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A CHARACTERIZATION OF CERTAIN RANK-ORDER TESTS WITH BOUNDS FOR THE ASYMPTOTIC RELATIVE EFFICIENCY

BY KONRAD BEHNEN[1]

University of California, Berkeley, and University of Münster

For the one-sample independence problem, the one-sample symmetry problem, and the two-sample problem it is shown that every one-sided rank test is asymptotically optimal for a certain nonparametric subclass of contiguous alternatives, provided the test and the associated subclass of alternatives are generated by certain square-integrable functions defined on the unit square. Then the asymptotic normality of the respective rank statistics under every alternative contiguous to the hypothesis is used in order to give necessary and sufficient conditions for local asymptotic unbiasedness of such tests. Finally, for locally asymptotically unbiased tests there are given necessary and sufficient conditions for having bounds for their asymptotic relative efficiency under contiguous alternatives.

1. Summary. In the paper [1] of the author it is shown that every sequence of rank tests of a certain form for testing independence against positive quadrant dependence is asymptotically optimal in the sense of Neyman [14] for a suitable subclass of alternatives. Also asymptotic normality of the respective test statistics was derived under every sequence of alternatives that is contiguous (cf. Hájek and Šidák [8] or LeCam [12]) to some hypothesis. Then sufficient conditions for two such tests having a lower bound larger than zero for their asymptotic relative efficiency (ARE) under alternatives contiguous to some hypothesis were given and applied to some well-known examples.

In this paper we will derive necessary and sufficient conditions for two tests of a certain class to have a strictly positive lower bound for their ARE under alternatives contiguous to some hypothesis. In order to get these results we must allow more general generating functions for the tests under consideration than was done in [1]. Therefore, at the beginning, we have to state assumptions which are somewhat similar to those stated in [1]. To make it a little bit more interesting we will do all things not only in the "independence case" but also in the "one-sample symmetry case" and the "two-sample case." (Possibly one can derive similar results in some other cases.) The results will give an immediate explanation for the behavior of the ARE in the examples stated in [1].

In Section 3 we have listed some technicalities on approximation by special step functions needed in the proofs of Section 2.

2. Main results. Let $X_{N1}, \cdots, X_{Nn}$ be independent k-dimensional random variables and suppose that the distribution of X_{Nj} is given by a k-dimensional continuous (cumulative) distribution function F_{Nj}. Under this basic assumption

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[1] Now at University of Freiburg, Germany.

we shall simultaneously consider the following three special "asymptotic test problems."

A. *The 2-dimensional one-sample problem "independence vs. positive quadrant dependence."* This means especially $k = 2$, $N = n$, $N \to \infty$, $X_{Nj} = (Y_{Nj}, Z_{Nj})$, $F_{Nj} = F_N$, $j = 1, \cdots, N$, with the hypothesis of independence

$$H\colon F_N(y, z) = F(y, z) = F(y, \infty)F(\infty, z) \qquad \forall y, z \in \mathbb{R} \;\; \forall N$$

and the alternative of positive quadrant dependence

$$K\colon F_N(y, z) \underset{\neq}{\geqq} F_N(y, \infty)F_N(\infty, z) \qquad \begin{array}{l}\forall y, z \in \mathbb{R}\\ \exists y, z \in \mathbb{R} \;\; \forall N .\end{array}$$

B. *The 1-dimensional one-sample problem "symmetry vs. positive unsymmetry."* This means $k = 1$, $N = n$, $N \to \infty$, $F_{Nj} = F_N$, $j = 1, \cdots, N$, with the hypothesis of symmetry

$$H\colon F_N(x) = F(x) = 1 - F(-x) \qquad \forall x \in \mathbb{R} \;\; \forall N$$

and the alternative of positive unsymmetry

$$K\colon F_N(x) \underset{\neq}{\leqq} 1 - F_N(-x) \qquad \begin{array}{l}\forall x \in \mathbb{R}\\ \exists x \in \mathbb{R} \;\; \forall N .\end{array}$$

C. *The 1-dimensional two-sample problem "randomness vs. positive stochastic deviation of the first sample."* This means $k = 1$, $N = (n_1, n_2)$, $n = n_1 + n_2$, $N \to (\infty, \infty)$, $F_{Nj} = F_{1N}$, $j = 1, \cdots, n_1$, $F_{Nj} = F_{2N}$, $j = n_1 + 1, \cdots, n$, with the hypothesis of randomness

$$H\colon F_{1N}(x) = F_{2N}(x) = F(x) \qquad \forall x \in \mathbb{R} \;\; \forall N$$

and the alternative of positive stochastic deviation of the first sample

$$K\colon F_{1N}(x) \underset{\neq}{\leqq} F_{2N}(x) \qquad \begin{array}{l}\forall x \in \mathbb{R}\\ \exists x \in \mathbb{R} \;\; \forall N .\end{array}$$

For these problems we can derive (similar to [1]) asymptotically optimal rank tests at level α, $0 < \alpha < 1$, for H against a suitable subclass of K. As generating functions b for the asymptotic test $\{\varphi_{bN}\}$ and for the subclass of alternatives K_b, for which $\{\varphi_{bN}\}$ will be asymptotically optimal (in the sense of Neyman [14]), we allow in the three cases real-valued Lebesgue-measurable functions on $(0, 1) \times (0, 1)$ and $(0, 1)$, respectively, which have the properties (2.1.A), (2.1.B), (2.1.C), respectively.

Notation. Throughout the paper we will denote the Lebesgue-measure on $(0, 1)$ and $(0, 1) \times (0, 1)$ by λ_1 and λ_2, respectively. For convenience we write for example $\int_0^s \int_0^t b(y, z)\, dy\, dz$ or $\int_0^s \int_0^t b\, dx$ instead of $\int_{(0,s)\times(0,t)} b\, d\lambda_2$, and shortly "a.e." instead of "Lebesgue-almost everywhere."

$$\text{(2.1.A)} \qquad \int_0^1 b(y, \cdot)\, dy = 0 \quad \text{a.e.}, \qquad \int_0^1 b(\cdot, z)\, dz = 0 \quad \text{a.e.},$$
$$\int b^2\, dx = 1\,, \qquad \int_0^s \int_0^t b\, dx \geqq 0 \qquad \forall s, t \in (0, 1)\,.$$

(2.1.B) $b(x) = -b(1-x)$ a.e., $\int b^2\,dx = 1$, $\int_0^t b\,dx \leqq 0$ $\forall t \in (0, 1)$ (especially: $\int b\,dx = 0$).

(2.1.C) $\int b\,dx = 0$, $\int b^2\,dx = 1$, $\int_0^t b\,dx \leqq 0$ $\forall t \in (0, 1)$.

Now we assume that we generate scores b_{Nij}, b_{Ni}, respectively, from b (using the notation $U_N^{(i)}$, $V_N^{(j)}$ as stated before formula (3.13)) by

(2.2.A) $$b_{Nij} = Eb(U_N^{(i)}, V_N^{(j)}), \qquad i, j = 1, \cdots, N,$$

(2.2.B) $$b_{Ni} = Eb(\tfrac{1}{2} + \tfrac{1}{2}U_N^{(i)}), \qquad i = 1, \cdots, N,$$

(2.2.C) $$b_{Ni} = Eb(U_n^{(i)}), \qquad i = 1, \cdots, n.$$

It will be seen from the proofs and from (3.19) that we can also choose as scores

$$b_{Nij} = N^2 \int_{(i-1)/N}^{i/N} \int_{(j-1)/N}^{j/N} b\,dx, \qquad b_{Ni} = 2N \int_{\frac{1}{2}+\frac{1}{2}(i-1)/N}^{\frac{1}{2}+\frac{1}{2}i/N} b\,dx,$$
$$b_{Ni} = n \int_{(i-1)/n}^{i/n} b\,dx,$$

respectively, or if b fulfills some additional conditions we can choose

$$b_{Nij} = b(i/(N+1), j/(N+1)), \qquad b_{Ni} = b(\tfrac{1}{2} + \tfrac{1}{2}i/(N+1)),$$
$$b_{Ni} = b(i/(n+1)),$$

respectively, or even more general scores.

By such choices one gets for example the Spearman or Wilcoxon tests for $b(y, z) = 12(y - \frac{1}{2})(z - \frac{1}{2})$, $b(x) = (12)^{\frac{1}{2}}(x - \frac{1}{2})$, respectively, the normal-scores test (Fisher-Yates or Van der Waerden version) for $b(y, z) = \Phi^{-1}(y)\Phi^{-1}(z)$, $b(x) = \Phi^{-1}(x)$, respectively, where Φ^{-1} denotes the inverse of the cumulative distribution function of the $N(0, 1)$-law. Finally one gets the quadrant, sign, or median test for $b(y, z) = \operatorname{sign}(y - \frac{1}{2}) \operatorname{sign}(z - \frac{1}{2})$, $b(x) = \operatorname{sign}(x - \frac{1}{2})$, respectively.

With these scores we define a sequence $\{\varphi_{bN}\}$ of rank-order tests $\varphi_{bN} = I_{\{T_{bN} > c_N\}}$ with $c_N \to u_\alpha = \Phi^{-1}(1-\alpha)$, $0 < \alpha < 1$, and

(2.3.A) $$T_{bN} = N^{-\frac{1}{2}} \sum_{i=1}^N b_{NR_{Ni}S_{Ni}},$$

$(R_{N1}, \cdots, R_{NN})$, $(S_{N1}, \cdots, S_{NN})$ ranks in $(Y_{N1}, \cdots, Y_{NN})$, $(Z_{N1}, \cdots, Z_{NN})$, resp.

(2.3.B) $$T_{bN} = N^{-\frac{1}{2}} \sum_{i=1}^N b_{NR_{Ni}} \operatorname{sign}(X_{Ni}),$$

$(R_{N1}, \cdots, R_{NN})$ ranks in $(|X_{N1}|, \cdots, |X_{NN}|)$,

(2.3.C) $$T_{bN} = (n_1 n_2/(n_1 + n_2))^{\frac{1}{2}}((1/n_1) \sum_{i=1}^{n_1} b_{NR_{Ni}} - (1/n_2) \sum_{i=n_1+1}^{n_1+n_2} b_{NR_{Ni}}),$$

$(R_{N1}, \cdots, R_{Nn})$ ranks in $(X_{N1}, \cdots, X_{Nn})$.

Before stating the main results, we have to go into the construction of the class of alternatives K_b for which $\{\varphi_{bN}\}$ will appear as asymptotically optimal. In Section 3 it is shown that the properties (2.1) imply the existence of an approximating sequence of functions b_N for b with the following properties:

(2.4.A) $$\int (b_N - b)^2\,dx \to 0, \qquad \int_0^1 b_N(y, \cdot)\,dy = \int_0^1 b_N(\cdot, z)\,dz = 0,$$
$$\int_0^s \int_0^t b_N\,dx \geqq 0 \quad \forall s, t \in (0, 1), \qquad \sup_{0<y,z<1} b_N^4(y, z)/N \to 0.$$
$$\neq \quad \exists s, t \in (0, 1)$$

(This implies especially the existence of some $D > 0$ such that $D \sup_{0<y,z<1} |b_N(y, z)|/N^{\frac{1}{2}} \leqq 1 \ \forall N$.)

$$\text{(2.4.B)} \qquad \int (b_N - b)^2\,dx \to 0\,, \qquad b_N(x) = -b_N(1-x) \qquad \forall x \in (0,1)$$
$$\int_0^t b_N\,dx \underset{\neq}{\leqq} 0 \quad \begin{matrix}\forall t \in (0,1)\,, \\ \exists t \in (0,1)\end{matrix} \qquad \sup_{0<x<1} b_N^4(x)/N \to 0\,.$$

(This implies especially $\int b_N\,dx = 0 \ \forall N$ and the existence of some $D > 0$ such that $D \sup_{0<x<1} |b_N(x)|/N^{\frac{1}{2}} \leqq 1 \ \forall N$.)

$$\text{(2.4.C)} \qquad \int (b_N - b)^2\,dx \to 0\,, \qquad \int b_N\,dx = 0\,,$$
$$\int_0^t b_N\,dx \underset{\neq}{\leqq} 0 \quad \begin{matrix}\forall t \in (0,1)\,, \\ \exists t \in (0,1)\end{matrix} \qquad \sup_{0<x<1} b_N^4(x)/\min\{n_1, n_2\} \to 0\,.$$

(This implies especially the existence of some $D > 0$ such that $D \sup_{0<x<1} |b_N(x)|/\min\{n_1^{\frac{1}{2}}, n_2^{\frac{1}{2}}\} \leqq 1 \ \forall N$.)

Therefore we can define for each $F \in H$ sequences of densities with respect to F (i.e. with respect to P_F) in the following way:

$$\text{(2.5.A)} \qquad f_{Nd}^F(y,z) = 1 + dN^{-\frac{1}{2}} b_N(F(y,\infty), F(\infty,z))\,, \qquad y, z \in \mathbb{R},\ 0 < d \leqq D\,,$$

$$\text{(2.5.B)} \qquad f_{Nd}^F(x) = 1 + dN^{-\frac{1}{2}} b_N(F(x))\,, \qquad x \in \mathbb{R},\ 0 < d \leqq D\,,$$

$$\text{(2.5.C)} \qquad f_{1Nd}^F(x) = 1 + d(n_1 n_2/(n_1+n_2))^{\frac{1}{2}}(1/n_1) b_N(F(x))\,,$$
$$f_{2Nd}^F(x) = 1 - d(n_1 n_2/(n_1+n_2))^{\frac{1}{2}}(1/n_2) b_N(F(x))\,, \qquad x \in \mathbb{R},\ 0 < d \leqq D\,.$$

From (2.4) it can easily be seen that the sequence of corresponding cumulative distribution functions $F_{Nd}^F(y,z)$, $F_{Nd}^F(x)$, $F_{1Nd}^F(x)$, $F_{2Nd}^F(x)$, respectively in each case belongs to the corresponding alternative K.

Thus we can define the subclass K_b of K as the set of all such sequences $\{F_{Nd}^F\}$, $\{F_{Nd}^F\}$, $\{(F_{1Nd}^F, F_{2Nd}^F)\} = \{F_{Nd}^F\}$, respectively, for all $F \in H$ and all sequences $\{b_N\}$ with (2.4).

REMARK. It can be shown that for each $F \in H$ and $\{F_{Nd}^F\} \in K_b$ the sequence $\{Q_N\}$, $Q_N = \mathscr{L}[(X_{N1}, \cdots, X_{Nn}) \mid F_{Nd}^F]$, is contiguous (in the sense of Hájek [6]) to the sequence $\{P_N\}$, $P_N = \mathscr{L}[(X_{N1}, \cdots, X_{Nn}) \mid F]$, i.e. $P_N(A_N) \to 0$ implies $Q_N(A_N) \to 0$. Shortly: $\{F_{Nd}^F\}$ is contiguous to F.

Now we are in the position to state the first result (corresponding to Theorem 1 in [1]) simultaneously for the three considered cases.

THEOREM 2.1. (a) $F \in H \Rightarrow \mathscr{L}[T_{bN} \mid F] \to N(0,1)$.

(b) $\{\varphi_{bN}\}$ *is an asymptotically optimal test for* H *against* K_b *at level* α.

(c) $F \in H$ *and* $\{F_N\}$ *contiguous to* F *imply*

$$\mathscr{L}[T_{bN} - \delta_N(b_N, F, F_N) \mid F_N] \to N(0,1)$$

with

$$\text{(2.6.A)} \qquad \delta_N(b_N, F, F_N) = N^{\frac{1}{2}} \int b_N(F(y,\infty), F(\infty,z))\,dF_N(y,z)\,,$$

$$\text{(2.6.B)} \qquad \delta_N(b_N, F, F_N) = N^{\frac{1}{2}} \int b_N(F(x))\,dF_N(x)\,,$$

$$\text{(2.6.C)} \qquad \delta_N(b_N, F, F_N) = (n_1 n_2/(n_1+n_2))^{\frac{1}{2}} \int b_N(F(x))(dF_{1N}(x) - dF_{2N}(x))\,.$$

The proof is even in this more general case very similar to the proof of Theorem 1 in [1] and shall therefore mainly be omitted. It is again based on the comparison of T_{bN}, for each fixed $F \in H$ and each sequence $\{F_N\}$ contiguous to F, with

$$(2.7.A) \qquad T_N' = N^{-\frac{1}{2}} \sum_{i=1}^N b_N(F(Y_{Ni}, \infty), F(\infty, Z_{Ni})) ,$$

$$(2.7.B) \qquad T_N' = N^{-\frac{1}{2}} \sum_{i=1}^N b_N(F(X_{Ni})) = N^{-\frac{1}{2}} \sum_{i=1}^N b_N(\tfrac{1}{2} + \tfrac{1}{2}(2F(|X_{Ni}|) - 1)) \operatorname{sign}(X_{Ni}) ,$$

$$(2.7.C) \qquad T_N' = (n_1 n_2/(n_1 + n_2))^{\frac{1}{2}}((1/n_1) \textstyle\sum_{i=1}^{n_1} b_N(F(X_{Ni})) - (1/n_2) \sum_{i=n_1+1}^{n_1+n_2} b_N(F(X_{Ni}))) ,$$

respectively.

(Therefore part (a) and part (c) of Theorem 2.1 are also true for a suitable subclass H' of H for more general statistics which are asymptotically equivalent to such an T_N' for each $F \in H'$. One example of this kind is the Student-t-statistic, if we take H' as the subset of H having second moments.)

On the other hand we compare T_N' for sequences $\{F_N\} \in K_b$, derived from F according to (2.5), with the log-likelihood ratio $L_N = \log (dQ_N/dP_N)$ of the corresponding Q_N and P_N (cf. remark before Theorem 2.1).

The only difference to the proof of Theorem 1 in [1] is the use of (3.18) to (3.21) of Section 3 instead of Hájek and Šidák [8], Theorem V.1.4.a.

This theorem will enable us to examine the ARE of two such tests. The first step is the following theorem which characterizes the asymptotically locally unbiased tests in terms of the generating function b. Therefore we need the following *definitions*:

We call a Lebesgue-measurable function b on $(0, 1) \times (0, 1)$ to $\mathbb{R}$ positive Δ-monotone [almost everywhere (a.e.)] iff

$$M_4 = \{(x, x', y, y') : 0 < x < x' < 1, 0 < y < y' < 1, \Delta_b(x, x', y, y') = b(x', y') + b(x, y) - b(x', y) - b(x, y') < 0\}$$

is empty [has measure zero with respect to Lebesgue-measure λ_4 on $(0, 1)^{(4)}$].

We call a Lebesgue-measurable function b on $(0, 1)$ to $\mathbb{R}$ monotone increasing [almost everywhere (a.e.)] iff

$$M_2 = \{(x, x') : 0 < x < x' < 1, b(x') - b(x) < 0\}$$

is empty [has measure zero with respect to Lebesgue-measure λ_2 on $(0, 1)^{(2)}$].

THEOREM 2.2. *The following three statements are equivalent*:

(I) $\{\varphi_{bN}\}$ *is asymptotically unbiased for each* $\{F_N\} \in K$ *that is contiguous to some* $F \in H$.

(II) *There exists a sequence* $\{b_N\}$ *with* (2.4) *such that* $F \in H$ *and* $\{F_N\} \in K$, $\{F_N\}$ *contiguous to* F, *imply* $\liminf_{N\to\infty} \delta_N(b_N, F, F_N) \geq 0$.

(III) b *is positive* Δ-*monotone* a.e. *in case* A *and monotone increasing* a.e. *in the cases* B *and* C, *respectively*.

PROOF. The equivalence of (I) and (II) is obvious because of Theorem 2.1. As the proofs of the other cases are quite similar, we show the equivalence of (II) and (III) only in the case A.

First assume b is not positive Δ-monotone a.e. Then (3.10.A) and (3.1.A) imply the existence of an $r > 0$ and of intervals $J_{x_0} = (x_0, x_0 + r)$, $J_{x_1} = (x_1, x_1 + r)$, $J_{y_0} = (y_0, y_0 + r)$, $J_{y_1} = (y_1, y_1 + r)$ with $0 \leqq x_0 < x_0 + r \leqq x_1 < x_1 + r \leqq 1$, $0 \leqq y_0 < y_0 + r \leqq y_1 < y_1 + r \leqq 1$ and

$$(2.8) \qquad \int_{J_{x_0} \times J_{y_0}} b\,dx + \int_{J_{x_1} \times J_{y_1}} b\,dx - \int_{J_{x_0} \times J_{y_1}} b\,dx - \int_{J_{x_1} \times J_{y_0}} b\,dx < 0\,.$$

Now we define b_0: $(0, 1) \times (0, 1) \to \mathbb{R}$ by setting

$$\begin{aligned} 2rb_0(x, y) &= I_{J_{x_0} \times J_{y_0}}(x, y) + I_{J_{x_1} \times J_{y_1}}(x, y) - I_{J_{x_0} \times J_{y_1}}(x, y) - I_{J_{x_1} \times J_{y_0}}(x, y) \\ &= (-I_{J_{x_0}}(x) + I_{J_{x_1}}(x))(-I_{J_{y_0}}(y) + I_{J_{y_1}}(y))\,. \end{aligned}$$

Then, obviously, b_0 and $b_{0N} = b_0\ \forall N$ fulfill the conditions (2.1.A) and (2.4.A); and we can define the class of alternatives $K_{b_0} \subset K$ according to (2.5.A). From the remark before Theorem 2.1 we get to each $F \in H$ and $\{F_{Nd}^F\} \in K_{b_0}$ the contiguity of $\{F_{Nd}^F\}$ to F and therefore by Theorem 2.1

$$\begin{aligned} \delta_N(b_N, F, F_{Nd}^F) &= N^{\frac{1}{2}} \int b_N(F(y, \infty), F(\infty, z)) \\ &\qquad \times (1 + dN^{-\frac{1}{2}} b_0(F(y, \infty), F(\infty, z)))\,dF(y, \infty)\,dF(\infty, z) \\ &= d \int b_N b_0\,dx\,. \end{aligned}$$

Thus (2.4.A) and (2.8) imply $\lim_{N\to\infty} \delta_N(b_N, F, F_{Nd}^F) = d \int bb_0\,dx < 0$.

Now assume that b is positive Δ-monotone a.e. If we choose b_N as b_{m_N} according to (3.1.A) with $m_N = [N^{\frac{1}{8}}]$, then b_N fulfills (2.4.A) and is positive Δ-monotone because of (3.4), (3.6.A), (3.8.A), (3.7), and (3.9.A). From this, (3.12.A), and (2.6.A) we get therefore for each $\{F_N\} \in K$ and $F \in H$ with $\{F_N\}$ contiguous to F

$$\begin{aligned} &\delta_N(b_N, F, F_N) - N^{\frac{1}{2}} \int b_N(F(y, \infty), F(\infty, z))\,dF_N(y, \infty)\,dF_N(\infty, z) \\ &\qquad = N^{\frac{1}{2}} \sum_{i=1}^{m_N-1} \sum_{j=1}^{m_N-1} \Delta_{b_{m_N}}^{ij} (F_N(y_{Ni}, z_{Nj}) - F_N(y_{Ni}, \infty) F_N(\infty, z_{Nj})) \geqq 0\,, \end{aligned}$$

with y_{Ni} and z_{Nj} defined by $(-\infty, y_{Ni}] = [F(\cdot, \infty) \leqq i/m_N]$ and $(-\infty, z_{Nj}] = [F(\infty, \cdot) \leqq j/m_N]$, respectively. With $F_{1N} = F_N(\cdot, \infty)$, $F_1 = F(\cdot, \infty)$, $F_{2N} = F_N(\infty, \cdot)$, $F_2 = F(\infty, \cdot)$ we get on the other side

$$\begin{aligned} &|N^{\frac{1}{2}} \int b_N(F_1(y), F_2(z))\,dF_{1N}(y)\,dF_{2N}(z)| \\ &\qquad = |N^{\frac{1}{2}} \int b_N(F_1(y), F_2(z))(dF_{1N}(y) - dF_1(y))(dF_{2N}(z) - dF_2(z))| \\ &\qquad \leqq N^{\frac{1}{2}} \sup_{0<y,z<1} |b_N(y, z)| 2\|P_{F_{1N}} - P_{F_1}\| 2\|P_{F_{2N}} - P_{F_2}\| \\ &\qquad \leqq 4(N^{\frac{1}{2}}\|P_{F_N} - P_F\|)^2 \sup_{0<y,z<1} |b_N(y, z)|/N^{\frac{1}{2}} \to 0 \end{aligned}$$

according to (4) in [1]. Thus $\liminf_{N\to\infty} \delta_N(b_N, F, F_N) \geqq 0$. This completes the proof.

Under some restrictions on the choice of the scores b_{Nij} and b_{Ni}, i.e. $b_{N[Ny+1][Nz+1]}$, $b_{N[Nx+1]}$, and $b_{N[nx+1]}$ have to be positive Δ-monotone or monotone increasing, respectively, one can even show the unbiasedness of φ_{bN} for every fixed N. In the independence case, however, this is only known for the smaller class of regression dependent alternatives, whereas we need it here (at least asymptotically) for quadrant dependent alternatives. For such questions one may consult for example Lehmann [13], Yanagimoto and Okamoto [17], or Witting and Noelle [16], Theorems 3.13 and 3.14.

From Theorem 2.2 it is clear that the question on general bounds for the ARE of two such tests only has meaning if the generating functions b_1 and b_2 are positive Δ-monotone a.e. (or monotone increasing a.e. in case B or C).
Because of Theorem 2.1 we use

$$\liminf_{N\to\infty} (\delta_N(b_{1N}, F, F_N)/\delta_N(b_{2N}, F, F_N))^2$$

as the definition of ARE $(\{\varphi_{b_1N}\}: \{\varphi_{b_2N}\} \mid \{F_N\})$ iff

$$0 \leqq \liminf_{N\to\infty} \delta_N(b_{1N}, F, F_N), \qquad 0 < \liminf_{N\to\infty} \delta_N(b_{2N}, F, F_N),$$
$$\limsup_{N\to\infty} \delta_N(b_{1N}, F, F_N) < \infty .$$

For the well-known connections of this definition with the original definition of Pitman-efficiency see for example Noether [15] or Witting and Noelle [16], page 161. Even if all results are just statements on asymptotic normality and asymptotic power we take this definition of the ARE (which is not in general equivalent to the definition by means of sample sizes), in order to get comparability with the well-known results on lower bounds for the ARE by Hodges and Lehmann [9], [10], Chernoff and Savage [3], and others.

THEOREM 2.3. *Let b_1 and b_2 positive Δ-monotone* a.e. *generating functions in case* A (*or monotone increasing* a.e. *in case* B *or* C). *Then, for each constant $c > 0$, we have the equivalence of the following two statements*:

(I) $b_1 - cb_2$ *is positive Δ-monotone* a.e. (*monotone increasing* a.e.).

(II) ARE $(\{\varphi_{b_1N}\}: \{\varphi_{b_2N}\} \mid \{F_N\}) \leqq c^2$ *for all* $\{F_N\} \in K$ *which are contiguous to some* $F \in H$ *and for which*

$$\limsup_{N\to\infty} E(\varphi_{b_1N} \mid F_N) < 1 , \qquad \liminf_{N\to\infty} E(\varphi_{b_2N} \mid F_N) > \alpha .$$

PROOF. Given $c > 0$ we define $b = b_1 - cb_2$. Because of the similarity of the proofs we give the proof only in the case A.

If b is not positive Δ-monotone a.e., then we construct b_0 and K_{b_0} as in the proof of Theorem 2.2. Thus we get the existence of $F \in H$, $\{F_{Nd}^F\} \in K_{b_0}$, $\{F_{Nd}^F\}$ contiguous to F, such that

$$0 > \int bb_0\,dx = \int b_1 b_0\,dx - c \int b_2 b_0\,dx$$

and

$$\lim_{N\to\infty} \delta_N(b_{jN}, F, F_{Nd}^F) = d \int b_j b_0\,dx \geqq 0 , \qquad j = 1, 2;\ d > 0 ,$$

because of Theorem 2.2. Therefore we have $c \int b_2 b_0\,dx > \int b_1 b_0\,dx \geqq 0$ and thus

$$\text{ARE}\,(\{\varphi_{b_1N}\}: \{\varphi_{b_2N}\} \mid \{F_{Nd}^F\}) = (d \int b_1 b_0\,dx / d \int b_2 b_0\,dx)^2 < c^2$$

with

$$\limsup_{N\to\infty} E(\varphi_{b_1N} \mid F^F_{Nd}) < 1 , \qquad \liminf_{N\to\infty} E(\varphi_{b_2N} \mid F^F_{Nd}) > \alpha$$

because of Theorem 2.1.

For the other direction of the proof we assume in a first step that b equals a constant $a \in \mathbb{R}$ almost everywhere. Then we get $a = \int b_1\,dx - c\int b_2\,dx = 0$, $1 = \int b_1^2\,dx = c^2 \int b_2^2\,dx = c^2$. Thus $c = 1$ and $b_1 = b_2$ a.e. This implies (II) trivially.

To complete the proof we may now assume $0 < \int b^2\,dx$. This and (2.1.A) for b_1 and b_2 imply

$$\int_0^1 b(y, \cdot)\,dy = 0 \quad \text{a.e.}, \qquad \int_0^1 b(\cdot, z)\,dz = 0 \quad \text{a.e.}, \qquad 0 < \int b^2\,dx = \sigma^2 < \infty ,$$

and

$$\int_0^s \int_0^t b\,dx = -\int_0^s \int_t^1 b\,dx = -\int_s^1 \int_0^t b\,dx = \int_s^1 \int_t^1 b\,dx \qquad \forall s, t \in (0, 1) .$$

By positive Δ-monotonicity a.e. of b we get therefore

$$\begin{aligned}\int_0^s \int_0^t b\,dx &= (1-s)(1-t)\int_0^s \int_0^t b\,dx - (1-s)t\int_0^s \int_t^1 b\,dx \\ &\quad - s(1-t)\int_s^1 \int_0^t b\,dx + st\int_s^1 \int_t^1 b\,dx \\ &= \int_0^s \int_s^1 \int_0^t \int_t^1 \Delta_b(x, x', y, y')\,d\lambda_4(x, x', y, y') \geqq 0 \qquad \forall s, t \in (0, 1) ,\end{aligned}$$

Thus condition (2.1.A) is also true for b/σ, and (3.1.A) etc. imply the existence of approximations b_N, b_{1N}, b_{2N} of b, b_1, b_2, respectively, with (2.4.A) and

$$b_N = b_{1N} - cb_{2N} \qquad \forall N . \tag{2.9}$$

Now we can apply Theorem 2.1 and Theorem 2.2 to the generating function b/σ and the approximation b_N and get

$$\liminf_{N\to\infty} \sigma^{-1}N^{\frac{1}{2}} \int b_N(F(y, \infty), F(\infty, z))\,dF_N(y, z) \geqq 0 \tag{2.10}$$

for all $\{F_N\} \in K$ with $\{F_N\}$ contiguous to some $F \in H$.

On the other side we get from Theorem 2.1

$$\liminf_{N\to\infty} \delta_N(b_{2N}, F, F_N) > 0 , \qquad \limsup_{N\to\infty} \delta_N(b_{1N}, F, F_N) < \infty$$

if in addition it holds that

$$\liminf_{N\to\infty} E(\varphi_{b_2N} \mid F_N) > \alpha , \qquad \limsup_{N\to\infty} E(\varphi_{b_1N} \mid F_N) < 1 .$$

This, together with (2.9) and (2.10), implies $\liminf_{N\to\infty} \delta_N(b_{1N}, F, F_N) > 0$ and

$$\text{ARE}(\{\varphi_{b_1N}\} : \{\varphi_{b_2N}\} \mid \{F_N\}) = \liminf_{N\to\infty} (\delta_N(b_{1N}, F, F_N)/\delta_N(b_{2N}, F, F_N))^2 \geqq c^2 .$$

This completes the proof.

From these general results one gets immediately the statements (on the behavior of the ARE for some well-known tests in all three cases) which were given for the independence case at the end of [1]. Now it is clear, however, what the reason for such behavior is.

Another application is the construction of a rank test that has lower bounds larger than zero for the ARE with respect to every test out of a finite set of such given tests. (One can take for example as the new generating function b the standardization of a suitable convex linear combination of the given generating functions.)

Of even more interest may be the construction of simple tests which have a "rather good" overall behavior for classes of alternatives with some reasonable restrictions. Such and similar questions will possibly be treated in an additional paper.

3. Technicalities. In this section we will list some of the more technical results needed in the proofs of Section 2.

Let b be a Lebesgue-measurable function on $(0, 1) \times (0, 1)$ or $(0, 1)$ with the property (2.1.A), (2.1.B), or (2.1.C), respectively. Then we define for each $m \in \mathbb{N}$ a real-valued step function b_m by

$$(3.1.A) \qquad b_m(x) = \sum_{i=1}^m \sum_{j=1}^m (m^2 \int_{(i-1)/m}^{i/m} \int_{(j-1)/m}^{j/m} b\,dx) I_{((i-1)/m, i/m] \times ((j-1)/m, j/m]}(x) ,$$

$$(3.1.B) \qquad b_m(x) = \sum_{i=1}^m ((2m+1) \int_{(i-1)/(2m+1)}^{i/(2m+1)} b\,dx)(I_{((i-1)/(2m+1), i/(2m+1)]}(x) - I_{[1-i/(2m+1), 1-(i-1)/(2m+1))}(x)) ,$$

$$(3.1.C) \qquad b_m(x) = \sum_{i=1}^m (m \int_{(i-1)/m}^{i/m} b\,dx) I_{((i-1)/m, i/m]}(x) .$$

With these notations the following statements are true:

$$(3.2) \qquad \int b_m{}^2\,dx \leqq \int b^2\,dx ,$$

$$(3.3) \qquad b_m(x) \to b(x) \quad \text{a.e.},$$

$$(3.4) \qquad \int (b_m - b)^2\,dx \to 0 ,$$

$$(3.5) \qquad \int b_m\,dx = \int b\,dx = 0 ,$$

$$(3.6.A) \qquad \int_0^1 b_m(y, \cdot)\,dy = \int_0^1 b_m(\cdot, z)\,dz = 0 ,$$

$$(3.7) \qquad \sup_x |b_m(x)| \leqq k_m \int |b|\,dx ,$$

$k_m = m^2$, $k_m = 2m + 1$, $k_m = m$, respectively,

$$(3.8.A) \qquad \int_0^s \int_0^t b_m\,dx \geqq 0 \qquad \forall s, t \in (0, 1) ,$$

$$(3.8.B,C) \qquad \int_0^t b_m\,dx \leqq 0 \qquad \forall t \in (0, 1) ,$$

(3.9.A) b positive Δ-monotone a.e. $\Rightarrow b_m$ positive Δ-monotone,

(3.9.B,C) b monotone increasing a.e. $\Rightarrow b_m$ monotone increasing,

(3.10.A) b not positive Δ-monotone a.e. $\Rightarrow b_m$ not positive Δ-monotone, for all sufficiently large m,

(3.10.B,C) b not monotone increasing a.e. $\Rightarrow b_m$ not monotone increasing, for all sufficiently large m,

$$(3.11.B) \qquad b_m(x) = -b_m(1 - x) \qquad \forall x \in (0, 1) .$$

(3.12.A) With the notation $\Delta_{b_m}^{i0} = b_m((i+1)/m, 1) - b_m(i/m, 1)$,

$$\Delta_{b_m}^{0j} = b_m(1, (j+1)/m) - b_m(1, j/m) ,$$

and

$$\Delta_{b_m}^{ij} = b_m((i+1)/m, (j+1)/m) - b_m((i+1)/m, j/m) - b_m(i/m, (j+1)/m) + b_m(i/m, j/m)$$

we have

$$b_m(y, z) = b_m(1, 1) - \sum_{i=1}^{m-1} \Delta_{b_m}^{i0} I_{(0,i/m]}(y) - \sum_{j=1}^{m-1} \Delta_{b_m}^{0j} I_{(0,j/m]}(z) + \sum_{i=1}^{m-1} \sum_{j=1}^{m-1} \Delta_{b_m}^{ij} I_{(0,i/m]}(y) I_{(0,j/m]}(z) .$$

(3.12.B) $$b_m(x) = -\sum_{i=1}^{m} \left(b_m\left(\frac{i+1}{2m+2}\right) - b_m\left(\frac{i}{2m+2}\right)\right) \times (I_{(0,i/(2m+1)]}(x) - I_{[1-i/(2m+1),1)}(x)) ,$$

(3.12.C) $$b_m(x) = b_m(1) - \sum_{i=1}^{m-1} (b_m((i+1)/m) - b_m(i/m)) I_{(0,i/m]}(x) .$$

The proofs of these relations are either well-known or just computational. For (3.3) see for example Dunford and Schwartz [4], Theorem III. 12.8, page 217. (3.4) follows directly from (3.2) and (3.3) cf. for example Hájek and Šidák [8], Theorem V.1.3, page 154. For (3.9) and (3.10) see the definitions of positive Δ-monotone a.e. and of monotone increasing a.e. just before Theorem 2.2 and apply (3.3). The rest is computation.

Now we consider another possibility of approximating b in quadratic mean. Let $U, U_1, U_2, \cdots, V, V_1, V_2, \cdots$ be independent identically distributed random variables, each of them having a rectangular distribution over (0, 1). For each $n \in \mathbb{N}$ let $(U_n^{(1)}, \cdots, U_n^{(n)})$ and $(V_n^{(1)}, \cdots, V_n^{(n)})$ be the order statistics of $(U_1, \cdots, U_n)$ and $(V_1, \cdots, V_n)$, respectively. Furthermore let $(R_{n1}, \cdots, R_{nn})$ and $(S_{n1}, \cdots, S_{nn})$ be the rank statistics in $(U_1, \cdots, U_n)$ and $(V_1, \cdots, V_n)$, respectively. With these notations we define three sequences $\{b_{kn}\}$, $k = 0, 1, 2$, of functions b_{kn}: $(0, 1) \times (0, 1) \to \mathbb{R}$ by

(3.13) $$b_{0n}(x, y) = Eb(U_n^{([nx+1])}, V_n^{([ny+1])}) \qquad \forall x, y \in (0, 1) ,$$

(3.14) $$b_{1n}(x, y) = Eb(x, V_n^{([ny+1])}) \quad \text{a.e.},$$

(3.15) $$b_{2n}(x, y) = Eb(U_n^{([nx+1])}, y) \quad \text{a.e.},$$

where $[z]$ denotes as usual the largest integer less or equal to z. Then the following statements hold:

(3.16) $$E(b_{jn}(U, V) - b(U, V))^2 \to 0 , \qquad j = 1, 2 ,$$

(3.17) $$E(b_{0n}(U, V) - b(U, V))^2 \to 0 ,$$

(3.18) $$E(b_{0n}(R_{n1}/(n+1), S_{n1}/(n+1)) - b(U_1, V_1))^2 \to 0 ,$$

(3.19) Let $\{b_n\}$ be any sequence of functions $b_n : (0, 1) \times (0, 1) \to \mathbb{R}$, each constant over each square $((i-1)/n, i/n) \times ((j-1)/n, j/n)$ of $(0, 1)^{(2)}$, $n \in \mathbb{N}$, with $E(b_n(U, V) - b(U, V))^2 \to 0$, then $E(b_n(R_{n1}/(n+1), S_{n1}/(n+1)) - b(U_1, V_1))^2 \to 0$.

(3.20) $Eb(U, \cdot) = 0$ a.e. implies $\sum_{i=1}^{n} b_{0n}(i/(n+1), j/(n+1)) = 0 \quad \forall j$.

(3.21) $Eb(\cdot, V) = 0$ a.e. implies $\sum_{j=1}^{n} b_{0n}(i/(n+1), j/(n+1)) = 0 \quad \forall i$.

The proof is based on Hájek and Šidák [8], Theorem V.1.4.b. (It could also be done by some generalization of Faddeev's theorem (cf. Hájek and Šidák [8], Lemma V.1.4.b) to more than one dimension, even in more general cases.) Because of the independence of U and V we get

$$E(b_{1n}(U, V) - b(U, V))^2 = \int_0^1 (E(b_{1n}(x, V) - b(x, V))^2)\, dx .$$

On the other side we have $Eb^2(x, V) < \infty$ for almost all $x \in (0, 1)$ and therefore by Hájek and Šidák [8], Theorem V.1.4.b,

$$0 \leqq B_n(x) = E(b_{1n}(x, V) - b(x, V)) \to 0 \qquad \text{for almost all } x \in (0, 1) .$$

Finally we have for all n

$$\begin{aligned} B_n(x) &\leqq 2Eb_{1n}^2(x, V) + 2Eb^2(x, V) = (2/n) \textstyle\sum_{j=1}^n (Eb(x, V_n^{(j)}))^2 + 2Eb^2(x, V) \\ &\leqq (2/n)E \textstyle\sum_j b^2(x, V_j) + 2Eb^2(x, V) = 4Eb^2(x, V) \end{aligned}$$

with $\int |4Eb^2(x, V)|\, dx = 4Eb^2(U, V) \leqq \infty$.

Thus (3.16) for $j = 1$ by the dominated convergence theorem. The result for $j = 2$ holds by symmetry. (3.17) follows from (3.16) because of the following inequalities:

$$\begin{aligned} E(b_{0n}(U, V) - b(U, V))^2 \leqq\ & 2E(b_{0n}(U, V) - b_{1n}(U, V))^2 \\ & + 2E(b_{1n}(U, V) - b(U, V))^2 \end{aligned}$$

and

$$\begin{aligned} E(b_{0n}(U, V) - b_{1n}(U, V))^2 &= \int (E(b_{0n}(x, V) - b_{1n}(x, V))^2)\, dx \\ &= \int ((1/n) \textstyle\sum_{j=1}^n (Eb(U_n^{([nx+1])}, V_n^{(j)}) - Eb(x, V_n^{(j)}))^2)\, dx \\ &\leqq \int ((1/n)E \textstyle\sum_j (b(U_n^{([nx+1])}, V_j) - b(x, V_j))^2)\, dx \\ &= \int (E(b(U_n^{([nx+1])}, V) - b(x, V))^2)\, dx \\ &= \int (Eb^2(U_n^{([nx+1)]}, V) - 2Eb(U_n^{([nx+1])}, V)b(x, V) + Eb^2(x, V))\, dx \\ &= \iint (Eb^2(U_n^{([nx+1])}, y))\, dx\, dy - 2 \iint b_{2n}(x, y)b(x, y)\, dy\, dx + Eb^2(U, V) \\ &= \int ((1/n) \textstyle\sum_{i=1}^n Eb^2(U_n^{(i)}, y))\, dy - 2Eb_{2n}(U, V)b(U, V) + Eb^2(U, V) \\ &= Eb^2(U, V) - 2Eb_{2n}(U, V)b(U, V) + Eb^2(U, V) \\ &= 2Eb(U, V)(b(U, V) - b_{2n}(U, V)) \\ &\leqq 2(Eb^2(U, V))^{\frac{1}{2}}(E(b_{2n}(U, V) - b(U, V))^2)^{\frac{1}{2}} . \end{aligned}$$

Now (3.17) and (3.19) imply (3.18) immediately. And (3.19) is true because of (3.17) and the following identity:

$$\begin{aligned} & E(b_n(R_{n1}/(n + 1), S_{n1}/(n + 1)) - b(U_1, V_1))^2 \\ &\quad = Eb_n^2(R_{n1}/(n + 1), S_{n1}/(n + 1)) + Eb^2(U_1, V_1) \\ &\qquad - 2Eb_n(R_{n1}/(n + 1), S_{n1}/(n + 1))b(U_1, V_1) \\ &\quad = \frac{1}{n^2} \textstyle\sum_i \sum_j b_n^2\left(\dfrac{i}{n + 1}, \dfrac{j}{n + 1}\right) + Eb^2(U, V) \\ &\qquad - 2Eb_n\left(\dfrac{R_{n1}}{n + 1}, \dfrac{S_{n1}}{n + 1}\right) b(U_1, V_1) \textstyle\sum_{i,j} I_{\{i\}}(R_{n1})I_{\{j\}}(S_{n1}) \end{aligned}$$

$$= Eb_n^2(U, V) + Eb^2(U, V)$$
$$- 2 \sum_{i,j} Eb_n\left(\frac{R_{n1}}{n+1}, \frac{S_{n1}}{n+1}\right) b(U_1, V_1) I_{\{i\}}(R_{n1}) I_{\{j\}}(S_{n1})$$
$$= Eb_n^2(U, V) + Eb^2(U, V)$$
$$- 2 \sum_{i,j} b_n\left(\frac{i}{n+1}, \frac{j}{n+1}\right) Eb(U_n^{(i)}, V_n^{(j)}) EI_{\{i\}}(R_{n1}) EI_{\{j\}}(S_{n1})$$
$$= Eb_n^2(U, V) + Eb^2(U, V)$$
$$- (2/n^2) \sum_i \sum_j b_n\left(\frac{i}{n+1}, \frac{j}{n+1}\right) b_{0n}\left(\frac{i}{n+1}, \frac{j}{n+1}\right)$$
$$= Eb_n^2(U, V) + Eb^2(U, V) - 2Eb_n(U, V) b_{0n}(U, V) .$$

Finally (3.20) (and by symmetry also (3.21)) holds, since the distribution of $V_n^{(j)}$ possesses a Lebesgue density (denoted by f_{nj}):

$$\sum_{i=1}^n b_{0n}(i/(n+1), j/(n+1))$$
$$= \sum_i Eb(U_n^{(i)}, V_n^{(j)}) = E \sum_i b(U_i, V_n^{(j)})$$
$$= nEb(U, V_n^{(j)}) = n \int (Eb(U, y)) f_{nj}(y)\, dy = 0 .$$

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