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A NOTE ON RANDOM MEASURES AND MOVING AVERAGES ON NON-DISCRETE GROUPS[1]

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The Fourier transform of a stationary random measure on an arbitrary locally compact Abelian group is defined and used to extend some results on the representation of homogeneous random fields as moving averages.

0. Introduction. In a recent paper of Bruckner [1971] it is shown that a homogeneous random field X_g on a discrete Abelian group G is a moving average $X_g = \sum_{g'} C_{g'} Z_{g-g'}$, where Z_g are orthogonal random variables, if and only if the spectral measure of the process X is absolutely continuous with respect to Haar measure on the character group $\hat{G}$ of G. For terminology, see Bruckner [1971].

This extends well-known results for processes on the line and integers (Doob (1953) page 498–499 and 532–533).

The difficulty when G is not discrete is that $\hat{G}$ is not compact and Haar measure on $\hat{G}$ is infinite. Thus a stationary random measure on $\hat{G}$ does not possess a Fourier transform in the ordinary sense. In this paper an analogue of the Fourier transform of a process with weakly stationary orthogonal increments (Doob (1953) page 434–436) is introduced and used to complete the extension of Bruckner to processes parameterized by arbitrary locally compact Abelian groups.

1. Results. A stationary random measure ξ on a locally compact Abelian group G is an orthogonal measure on G such that $E(\xi(A)\overline{\xi(A')}) = \mu(A \cap A')$, where μ is Haar measure on G. If G is not compact, the Fourier transform $\int_G \langle g, \alpha \rangle d\xi(g)$ will not exist since $\langle g, \alpha \rangle$ is not in $L^2(\mu)$. In place of the ordinary Fourier transform define

$$\xi^*(B) = \int_G \left(\int_B \langle g, \alpha \rangle \, d\nu(\alpha)\right) d\xi(g)\,,$$

where ν is Haar measure in $\hat{G}$ and B is a set of finite ν measures in $\hat{G}$. Formally, $\xi^*(B)$ is the Fourier transform of $d\xi$ integrated over B with respect to ν.

Because of the L^2 isometries of the Plancherel theorem:

$$f(\alpha) \to \int f(\alpha)\langle g', \alpha \rangle \, d\nu(\alpha) = \hat{f}(g)$$

and the definition of the stochastic integral: $f(g) \to \int f(g)\, d\xi(g)$, $\xi^*(B)$ is a well-defined random variable and, in fact, ζ^* is a stationary random measure on $\hat{G}$. Namely, $C_B \to \xi^*(B)$ is an isometry, where C_B is the characteristic function of

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B. The definition can thus be written $\int_{\hat{G}} C_B \, d\xi^* = \int_G \hat{C}_B \, d\xi$. By linearity this relation can be extended to say $\int_{\hat{G}} f \, d\xi^* = \int_G \hat{f} \, d\xi$.

By letting $\hat{f}$ be the characteristic function of some set A in G we see that

$$\int_{\hat{G}} (\int_A \overline{\langle g, \alpha \rangle} \, d\mu(g)) \, d\xi^* = \xi(A)$$

so that the relation between ξ and ξ^* is like that of a Fourier transform and its inverse.

THEOREM. *A homogeneous random field X_g on a locally compact Abelian group can be written $X_g = \int_G \omega(g + g') \, d\xi(g')$ for $\omega \in L^2(d\mu)$, where ξ is a stationary random measure on G if and only if the special measure of X_g is absolutely continuous with respect to Haar measure on $\hat{G}$.*

PROOF. If $X_g = \int_G \omega(g + g') \, d\xi(g')$ then

$$\begin{aligned} E(X_g \bar{X}_h) &= \int \omega(g + g')\overline{\omega(h + g')} \, d\mu(g') \\ &= \int |\check{\omega}(\alpha)|^2 \langle g - h, \alpha \rangle \, d\nu(\alpha) \,, \end{aligned}$$

where $\int \langle g, \alpha \rangle \, \check{\omega}(\alpha) \, d\nu(\alpha) = \omega(g)$. Thus X_g has absolutely continuous spectral measure with spectral density $|\check{\omega}(\alpha)|^2$.

Conversely, assume X_g has absolutely continuous spectral measure $dF = f^2 d\nu$. $X_g = \int \langle g, \alpha \rangle \, dZ(\alpha)$; where Z is an orthogonal measure with $E(|dZ|^2) = dF$. Let $Y(B) = \int_B (f(\alpha))^{-1} \, dZ(\alpha)$. Then Y is almost a stationary random measure.

The only difficulty is that on the set where f is 0, $E(|dY|^2) = 0 \neq d\nu$. To remedy this let A be the set where $f = 0$. Let Z_1 be a stationary random measure on $\hat{G}$ independent of Z. Define

$$\xi^*(B) = \int_B \frac{1}{f(\alpha)} \, dZ(a) + \int_B C_A(\alpha) \, dZ_1(\alpha) \,.$$

Then $E(|\xi^*(B)|^2) = \nu(B)$ and ξ^* is a stationary random measure. Moreover,

$$X_g = \int_{\hat{G}} \langle g, \alpha \rangle \, dZ(\alpha) = \int_{\hat{G}} \langle g, \alpha \rangle f(\alpha) \, d\xi^*(\alpha) \,.$$

By the fundamental identity between ξ and ξ^*,

$$X_g = \int_G \hat{f}(g' + g) \, d\xi(g') \,. \qquad \square$$

It is seen from the proof that the random variables $\xi^*(B)$ and hence the random variables $\xi(A)$ lie in $H(X)$, the Hilbert space spanned by X_g, g in G, if and only if the spectral density is positive almost everywhere with respect to Haar measure on $\hat{G}$.

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