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SPECIAL INVITED PAPERS

GEOMETRY OF DIFFERENTIAL SPACE[1]

DEDICATED TO THE MEMORY OF WILL FELLER

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The purpose of this paper is to explain why it is fruitful to think of Wiener space as an infinite—dimensional sphere of radius $\infty^{\frac{1}{2}}$. The idea goes back to Lévy and Wiener and has recently been employed to advantage by Hida; Hida, Kubo, Nomoto and Yosizawa; Kono; Orihara and Umemura; their results will be reported upon below.

1. Introduction. The purpose of this paper is to explain why it is a fruitful idea to think of Wiener measure as the uniform distribution on an infinite-dimensional spherical surface $S^\infty(\infty^{\frac{1}{2}})$ of radius $\infty^{\frac{1}{2}}$. This picture stems from an observation of Poincaré [12] and has recently been employed by Hida [4], Hida, Kubo, Nomoto and Yosizawa [5], Kono [6], Orihara [11], and Umemura [12], with entertaining consequences to be reported upon below. Poincaré noticed that if $x = (x_1, \cdots, x_n)$ *is uniformly distributed on the* $(n-1)$*-dimensional spherical surface* $S^{n-1}(n^{\frac{1}{2}})$ *of radius* $n^{\frac{1}{2}}$, *then for fixed* $m < \infty$,

$$\lim_{n\uparrow\infty} P[\textstyle\bigcap_{i=1}^m (a_i \leqq x_i \leqq b_i)] = \int_{a_1}^{b_1} \frac{e^{-x^2/2}}{(2\pi)^{\frac{1}{2}}}\,dx \cdots \int_{a_m}^{b_m} \frac{e^{-x^2/2}}{(2\pi)^{\frac{1}{2}}}\,dx\,.$$

The proof is elementary. Pick $-n^{\frac{1}{2}} < a < b < n^{\frac{1}{2}}$. Then the uniform measure of the spherical zone $a \leqq x_1 < b$ is

$$\frac{\int_a^b (n-x^2)^{n/2-1}\,dx}{\int_{-n^{\frac{1}{2}}}^{n^{\frac{1}{2}}} (n-x^2)^{n/2-1}\,dx} = \frac{\int_a^b (1-x^2/n)^{n/2-1}\,dx}{\int_{-n^{\frac{1}{2}}}^{n^{\frac{1}{2}}} (1-x^2/n)^{n/2-1}\,dx}$$

and as $n \uparrow \infty$, this ratio approximates

$$\int_a^b \frac{e^{-x^2/2}\,dx}{(2\pi)^{\frac{1}{2}}}\,.$$

The proof is finished by making a similar computation for $x_1' = y_1x_1 + \cdots + y_px_p$ with $\sum y_i^2 = 1$.

Poincaré's observation is connected to the Wiener measure as follows: if $\mathfrak{x}: [0, \infty) \to R^1$ is the sample path of a standard 1-dimensional Brownian motion

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and if $e_1, e_2, \cdots$ is a unit perpendicular basis of $L^2[0, \infty)$, then the map

$$\mathfrak{x} \to (\mathfrak{x}_1, \mathfrak{x}_2, \cdots)$$

defined by

$$\mathfrak{x}_n = \int_0^\infty e_n(t)\, d\mathfrak{x}(t)$$

effects an isomorphism between the Brownian motion and the space R^∞ provided with the limiting distribution of Poincaré:

$$P[\bigcap_{i=1}^m (a_i \leqq \mathfrak{x}_i \leqq b_i)] = \int_{a_1}^{b_1} \frac{e^{-x^2/2}\, dx}{(2\pi)^{\frac{1}{2}}} \cdots \int_{a_m}^{b_m} \frac{e^{-x^2/2}\, dx}{(2\pi)^{\frac{1}{2}}} .$$

The optimist will now hope that objects such as the rotation group $O(n)$, the spherical Laplacian Δ, the eigenfunctions of the latter (spherical harmonics), and the irreducible representations of $O(n)$ implemented by them will "stabilize" as the dimension n approaches ∞ and thereby make sense for the Brownian motion viewed as a uniformly distributed point of $S^\infty(\infty^{\frac{1}{2}})$. *This turns out to be so:* $O(\infty)$ is just the orthogonal group of $L^2[0, \infty)$ acting upon $\mathfrak{x}$ according to the rule

$$\mathfrak{x}(t) \to \mathfrak{x}'(t) = \int_0^\infty (\text{image of the indicator of } \quad 0 \leqq t' \leqq t)\, d\mathfrak{x} ;$$

the ∞-dimensional Laplacian is a sum of uncoupled Hermite operators

$$\Delta = \sum (\partial^2/\partial x_i^2 - x_i \partial/\partial x_i) ;$$

the ∞-dimensional spherical harmonics are products of Hermite polynomials

$$H_p(x) = H_{p_1}(x_1) H_{p_2}(x_2) \cdots \qquad |p| = p_1 + p_2 + \cdots < \infty ;$$

and the action of $O(\infty)$ upon the span of Hermite polynomials of the several total degrees or weights, $n = |p| = 0, 1, 2, 3, \cdots$, produces the irreducible representations of that group. For fixed $n = |p|$, the latter span is just the polynomials chaos of degree n of Wiener [15], so from the present standpoint Cameron-Martin-Wiener's functional power series

$$\mathfrak{f}(\mathfrak{x}) = \sum_{n=0}^\infty \sum_{|p|=n} H_p(\mathfrak{x})(p!)^{-1} E[\mathfrak{f}\, H_p(\mathfrak{x})]$$

for the general Brownian functional $\mathfrak{f}$ with $E(\mathfrak{f}^2) < \infty$ appears as an expansion into spherical harmonics.

I take the point of view that the statement $\mathfrak{x}$ *is uniformly distributed on* $S^\infty(\infty^{\frac{1}{2}})$ is justified by these facts. Additional confirmation of an elementary nature is provided by the strong law of large numbers:

$$P[(\mathfrak{x}_1^2 + \cdots + \mathfrak{x}_n^2)^{\frac{1}{2}} = n^{\frac{1}{2}}[1 + o(1)] \quad \text{as} \quad n \uparrow \infty] = 1$$

and by the strong law of P. Lévy [8]:

$$P\left[\lim_{n \uparrow \infty} \sum_{k2^{-n} \leqq t} \left| \mathfrak{x}\left(\frac{k}{2^n}\right) - \mathfrak{x}\left(\frac{k-1}{2^n}\right)\right|^2 = t \quad \text{for every} \quad t \geqq 0\right] = 1 .$$

The first should be viewed as a clear statement of the Pythagorean rule in the coordinates $\mathfrak{x}_n : n \geqq 1$, the second as a restatement thereof in the new "coordinates" $d\mathfrak{x}(t): 0 \leqq t < \infty$. The latter is the (formal but suggestive) point of view of "differential space" adopted by Lévy [7], [8] and by Wiener in his early paper [14] and will explain the title of this paper.

The details of the spherical picture will be explained below; as above, the point of view is determinedly informal; it is also purely expository. The reader is assumed to be familiar with the elementary facts about Brownian motion itself and the calculus of Brownian differentials and integrals, as presented, for example, in McKean [10].

But first a simple caricature may clarify the spherical picture of Brownian motion. Replace the Brownian motion (white noise) by independent fair Bernouilli trials $\mathfrak{x}_1, \mathfrak{x}_2, \cdots = \pm 1$ and identify $(\mathfrak{x}_1, \mathfrak{x}_2, \cdots)$ with a point of the unit interval $0 \leqq x \leqq 1$ in the customary way:

$$\mathfrak{x} = \sum_{n=1}^{\infty} \tfrac{1}{2}(1 + \mathfrak{x}_n)2^{-n} .$$

The Bernouilli distribution is mapped thereby onto the standard Lebesgue measure dx, and the trials $\mathfrak{x}_1, \mathfrak{x}_2, \cdots$ are identified as the Rademacher functions. The point $(\mathfrak{x}_1, \mathfrak{x}_2, \cdots)$ is now viewed as an element of the (commutative) infinite dyadic group $G = Z_2 \times Z_2 \times \cdots$, and you observe that this group acts upon $0 \leqq x \leqq 1$ (which happens to be a copy of G) in a self-evident way. The irreducible representations of the group are its nontrivial characters

$$\mathfrak{y}_p = \mathfrak{x}_{p_1}\mathfrak{x}_{p_2}\mathfrak{x}_{p_3}\cdots \qquad 0 < p_1 < p_2 < p_3 < \cdots ,$$

alias the Walsh functions, augmented by the unit character, and there are enough of these to span out the whole of $L^2[0, 1]$. The moral is that the Walsh expansion stands in the same relation to Bernouilli trials as the Cameron-Martin-Wiener expansion does to the white noise (=Brownian motion).

2. Polynomial chaos. The first item of business is the space $Z = L^2(W)$. Here W is the space of Brownian paths $\mathfrak{x}: [0, \infty) \to R^1$ provided with the customary Wiener measure. Wiener [15] discovered an important perpendicular splitting of this space:

$$Z = Z^0 \oplus Z^1 \oplus Z^2 \oplus \cdots$$

Z^0 is the constants, and for $n > 0$, Z^n is the polynomial chaos of degree n, so-called, populated by those Brownian functionals $\mathfrak{f} \in Z$ expressible as an n-fold Brownian integrals:

$$\mathfrak{f}(\mathfrak{x}) = \int_0^{\infty} d\mathfrak{x}(t_1) \int_0^{t_2} d\mathfrak{x}(t_2) \cdots \int_0^{t_{n-1}} d\mathfrak{x}(t_n) f(t_1, \cdots, t_n) \equiv \int_0^{\infty} f\, d^n\mathfrak{x}$$

with a sure integrand f subject to

$$\|f\|^2 = \int_0^{\infty} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n f^2(t_1, \cdots, t_n) < \infty .$$

The corresponding expansion of the general Brownian functional $\mathfrak{f} \in Z$:

$$\mathfrak{f}(x) = E(\mathfrak{f}) + \sum_{n=1}^{\infty} \int_0^{\infty} f_n d^n\mathfrak{x}$$

is to be thought of as a functional power series. This is the starting point of Wiener's study of non-linear devices; see, for example, [16]. Wiener liked to think of $\mathfrak{f}$ as a "black box" into which you feed the "white noise" $\dot{\mathfrak{x}}$:

$$\dot{\mathfrak{x}} \to \boxed{\mathfrak{f}} \to \sum \int_0^\infty f_n d\mathfrak{x}^n ,$$

the (sure) coefficient functions $f_0, f_1, f_2, \cdots$ being inner products of a sort, computed by the (formal) recipe

$$f_n(t_1, t_2, \cdots, t_n) = E[\mathfrak{f}\dot{\mathfrak{x}}(t_1)\dot{\mathfrak{x}}(t_2) \cdots \dot{\mathfrak{x}}(t_n)] .$$

This expansion (but not the recipe just above) is now established by a series of easy steps.

STEP 1. *If* $\|f\| < \infty$, *then* $\mathfrak{f} = \int_0^\infty f d^n\mathfrak{x} \in Z$ *and* $E(\mathfrak{f}^2) = \|f\|^2$.

PROOF. For $n = 3$.

$$\begin{aligned} E(\mathfrak{f}^2) &= E[\textstyle\int_0^\infty d\mathfrak{x}(t_1) \int_0^{t_1} d\mathfrak{x}(t_2) \int_0^{t_2} d\mathfrak{x}(t_3) f(t_1, t_2, t_3) \\ &\qquad \times \textstyle\int_0^\infty d\mathfrak{x}(t_1') \int_0^{t_1'} d\mathfrak{x}(t_2') \int_0^{t_2'} d\mathfrak{x}(t_3') f(t_1', t_2', t_3')] \\ &= \textstyle\int_0^\infty dt_1 E[\int_0^{t_1} d\mathfrak{x}(t_2) \int_0^{t_2} d\mathfrak{x}(t_3) f(t_1, t_2, t_3) \\ &\qquad \times \textstyle\int_0^{t_1} d\mathfrak{x}(t_2') \int_0^{t_2'} d\mathfrak{x}(t_3') f(t_1, t_2', t_3')] \\ &= \textstyle\int_0^\infty dt_1 \int_0^{t_1} dt_2 E[\int_0^{t_2} d\mathfrak{x}(t_3) f(t_1, t_2, t_3) \\ &\qquad \times \textstyle\int_0^{t_2} d\mathfrak{x}(t_3') f(t_1, t_2, t_3')] \\ &= \textstyle\int_0^\infty dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 f^2(t_1, t_2, t_3) \\ &= \|f\|^2 . \end{aligned}$$

STEP 2. Z^i *is perpendicular to* Z^j if $i \neq j$. The proof for, *e.g.*, $i = 2$ and $j = 3$ runs along the lines of Step 1.

STEP 3. Is to introduce the Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2/2}(d/dx)^n e^{-x^2/2}$$

and prove (for use in Step 5) the *addition formula*:

$$\sum \binom{n}{p} H_{p_1}(x_1) H_{p_2}(x_2) \cdots y^p = H_n(x \cdot y) .$$

The summation extends over $p = (p_1, p_2, \cdots)$ with $|p| = p_1 + p_2 + \cdots = n$, $\binom{n}{p} = n!\,(p_1!\,p_2! \cdots)^{-1}$, $x = (x_1, x_2, \cdots) \in R^\infty$, $y = (y_1, y_2, \cdots)$ likewise, $y^p = y_1^{p_1} y_2^{p_2} \cdots$, $\sum y_i^2 = 1$, and $x \cdot y = \sum x_i y_i$.

AMPLIFICATION. For the interpretation of $x \cdot y$, replace x by $\mathfrak{x} = (\mathfrak{x}_1, \mathfrak{x}_2, \cdots) \in S^\infty(\infty^{\frac{1}{2}})$; for the proof, think of x_i and/or y_i as being 0 from some $i = j$ on.

PROOF. By the definition of the Hermite polynomials, the left-hand side is

$$\begin{aligned} &\textstyle\sum_{|p|=n} \binom{n}{p} \prod (-1)^{p_i} e^{x_i^2/2} \left(y_i \dfrac{\partial}{\partial x_i}\right)^{p_i} e^{-x_i^2/2} \\ &\qquad = (-1)^n \exp[\tfrac{1}{2} \textstyle\sum x_i^2] \left[\sum y_i \dfrac{\partial}{\partial x_i}\right]^n \exp[-\tfrac{1}{2} \textstyle\sum x_i^2] \\ &\qquad = (-1)^n e^{(x\cdot y)^2/2} [\partial^n/\partial(x \cdot y)^n] e^{-(x\cdot y)^2/2} \\ &\qquad = H_n(x \cdot y) , \end{aligned}$$

as you can see from the formula

$$x^2 = \sum x_i^2 = (x \cdot y)^2 + |\text{coprojection of } x \text{ upon } y|^2 .$$

STEP 4. The formula

$$\textstyle\int_0^\infty e(t_1)\, d\mathfrak{x}(t_1) \int_0^{t_1} e(t_2)\, d\mathfrak{x}(t_2) \cdots \int_0^{t_{n-1}} e(t_n)\, d\mathfrak{x}(t_n) = (n!)^{-1} H_n(\int_0^\infty e\, d\mathfrak{x})$$

for a sure function e with $||e||^2 = 1$ is also needed for Step 5. The moral is that in the present game, it is H_n that plays the role of the customary power x^n.

PROOF. Pick a real number γ. Then

$$\mathfrak{y}(t) = \exp [\gamma \textstyle\int_0^t e\, d\mathfrak{x} - \frac{1}{2}\gamma^2 \int_0^t e^2\, ds]$$

is a solution of $d\mathfrak{y} = e\mathfrak{y}\, d\mathfrak{x}$, and $\mathfrak{y}(0) = 1$, so by a self-evident iteration,

$$\mathfrak{y}(\infty) = 1 + \textstyle\sum_{n=1}^\infty \gamma^n \int_0^\infty e(t_1)\, d\mathfrak{x}(t_1) \int_0^{t_1} e(t_2)\, d\mathfrak{x}(t_2) \cdots \int_0^{t_{n-1}} e(t_n)\, d\mathfrak{x}(t_n) .$$

But also, by the identity

$$\textstyle\sum_{n=0}^\infty \gamma^n (n!)^{-1} H_n(x) = e^{x^2/2} e^{-(x-\gamma)^2/2} = e^{\gamma x - \gamma^2/2} ,$$

you have

$$\mathfrak{y}(\infty) = \textstyle\sum_{n=0}^\infty \gamma^n (n!)^{-1} H_n(\int_0^\infty e\, d\mathfrak{x}) .$$

Now match like powers of γ.

STEP 5. $\bigoplus_{n=0}^\infty Z^n = Z$.

PROOF. Pick $y = (y_1, y_2, \cdots) \in R^\infty$ with $y^2 = \sum y_i^2 = 1$, let $\mathfrak{x}_i = \int_0^\infty e_i d\mathfrak{x}$ with sure functions $e_1, e_2, \cdots$ making a unit perpendicular basis of $L^2[0, \infty)$, and put

$$H_p(\mathfrak{x}) = H_{p_1}(\mathfrak{x}_1) H_{p_2}(\mathfrak{x}_2) \cdots \qquad \text{for} \quad |p| = p_1 + p_2 + \cdots < \infty .$$

Then by the addition formula of Step 3 and the formula of Step 4,

$$\begin{aligned} \textstyle\sum_{|p|=n} \binom{n}{p} H_p(\mathfrak{x}) y^p &= H_n(\mathfrak{x} \cdot y) \\ &= H_n(\textstyle\int_0^\infty e\, d\mathfrak{x}) \\ &= n! \textstyle\int_0^\infty e(t_1) e(t_2) \cdots e(t_n)\, d^n \mathfrak{x} , \end{aligned}$$

in which $e = \sum y_i e_i$ and the fact that $||e||^2 = y^2 = 1$ is used for the application of Step 4 to line 3. This proves that $H_p(\mathfrak{x}) \in Z^n$ provided only that the total degree $|p|$ equals n; *especially*, since the Hermite polynomials span $L^2(R^1, e^{-x^2/2})$ and the coordinates $\mathfrak{x}_i$ are independent with common density $(2\pi)^{-\frac{1}{2}} e^{-x^2/2}$, you see by specializing the functions e_i that $\oplus Z^n$ contains every reasonable tame function of differences $\mathfrak{x}(t_i) - \mathfrak{x}(t_{i-1})$ with $t_0 = 0 < t_1 < t_2 \cdots$. But such tame functions span Z. The proof is finished.

By Step 5, the expansion can be put into the more explicit form due to Cameron and Martin [3]:

$$\mathfrak{f}(\mathfrak{x}) = \textstyle\sum_{n=0}^\infty \sum_{p=n} H_p(\mathfrak{x}) (p!)^{-1} E[\mathfrak{f} H_p(\mathfrak{x})] .$$

The only new ingredient is the easy evaluation

$$\int_{-\infty}^{\infty} [H_n(x)]^2 \frac{e^{-x^2/2}}{(2\pi)^{\frac{1}{2}}}\, dx = n!\,.$$

An alternative (somewhat more formal) expression is

$$\mathfrak{f}(\mathfrak{x}) = \sum_{|p|\geq 0} H_p(\mathfrak{x})(p!)^{-1} \frac{\partial^p \mathfrak{f}}{\partial \mathfrak{x}^p}(0)\,,$$

in which $\partial^p/\partial x^p$ stands for $(\partial/\partial x_1)^{p_1}(\partial/\partial x_2)^{p_2}\cdots$; the derivation is immediate from

$$\begin{aligned}[\partial^p H_q/\partial x^p](0) &= p! &&\text{if } q = p\\ &= 0 &&\text{if } q \neq p\,,\end{aligned}$$

and the formula confirms the moral Step 4 above: that it is $H_p(x)$ that plays the role of the customary power x^p. From this standpoint, the sum for $\mathfrak{f}$ is a bona fide power series.

A third expression for the power series is obtained by means of operators of creation and annihilation:

$$\begin{aligned}-\partial/\partial x + x&: H_n \to H_{n+1}\\ \partial/\partial x&: H_n \to nH_{n-1}\,.\end{aligned}$$

These are dual in $L^2(R^1, e^{-x^2/2})$, so $(\partial/\partial x)^{*n}1 = H_n$, and you obtain the formula

$$\mathfrak{f}(\mathfrak{x}) = \sum_{|p|\geq 0}\left(\frac{\partial^p}{\partial \mathfrak{x}^p}\right)^*\left[\frac{\partial^p \mathfrak{f}}{\partial \mathfrak{x}^p}(0)\right].$$

You may readily believe that such expansions are difficult to compute. The following cute example is from Wiener [16]:

$$\exp\left[\gamma\left(\int_0^\infty e\, d\mathfrak{x}\right)^2\right] = \sum_{n=0}^{\infty} (n!)^{-1}\gamma^n(1-2\gamma)^{-n-\frac{1}{2}} H_{2n}\left(\int_0^\infty e\, d\mathfrak{x}\right)$$

for $||e|| = 1$ and $-\infty < \gamma < \frac{1}{4}$.

3. The rotation group. An important fact about $S^{n-1}(n^{\frac{1}{2}})$ for $n < \infty$ is that the orthogonal group $O(n)$ acts upon it. The map

$$\mathfrak{x} \to (\mathfrak{x}_1, \mathfrak{x}_2, \cdots) = \left(\int_0^\infty e_1\, d\mathfrak{x}, \int_0^\infty e_2\, d\mathfrak{x}, \cdots\right)$$

suggests that the counterpart for the Brownian motion is (or ought to be) the group $O(\infty)$ of orthogonal transformations of $L^2[0, \infty)$ acting upon $\mathfrak{x}$ according to the rule

$$\int_0^\infty e\, d\mathfrak{x} \to \int_0^\infty (\text{image of } e)\, d\mathfrak{x}\,;$$

in fact, this recipe induces a measure-preserving map (automorphism) of the Brownian motion, as first noted by Akutowicz and Wiener [1]. The proof is to remark that for any orthogonal transformation of $L^2[0, \infty)$,

$$\mathfrak{x}'(t) = \int_0^\infty (\text{image of the indicator of } 0 \leq t' \leq t)\, d\mathfrak{x}$$

is also a Brownian motion, as you may verify by observing that $\mathfrak{x}'$ is a Gaussian process with mean 0 and

$$E[\mathfrak{x}'(t_1)\mathfrak{x}'(t_2)] = \text{the smaller of} \quad t_1 \quad \text{and} \quad t_2 .$$

The polynomial chaos Z^n is invariant under the action of $O(\infty)$ so induced; this is obvious from the fact that $H_p(\mathfrak{x})$: $|p| = n$ spans Z^n for *any* choice of the unit perpendicular basis $e_1, e_2, \cdots$ of $L^2[0, \infty)$. What is less obvious is that *the action of $O(\infty)$ upon Z^n is an irreducible representation of that group.*

PROOF. Pick $\mathfrak{f} \neq 0$ from Z^n and look at $E(\mathfrak{f} \mid \int_0^\infty e\, d\mathfrak{x})$ for a fixed direction $e\,[\|e\| = 1]$; this cannot vanish for every direction e, as that would make

$$\begin{aligned} E[\mathfrak{f}H_n(\textstyle\int_0^\infty \sum_{j=1}^\infty o_{ij} e_j\, d\mathfrak{x})] &= \textstyle\sum_{|p|=n} \binom{n}{p} (o_{i1})^{p_1} (o_{i2})^{p_2} \cdots E[\mathfrak{f}H_p(\mathfrak{x})] \\ &= 0 \end{aligned}$$

for every orthogonal transformation (o_{ij}), contradicting $\mathfrak{f} \neq 0$. Pick such a direction e and a basis $e_1, e_2, \cdots$ of $L^2[0, \infty)$ beginning with $e_1 = e$. Then

$$E(\mathfrak{f} \mid \textstyle\int_0^\infty e\, d\mathfrak{x}) = E(\mathfrak{f} \mid \mathfrak{x}_1)$$

is nontrivial multiple of $H_n(\mathfrak{x}_1)$ since the expectation

$$E[E(\mathfrak{f} \mid \mathfrak{x}_1) H_m(\mathfrak{x}_1)] = E[\mathfrak{f} H_m(\mathfrak{x}_1)]$$

vanishes for $m \neq n$. Because $H_n(\mathfrak{x}_1)$ spans Z^n under the action of $O(\infty)$, it is now enough to check that $E(\mathfrak{f} \mid \mathfrak{x}_1)$ belongs to the span of $\mathfrak{f}$ under that same action. But the latter statement is obvious from the formula

$$E(\mathfrak{f} \mid \mathfrak{x}_1) = \lim_{m \uparrow \infty} \int_{O(m-1)} \mathfrak{f}(x_1, x_2', \cdots x_m', \mathfrak{x}_{m+1}, \cdots)\, do$$

in which $O(m-1)$ acts in the natural way upon $(x_2, \cdots, x_m) \in R^{m-1}$ fixing $\mathfrak{x}_1$ and $\mathfrak{x}_i$ for $i > m$, and *do* is the invariant volume element of that group. This you check, beginning with tame functions $\mathfrak{f}$, using Poincaré's estimation of the spherical average on $S^{m-1}(m^{\frac12})$ and the strong law of large numbers

$$(\mathfrak{x}_2^2 + \cdots + \mathfrak{x}_m^2)^{\frac12} = m[1 + o(1)]$$

for $m \uparrow \infty$. The proof is finished.

AMPLIFICATION. $O(\infty) : Z^n \to Z^n$ should be a complete list of the irreducible representations of $O(\infty)$. For a completely satisfactory proof, you would have to specify more closely the kind of representations permitted, but this has not been done.

4. The spherical Laplace operator. By the previous development, you may think of Z^n as "spherical harmonics of weight n," which suggests that you look for an infinite-dimensional spherical Laplace operator commuting with the action of $O(\infty)$ on $S^\infty(\infty^{\frac12})$ by studing the customary spherical Laplacian Δ on the $(n-1)$-dimensional spherical surface $S^{n-1}(n^{\frac12})$ for $n \uparrow \infty$. To do this, pick a smooth function f on $S^{n-1}(n^{\frac12})$ and extend it nearby by the rule $f = f(n^{\frac12}|x|^{-1}x)$. On the surface $|x| = n^{\frac12}$,

$$\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right) f = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\right) f + n^{-1}\Delta f ,$$

and if you work out the left-hand side by hand, you obtain

$$n^{-1}\Delta = \sum_{i\leqq n}\frac{\partial^2}{\partial x_i^2} - \frac{1}{n}\sum_{i,j\leqq n} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} - \frac{n-1}{n}\sum_{i\leqq n} x_i \frac{\partial}{\partial x_i}$$

still with $|x| = n^{\frac{1}{2}}$; in particular,

$$\lim_{n\uparrow\infty} n^{-1}\Delta = \sum (\partial^2/\partial x_i^2 - x_i \partial/\partial x_i)$$

since the cross partials $\partial^2/\partial x_i \partial x_j$ (acting on tame functions) are small in number the comparison to n. This operator is now declared to be the infinite-dimensional Laplacian Δ, and you conclude from

$$H_n'' - xH_n' = -nH_n$$

that

$$\Delta : H_p(x) = H_{p_1}(x_1)H_{p_2}(x_2)\cdots \to -nH_p(x)$$

for $|p| = n$, which is to say that Z^n is an eigenspace of Δ, as it should be. Δ may now be viewed as a bona fide self-adjoint operator on Z:

$$-\Delta = 0 \oplus 1 \oplus 2 \oplus 3 \oplus \cdots .$$

The fact that Δ commutes with the action of $O(\infty)$ is self-evident from this splitting.

Δ can also be identified as a kind of Casimir operator: you take the infinitesimal rotations

$$x_i \partial/\partial x_j - x_j \partial/\partial x_i \qquad i < j$$

acting upon tame functions of $x_1, x_2, \cdots$, and you average the sum of their squares on the spherical surface $|x| = n^{\frac{1}{2}}$ for $n \uparrow \infty$, so:

$$n^{-1}\sum_{i<j\leqq n}(x_i\partial/\partial x_j - x_j\partial/\partial x_i)^2 = \sum_{i=1}^n (1 - x_i^2/n)\partial^2/\partial x_i^2 - \frac{n-1}{n}\sum_{i=1}^n x_i\partial/\partial x_i .$$

The cross partials $\partial^2/\partial x_i \partial x_j$ (acting upon tame functions) have been left out as being small in number in comparison to n, and it is plain that this operator approximates Δ as $n \uparrow \infty$.

It is a source of added satisfaction to notice that the "spherical harmonics" $H_p(x)$ come from the n-dimensional ones by making $n \uparrow \infty$. For dimension $n \geqq 3$, fixed $|p| = p_1 + p_2 + \cdots = m$, and $|x| = n^{\frac{1}{2}}$, the functions

$$\frac{\partial^p |x|^{2-n}}{\partial x^p}$$

span out the class of spherical harmonics of weight m on $S^{n-1}(n^{\frac{1}{2}})$, a fact due to J. C. Maxwell [9] for $n = 3$. Fix k so large that $p_i = 0$ for $i > k$. Then, still with $|x| = n^{\frac{1}{2}}$,

$$\lim_{n\uparrow\infty} n^{n/2-1}\frac{\partial^p |x|^{2-n}}{\partial x^p} = (-1)^m H_p(x) .$$

PROOF. The left-hand side can be expressed as

$$\lim_{n\uparrow\infty} n^{n/2-1}(x_{k+1}^2+\cdots+x_n^2)^{1-n/2}\frac{\partial^p}{\partial x^p}\left[1+\frac{x_1^2+\cdots+x_k^2}{x_{k+1}^2+\cdots+x_n^2}\right]^{1-n/2}$$

$$=\lim_{n\uparrow\infty}\left(1-\frac{x_1^2+\cdots+x_k^2}{n}\right)^{1-n/2}\frac{\partial^p}{\partial x^p}\left[1+\frac{x_1^2+\cdots+x_k^2}{n+O(1)}\right]^{1-n/2}$$

$$=\exp[(x_1^2+\cdots+x_k^2)/2]\frac{\partial^p}{\partial x^p}\exp[-(x_1^2+\cdots+x_k^2)/2]$$

$$=(-1)^m H_p(x)\,,$$

as stated.

5. The translation group. A case will now be made for the statement that *despite its name "sphere," $S^\infty(\infty^{\frac{1}{2}})$ is pretty flat.* The precise result is that *if f is a sure absolutely continuous function with $f(0)=0$, and if $\mathfrak{x}'$ is the translated path*

$$\mathfrak{x}'(t)=\mathfrak{x}(t)+f(t)\,,$$

then

$$P(\mathfrak{x}'\in W)=1 \quad or \quad 0$$

according as

$$\|f^\bullet\|^2<\infty \quad or\ not^2\,;$$

in the first case, the distribution of $\mathfrak{x}'$ is expressed by the formula of Cameron and Martin [2]:

$$P(\mathfrak{x}'\in B)=E[\exp(\textstyle\int_0^\infty f^\bullet\,d\mathfrak{x}-\frac{1}{2}\int_0^\infty (f^\bullet)^2\,dt),B]\,,$$

which is to say that

$$\mathfrak{j}=\exp(\textstyle\int_0^\infty f^\bullet\,d\mathfrak{x}-\frac{1}{2}\int_0^\infty (f^\bullet)^2\,dt)$$

is the "Jacobian" of the translation $\mathfrak{x}\to\mathfrak{x}+f$. The geometrical content is that *a mild translation $[\|f^\bullet\|<\infty]$ cannot move you off the spherical surface $S^\infty(\infty^{\frac{1}{2}})$*; plainly, this is a purely infinite-dimensional phenomenon.

PROOF OF THE CAMERON-MARTIN FORMULA FOR $\|f^\bullet\|<\infty$: $\mathfrak{y}=\int_0^\infty f^\bullet\,d\mathfrak{x}$ is Gaussian distributed with mean 0 and $E(\mathfrak{y}^2)=\|f^\bullet\|^2$, so

$$P(\mathfrak{x}'\in W)=E(\mathfrak{j})=1\,,$$

which is hopeful. The rest of the proof proceeds by approximation: it is enough to check the formula for the special cylinder sets

$$B=\bigcap\nolimits_{0\le i\le k}\left(a_i\leqq\mathfrak{x}\left(\frac{i}{n}\right)-\mathfrak{x}\left(\frac{i-1}{n}\right)<b_i\right)$$

and for functions f of constant slope in every interval $(i-1)/n\leqq t<i/n$. But in that case, both sides of the formula split into simple (independent) factors

[2] $f^\bullet=df/dt$.

of which the first is typical:

$$
\begin{aligned}
P&\left(a_1 \leqq \mathfrak{x}'\left(\frac{1}{n}\right) < b_1\right) \\
&= P\left[a_1 \leqq \mathfrak{x}\left(\frac{1}{n}\right) + \frac{1}{n} f^{\bullet}(0) < b_1\right] \\
&= \int_{a_1}^{b_1} \frac{\exp[-(n/2)[x - n^{-1} f^{\bullet}(0)]^2]}{(2\pi/n)^{\frac{1}{2}}}\, dx \\
&= \int_{a_1}^{b_1} \frac{e^{-nx^2/2}}{(2\pi/n)^{\frac{1}{2}}} \exp[f^{\bullet}(0)x - [f^{\bullet}(0)]^2/2n]\, dx \\
&= E\left[a_1 \leqq \mathfrak{x}\left(\frac{1}{n}\right) < b_1, \exp\left[\int_0^{1/n} f^{\bullet}\, d\mathfrak{x} - \tfrac{1}{2}\int_0^{1/n} (f^{\bullet})^2\, dt\right]\right].
\end{aligned}
$$

The rest of the proof is plain sailing.

PROOF THAT $P(\mathfrak{x}' \in W) = 0$ IF $\|f^{\bullet}\| = \infty$: The rough idea is that if $P(\mathfrak{x}' \in W) > 0$, then $P(\int_0^{\infty} e\, d\mathfrak{x}'$ exists$) > 0$ for every $e \in L^2[0, \infty)$; as this can happen only if $\int e\, df$ exists, you conclude that $f^{\bullet}$ exists and belongs to $L^2[0, \infty)$.

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