R^*(x)$. The proof is complete.

Acknowledgments. Much of this work was done while the author was a summer guest of the Mathematics Institute of the University of Amsterdam and the Amsterdam Mathematics Center. The author thanks those institutions for their hospitality and support.

The initial draft of this paper was completed while the author was working at the Technion-Israel Institute of Technology.

Wim Vervaat made several helpful suggestions which were incorporated in the final version.

REFERENCES

[1] BARNDORFF-NIELSEN, O. (1963). On the limit behavior of extreme order statistics. *Ann. Math. Statist.* **34** 992–1002.

[2] DE HAAN, L. (1970). *On Regular Variation and Its Application to the Weak Convergence of Sample Extremes.* Mathematical Center Tracts 32, Amsterdam.

[3] DE HAAN, L. and HORDIJK, A. (1972). The rate of growth of sample maxima. *Ann. Math. Statist.* **43** 1185–1196.

[4] DE HAAN, L. (1972). Equivalence classes of regularly varying functions. Technical Report No. 41, Department of Statistics, Stanford Univ.

[5] DE HAAN, L. and BALKEMA, A. (1972). On R. Von Mises' condition for the domain of attraction of $\exp\{-e^{-x}\}$. *Ann. Math. Statist.* **43** 1352–1354.

[6] DE HAAN, L. and RESNICK, S. I. (1972). Almost sure limit points of record values. Technical Report No. 39, Department of Statistics, Stanford Univ. To appear *J Appl. Probability*.

[7] FELLER, W. (1946). The law of the iterated logarithm for identically distributed random variables. *Ann. of Math.* **47** 631–638.

[8] FELLER, W. (1966). *An Introduction to Probability Theory and Its Applications* **2**. Wiley, New York.

[9] FRANK, O. (1966). Generalization of an inequality of Hájek and Rényi. *Skand. Aktuarietidskr.* Haft 1–2, 85–89.

[10] GNEDENKO, B. V. (1943). Sur la distribution limite du terme maximum d'une serie aleatoire. *Ann. of Math.* **44** 423–453.

[11] GRENANDER, U. (1965). A limit theorem for sums of minima of stochastic variables. *Ann. Math. Statist.* **36** 1041–1042.

[12] HARTMAN, P. and WINTNOR, A. (1941) On the law of the iterated logarithm. *Amer. J. Math.* **63** 169–176.

[13] MARCUS, M. and PINSKY, M. (1969). On the domain of attraction of $\exp\{-e^{-x}\}$. *J. Math. Anal. Appl.* **28** 440–449.

[14] RÉNYI, A. (1962). Theorie des elements saillants d'une suite d'observations. Colloquium on Combinatorial Methods in Probability Theory, August 1–10, Mathematisk Institut, Aarhus Universitet, Denmark.

[15] RESNICK, S. I. (1971). Tail equivalence and its applications. *J. Appl. Probability* **8** 136–156.

[16] RESNICK, S. I. (1973). Limit laws for record values. *J. Stoch. Processes and Their Appl.* **1** 67–82.

[17] RESNICK, S. I. and TOMKINS, R. J. (1972). Almost sure stability of maxima. To appear *J. Appl. Probability*.

[18] TATA, M. N. (1969). On outstanding values in a sequence of random variables. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **12** 9–20.

[19] VON MISES, R. (1936). La distribution de la plus grande de *n* valeurs. *Selected Papers* **2**. American Mathematical Society, Providence, 271–294.

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305