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EXISTENCE OF LIMITS IN REGENERATIVE PROCESSES[1]

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Feller (1966) claims that a regenerative stochastic process with a non-lattice interarrival-time distribution has a limiting distribution. This is true if (i) the class of interarrival-time distributions is restricted, or (ii) regularity conditions are imposed on the sample paths of the process.

0. Introduction. Feller ((1966) page 365) defines a regenerative process $\{V_0(t), t \geqq 0\}$ as a stochastic process such that "with probability one there exists an epoch S_1 such that the continuation of the process beyond S_1 is a probabilistic replica of the whole process commencing at epoch 0," independent of $\{V_0(t), 0 \leqq t < S_1\}$. If S_1 is a non-lattice random variable with finite mean and if $V_0(t)$ has a countable state space, Feller asserts that $V_0(t) \rightarrow_{\mathscr{L}} V_*$ as $t \rightarrow \infty$. V_* is a proper random variable and $\rightarrow_{\mathscr{L}}$ denotes convergence in distribution. In his proof Feller assumes that a measurable function which is bounded by an integrable monotone function is directly Riemann integrable. In general this is not true and additional assumptions must be made for the theorem to hold. Witness the counterexample:

$$S_1 = 1/n\,, \qquad \text{with probability} \quad 2^{-n} \quad \text{for} \quad n = 1, 2, \cdots;$$
$$V_0(t) = 1_Q(t)\,, \qquad \text{where} \quad Q \quad \text{is the set of rationals.}$$

There are two forms which the forementioned additional assumptions may take: first, restrictions on the distribution of S_1; second, restrictions on the sample paths of $V_0(t)$.

Let $\mathscr{S}$ be the family of distributions F such that for some $n \geqq 1$, F^{n*} has a component which is absolutely continuous with respect to Lebesgue measure. This class was introduced by W. L. Smith (1958), and it follows from a theorem of Smith (1958), page 259, that if V_0 is regenerative with interarrival distribution $F \in \mathscr{S}$ then $V_0(t)$ necessarily converges in distribution as $t \rightarrow \infty$. Thus Feller's conclusion holds for $F \in \mathscr{S}$ but not necessarily for F non-lattice. $\mathscr{S}$ turns out to be the largest class for which Feller's conclusion holds in the sense that given F non-lattice, $F \notin \mathscr{S}$, one can construct a regenerative process V_0 with interarrival distribution F such that $V_0(t)$ does not converge in distribution as $t \rightarrow \infty$. Thus in the absence of any condition on sample path regularity one cannot conclude convergence in distribution of $V_0(t)$ for $F \notin \mathscr{S}$.

Let $\mathscr{D}$ be the set of real-valued functions on $[0, \infty)$ which are right continuous

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and for which left-hand limits exist. We demonstrate that if $\{V_0(t), t \geqq 0\}$ is a regenerative process having a measurable modification with sample paths in $\mathscr{D}$, and if F is non-lattice, then $V_0(t)$ converges in distribution as $t \to \infty$. Thus Feller's conclusion holds under this restriction on sample path behavior.

Actually the finite-dimensional joint distributions of $V_0(t)$ converge to those of a stationary regenerative process $\{V_*(t), -\infty < t < \infty\}$ when either of the above restrictions is imposed. Restriction to a countable state space is not necessary. Analogous results hold when S_1 is a lattice random variable.

1. Definitions and basic results. W. L. Smith introduced regenerative processes (1955). We use the concept of a "tour" introduced by him (1958) to define a real-valued regenerative process.

A random tour is a pair $(x(\cdot, \omega), X(\omega))$ defined on some probability space $(\Omega, \mathscr{F}, P_\Omega)$. X is a nonnegative random variable and for each $\omega \in \Omega$, $x(\cdot, \omega)$ is a real-valued function defined on $[0, X(\omega))$. We shall require that for $0 \leqq t_1 < t_2 < \cdots < t_n$ and $a_1, \cdots, a_n \in R^1$ the set

$$\{\omega \in \Omega : x(t_i, \omega) \leqq a_i, \quad i = 1, \cdots, n; \quad X(\omega) > t_n\}$$

be measurable and shall assume these sets generate $\mathscr{F}$. We shall assume that $x(t, \omega)$ is jointly Borel measurable relative to $B[0, \infty) \otimes \mathscr{F}$ (restricted, of course, to the subset of $[0, \infty) \times \Omega$ on which $x(t, \omega)$ is defined).

Now to define the regenerative process $V_0(t)$, we shall take an infinite sequence of independent, identically distributed tours:

$$(x_0(\cdot), X_0), (x_1(\cdot), X_1), (x_2(\cdot), X_2), \cdots.$$

Define:

$$\begin{aligned} S_n &= 0, & n &= 0 \\ &= \textstyle\sum_{i=0}^{n-1} X_i, & n &\geqq 1 \\ N_0(t) &= \max(n : S_n \leqq t) \\ Z_0(t) &= t - S_{N_0(t)} \\ V_0(t) &= x_{N_0(t)}(Z_0(t)). \end{aligned}$$

Let $(\tilde{\Omega}, G, P) = (\prod_0^\infty \Omega, \otimes_0^\infty \mathscr{F}, \prod_0^\infty P_\Omega)$ be the probability space on which the infinite sequence is defined. S_n, $N_0(t)$, and $Z_0(t)$ are all measurable processes and, because we required the tours to be jointly measurable, $V_0(t)$ is a measurable process. The points S_n are called "regeneration points" and $\{S_n, n \geqq 0\}$ is the embedded renewal process.

Now we define a stationary regenerative process, $\{V_*(t), -\infty < t < \infty\}$. It will be related to $\{V_0(t), t \geqq 0\}$ by virtue of arising from the same random tour $(x(\cdot), X)$. Let $F_X(x) = P_\Omega[X \leqq x]$ and $\mu = EX$. Throughout this paper we shall assume $\mu < \infty$. Define the random variables $Z_*(0)$ and X_0:

$$P(Z_*(0) \leqq a) = \frac{1}{\mu} \int_0^a (1 - F_X(s))\, ds$$

$$P[X_0 \geqq x \mid Z_*(0) = a] = 1 \qquad x < a$$
$$= \frac{1 - F_X(x)}{1 - F_X(a)} \qquad x \geqq a .$$

Note that the above definition implies $P[X_0 > Z_*(0)] = 1$. This joint distribution can be extended to $((x_0(\cdot), X_0), Z_*(0))$. If A is any set of the form $\{(x(\cdot), X): x(t_i) \leqq a_i, X > t_n\}$ where $0 \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_n$ and $a_1, a_2, \cdots, a_n$ are real then

$$P[(x_0(\cdot), X_0) \in A, Z_*(0) \leqq a]$$
$$= \frac{1}{\mu} \int_0^a \int_b^\infty P_\Omega[(x(\cdot), X) \in A \mid X = s] \frac{dF_X(s)}{1 - F_X(b)} (1 - F_X(b))\, db$$
$$= \frac{1}{\mu} \int_0^a P_\Omega[(x(\cdot), X) \in A \mid X > b](1 - F_X(b))\, db .$$

Let $(x_i(\cdot), X_i)$, $i = \pm 1, \pm 2, \cdots$ be independent identically distributed tours sampled from $(\Omega, \mathscr{F}, P_\Omega)$. Define for $-\infty < t < \infty$

$$S_{*,n} = -Z_*(0) + \textstyle\sum_{i=0}^{n-1} X_i , \qquad n > 0$$
$$= -Z_*(0) , \qquad n = 0$$
$$= -Z_*(0) - \textstyle\sum_{i=-1}^{n} X_1 , \qquad n < 0$$
$$N_*(t) = \max(n: S_{*,n} \leqq t)$$
$$Z_*(t) = t - S_{*,N_*(t)}$$
$$V_*(t) = x_{N_*(t)}(Z_*(t)) .$$

As before these are all measurable processes.

THEOREM 1.1. *$\{V_*(t), -\infty < t < \infty\}$ is strictly stationary.*

PROOF. See [4].

$Z_0(t)$ and $Z_*(t)$ are called the "backward recurrence times" for their respective processes. Smith (1958) proved a theorem which is quite useful when applied to backwards recurrence times:

THEOREM 1.2. (*Smith*). *If F_X is non-lattice and $\mu < \infty$ then $Z_0(t) \to_{\mathscr{L}} Z_*(0)$ as $t \to \infty$; furthermore, if $F_X \in \mathscr{S}$ then $P[Z_0(t) \in A] \to P[Z_*(0) \in A]$ as $t \to \infty$, where A is* any *Borel subset of* $[0, \infty)$.

Doob (1948) also proved the second half of this theorem.

2. Restrictions on the interarrival distributions.

THEOREM 2.1. *If $F_X \in \mathscr{S}$ and $\mu < \infty$, then $P[V_0(t + t_i) \in A_i; i = 1, \cdots, n] \to P[V_*(t_i) \in A_i; i = 1, \cdots, n]$ as $t \to \infty$, where A_i, $i = 1, \cdots, n$ are Borel subsets of R^1.*

PROOF. Assume $t_i \geqq 0$, $i = 1, 2, \cdots, n$. Let $g_t(z) = P[V_0(t + t_i) \in A_i; i = 1, \cdots, n \mid Z_0(t) = z]$ and $g_*(z) = P[V_*(t_i) \in A_i; i = 1, \cdots, n \mid Z_*(0) = z]$. For $t_1, \cdots, t_n, A_1, \cdots, A_n$, and z there corresponds a subset, $\tilde{A}_z$, of the space of infinite sequences of random tours such that

$$g_t(z) = P[((x_{N(t)}(\cdot), X_{N(t)}), (x_{N(t)+1}(\cdot), X_{N(t)+1}), \cdots) \in \tilde{A}_z \mid Z_0(t) = z]$$
$$g_*(z) = P[((x_0(\cdot), X_0), (x_1(\cdot), X_1), \cdots) \in \tilde{A}_z \mid Z_*(0) = z] .$$

But

$$P[X_{N(t)} \leqq b \mid Z_0(t) = z] = \frac{F_X(b) - F_X(z)}{1 - F_X(z)} \quad \text{a.s.} \qquad [P_{Z_0(t)}] ;$$

from before we know that

$$\frac{F_X(b) - F_X(z)}{1 - F_X(z)} = P[X_0 \leqq b \mid Z_*(0) = z] ,$$

so the conditional distributions of the "present" tours are the same. We know from the construction of V_0 and V_* that the "future" tours are independent of $Z_0(t)$ and $Z_*(0)$, respectively, and identically distributed. Thus $g_t(z) = g_*(z)$ a.s. $[P_{Z_0(t)}]$. Now consider:

$$\begin{aligned} P[V_0(t + t_i) \in A_i; i = 1, \cdots, n] &= E[P[V_0(t + t_i) \in A_i; i = 1, \cdots, n \mid Z_0(t)]] \\ &= Eg_t(Z_0(t)) \\ P[V_*(t_i) \in A_i; i = 1, \cdots, n] &= E[P[V_*(t_i) \in A_i; i = 1, \cdots, n \mid Z_*(0)]] \\ &= Eg_*(Z_*(0)) . \end{aligned}$$

Since $g_t(z) = g_*(z)$ a.s. $[P_{Z_0(t)}]$, we have

$$P[V_0(t + t_i) \in A_i; i = 1, \cdots, n] = Eg_*(Z_0(t)) .$$

But the fact that $0 \leqq g_* \leqq 1$ and the second half of Theorem 1.2 imply

$$Eg_*(Z_0(t)) \to Eg_*(Z_*(0)) \qquad \text{as} \quad t \to \infty .$$

The restriction $t_i \geqq 0$ is trivial to remove. □

In a weak sense $F_X \in \mathscr{S}$ is also a necessary condition for convergence: if $F_X \notin \mathscr{S}$ there exists a regenerative process, $\{V_0(t), t \geqq 0\}$, whose embedded renewal process has distribution F_X but $V_0(t)$ does not converge in distribution as $t \to \infty$. We shall construct such a process:

Pick any unbounded infinite sequence of points: $0 < t_1 < t_2 < \cdots$. Because $F_X \notin \mathscr{S}$, $Z_0(t_i)$, $i = 1, 2, \cdots$, are all singular with respect to Lebesgue measure. Thus there exist sets A_i, $i = 1, 2, \cdots$ such that $P[Z_0(t_i) \in A_i] = 1$ and $|A_i| = 0$. Let $A = \bigcup_{i=1}^{\infty} A_i$, thus $|A| = 0$. Define $V_0(t) = 1_A(Z_0(t))$ for $t \geqq 0$. It follows that $V_0(t_i) = 1$, $i = 1, 2, \cdots$. This implies that any limiting distribution of $V_0(t)$ as $t \to \infty$ must give all its mass to 1. However, it is known from renewal theory that, with probability one, there are only a finite number of renewal epochs in any finite interval. Therefore the Z_0-process assumes any real value at most a countable number of times, with probability one. Consequently

$$\int_0^\infty 1_A(Z_0(t, \omega))\, dt \leqq |A| + |A| + \cdots = 0$$

for almost all ω. Therefore

$$\begin{aligned} \int_0^\infty P[V_0(t) = 1]\, dt &= \int_0^\infty \int_\Omega 1_A(Z_0(t, \omega))\, dP(\omega)\, dt \\ &= \int_\Omega \int_0^\infty 1_A(Z_0(t, \omega))\, dt\, dP(\omega) = \int_\Omega 0 \cdot dP(\omega) = 0 . \end{aligned}$$

Thus $P[V_0(t) = 1] = 0$ for almost all (Lebesgue) t. Therefore it is impossible for $\lim_{t\to\infty} P[V_0(t) = 1]$ to exist.

A slightly less pathological example is $V_0(t) = Z_0(t) \cdot 1_A(Z_0(t))$, $t \geqq 0$.

3. Regularity conditions on sample paths.

THEOREM 3.1. *If F_X non-lattice and $\mu < \infty$ and $\{V_0(t), t \geqq 0\}$ has a measurable modification with paths in $\mathscr{D}$ then $V_0(t) \to_{\mathscr{L}} V_*(0)$ as $t \to \infty$.*

Before proving this theorem we present a definition and two preliminary lemmas.

DEFINITION. A δ-regular step function is a real-valued right continuous step function with only a finite number of steps occurring in each finite interval and the magnitude of each step being greater than or equal to δ.

LEMMA 3.1. *$f: [0, \infty) \to R^1$, $f \in \mathscr{D}$ then there exists a δ-regular step function, g, such that $|f(x) - g(x)| < \delta$ for all $x \geqq 0$.*

PROOF. Let $g(0) = f(0)$. The set $\{x: f(x) \geqq g(0) + \delta\}$ is closed on the left because of the right continuity of f. Let $a = \min\{x: f(x) \geqq g(0) + \delta\}$; then $a > 0$, again by right continuity of f. Similarly let $b = \min\{x: f(x) \leqq g(0) - \delta\}$ and let $x_1 = \min(a, b)$. Define $g(x) = g(0)$ for $0 \leqq x < x_1$ and $g(x_1) = f(x_1)$. Continue to define g inductively in the same way, getting a sequence x_1, x_2, $x_3, \cdots$. If $\{x_i\}$ is a bounded sequence, then it has a limit; but by definition $|f(x_i) - f(x_{i+1})| \geqq \delta$, contradicting the existence of the left-hand limits of f. Therefore there are only finitely many x_i's in each finite interval and g is a δ-regular step function.

LEMMA 3.2. *$(x(\cdot), X)$ is a random tour such that $x(\cdot, \omega)$ is a δ-regular step function for all $\omega \in \Omega$, for all a let $f_a(z) = P[x(z) \leqq a \mid X > z]$ if $P[X > z] > 0$; 0 otherwise, then $f_a(z) \in \mathscr{D}$.*

PROOF. Fix z. Since $x(\cdot, \omega)$ is a δ-regular step function, $P[\text{step in } [z, z+\varepsilon]] \to 0$ as $\varepsilon \to 0+$. Because $x(\cdot, \omega)$ is right continuous, we shall adopt the convention of saying "no jump occurs in $[z, z + \varepsilon]$" if no jump occurs in $(z, z + \varepsilon]$.

$$f_a(z) = \frac{P[x(z) \leqq a \quad \text{and} \quad X > z]}{P\{X > z\}} \qquad \text{if} \quad P\{X > z\} > 0$$
$$= 0 \qquad \text{otherwise}\,.$$

$$P[\{x(z) \leqq a\} \triangle \{x(z + \varepsilon) \leqq a\}] \leqq P[\text{step in} \quad [z, z + \varepsilon]] \to 0\,, \qquad \text{as} \quad \varepsilon \to 0\,.$$
$$P[\{z + \varepsilon \geqq X > z\}] \to 0\,, \qquad \text{as} \quad \varepsilon \to 0\,.$$

$$\{x(z) \leqq a; X > z\} \triangle \{x(z + \varepsilon) \leqq a; X > z + \varepsilon\}$$
$$\subset [\{x(z) \leqq a\} \triangle \{x(z + \varepsilon) \leqq a\}] \cup [\{X > z\} \triangle \{X > z + \varepsilon\}]\,.$$

($A \triangle B$ denotes the symmetric difference of the sets A and B.)

Thus

$$P[\{x(z) \leqq a; X > z\} \triangle \{x(z + \varepsilon) \leqq a; X > z + \varepsilon\}] \to 0\,, \qquad \text{as} \quad \varepsilon \to 0\,.$$

This implies that f_a is right continuous at z. Again fix z. P[step in $[z-\varepsilon, z)] \to 0$, as $\varepsilon \to 0$. Let $\{z_i\}$ be a monotone sequence converging to z from the left. Consequently,

$$P[\{x(z_m) \leqq a\} \triangle \{x(z_n) \leqq a\}] \to 0 \qquad \text{as} \quad m, n \to \infty$$
$$P[\{X > z_m\} \triangle \{X > z_n\}] \to 0 \qquad \text{as} \quad m, n \to \infty .$$

As before, this implies

$$P[\{x(z_m) \leqq a;\, X > z_m\} \triangle \{x(z_n) \leqq a;\, X > z_n\}] \to 0 \qquad \text{as} \quad m, n \to \infty .$$

Thus

$$|P[x(z_m) \leqq a;\, X > z_m] - P[x(z_n) \leqq a;\, X > z_n]| \to 0 \qquad \text{as} \quad m, n \to \infty .$$

Consequently $\lim_{z_n \uparrow z} P[x(z_n) \leqq a;\, X > z_n]$ exists and $f_a(z)$ has a left-hand limit at z.

Proof of Theorem 3.1. Suppose $\{V_0(t),\, t \geqq 0\}$ is the measurable modification with paths in $\mathscr{D}$. The tours of this modification all have paths in $\mathscr{D}$ and are jointly measurable. By Lemma 3.1, given $\delta > 0$, the path $x(\cdot, \omega)$ in each realization of the random tour can be approximated (within δ) by a δ-regular step function $\tilde{x}(\cdot, \omega)$. If we use the same construction as in the proof of Lemma 3.1, the joint-measurability of $x(\cdot, \cdot)$ will imply joint measurability of $\tilde{x}(\cdot, \cdot)$. Denote the process generated by these tours as $\{\tilde{V}_0(t),\, t \geqq 0\}$. For all $\omega \in \Omega$ and $t \geqq 0$, $|\tilde{V}_0(\omega, t) - V_0(\omega, t)| \leqq \delta$. $\{\tilde{V}_0(t),\, 0 \leqq t\}$ is obviously a regenerative process with the same regeneration epochs as $\{V_0(t),\, t \geqq 0\}$. $P[\tilde{V}_0(t) \leqq a] = E[P[\tilde{V}_0(t) \leqq a \,|\, Z_0(t)]]$. Let $f_t(z) = P[\tilde{V}_0(t) \leqq a \,|\, Z_0(t) = z]$ for z such that $P[X > z] > 0$, and 0 otherwise. Then, by an argument similar to that in the proof of Theorem 2.1, $f_t(z) = P[\tilde{V}_*(0) \leqq a \,|\, Z_*(0) = z] = P[\tilde{x}(z) \leqq a \,|\, X > z] = f_*(z)$ a.s. $[P_{Z_0(t)}]$. By Lemma 3.2, this version of $f_* \in \mathscr{D}$. But any $f \in \mathscr{D}$ is continuous a.e. with respect to Lebesgue measure; and the distribution function of $Z_*(0)$ is absolutely continuous with respect to Lebesgue measure. Therefore, by the Mann-Wald theorem, Theorem 1.2, and the boundedness of f_*

$$Ef_*(Z_0(t)) \to Ef_*(Z_*(0)) \qquad \text{as} \quad t \to \infty ,$$

i.e.

$$P[\tilde{V}_0(t) \leqq a] \to P[\tilde{V}_*(0) \leqq a] \qquad \text{as} \quad t \to \infty .$$

But

$$P[\tilde{V}_0(t) \leqq a - \delta] \leqq P[V_0(t) \leqq a] \leqq P[\tilde{V}_0(t) \leqq a + \delta] , \qquad t \geqq 0 \quad \text{and}$$
$$P[\tilde{V}_*(0) \leqq a - \delta] \leqq P[V_*(0) \leqq a] \leqq P[\tilde{V}_*(0) \leqq a + \delta] .$$

Since δ is arbitrary, if a is a point of continuity of $V_*(0)$, $P[V_0(t) \leqq a] \to P[V_*(0) \leqq a]$ as $t \to \infty$.

Corollary 3.1. *F_X non-lattice, $\mu < \infty$ and $\{V_0(t);\, t \geqq 0\}$ has a measurable modification with paths in $\mathscr{D}$, then $P[V_0(t + t_i) \leqq a_i;\, i = 1, \cdots, n] \to P[V_*(t_i) \leqq a_i;\, i = 1, \cdots, n]$ as $t \to \infty$ for a_i, $i = 1, \cdots, n$ points of continuity of $V_*(0)$.*

PROOF. Combines ideas of the proofs of Theorems 2.1 and 3.1.

REMARK 1. Theorem 3.1 can also be proved using the key renewal theorem directly. This is Feller's approach. In this case the restriction on the sample paths ($V_0(\omega, \cdot) \in \mathscr{D}$) will imply direct Riemann integrability of the necessary function, which allows application of the key renewal theorem.

REMARK 2. The restriction $V_0(\omega, \cdot) \in \mathscr{D}$ is actually stronger than necessary. Theorem 3.1 can be proved using only the existence of right- and left-hand limits, not necessarily right continuity. However, restricting to $\mathscr{D}$ is a standard procedure and further generality is of doubtful gain. The class $\mathscr{D}$ is extensively studied in the literature (see Billingsley (1968)).

REMARK 3. The existence of a limit for a regenerative process in many cases follows directly from a result of Smith ((1958) page 259, Condition B): whenever $\mu = EX < \infty$ and $\psi_A(t) = P[x(t) \in A \text{ and } X > t]$ is of bounded variation in every finite t-interval, $\lim_{t\to\infty} P[V_0(t) \in A]$ exists and equals $1/\mu \int_0^\infty \psi_A(t)\, dt$. Smith ((1955) page 17) gave a sufficient condition for ψ_A to be of bounded variation. Takács ((1963) page 94) observed that Smith's condition is usually automatically satisfied in queueing applications. In particular, if we restrict our attention to regenerative processes with finite state space (Feller's case), a limiting distribution will exist whenever the number of state changes in a finite interval has finite expected value.

4. The lattice case. If F_X is a lattice distribution then in general $V_0(t)$ will not have a limiting distribution, but statements can still be made about asymptotic behavior. The results are analogous to those derived for non-lattice interarrival times.

Define $Z_*(0)$ by $P(Z_*(0) = b\lambda) = \lambda/\mu(1 - F_X(b\lambda))$ for $b = 0, 1, 2, \cdots$ where λ = span of F_X. Define the process $\{V_*(t), -\infty < t < \infty\}$ as follows: The "zeroth" tour's lifetime has conditional distribution $P[X_{*,0} = j\lambda \mid Z_*(0) = k\lambda] = P[X_{*,0} = j\lambda \mid X_{*,0} > k\lambda] = P[X = j\lambda]/P[X > k\lambda]$ for $j > k$, 0 otherwise. The remainder of the construction of $\{V_*(t), -\infty < t < \infty\}$ is the same as the non-lattice case in Section 1.

THEOREM 4.1. *F_X lattice with span λ, then*

$$P[V_*(t_i) \leqq a_i;\, i = 1, \cdots, n] = P[V_*(h + t_i) \leqq a_i;\, i = 1, \cdots, n]$$

for h = multiple of λ.

PROOF. Similar to Theorem 1.1.

THEOREM 4.2. *F_X lattice with span λ, $\mu < \infty$, and $0 \leqq s < \lambda$, then $Z_0(n\lambda + s) \to_{\mathscr{L}} Z_*(s)$ as $n \to \infty$.*

PROOF. Follows from discrete key renewal theorem. See Feller ((1966) page 348).

THEOREM 4.3. *F_X lattice with span λ, $\mu < \infty$, then*

$$P[V_0(n\lambda + t_i) \leqq a_i;\ i = 1, \cdots, m] \to P[V_*(t_i) \leqq a_i;\ i = 1, \cdots, m]$$

as $n \to \infty$.

PROOF. Same idea as proof of Theorem 2.1.

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REFERENCES

[1] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York.

[2] DOOB, J. L. (1948). Renewal theory from the point of view of the theory of probability. *Trans. Amer. Math. Soc.* **63** 422–438.

[3] FELLER, WILLIAM (1966). *An Introduction to Probability Theory and its Applications* **2**. Wiley, New York.

[4] MILLER, D. R. (1971). Limit theorems for functionals of regenerative processes. Cornell Univ. Technical Report.

[5] SMITH, W. L. (1955). Regenerative stochastic processes. *Proc. Roy. Soc. Ser. A* **232** 6–31.

[6] SMITH, W. L. (1958). Renewal theory and its ramifications. *J. Roy. Statist. Soc. Ser. B* **20** 243–302.

[7] TAKÁCS, L. (1963). The limiting distribution of the virtual waiting time and the queue size for a singleserver queue with recurrent input and general service times. *Sankhyā Ser. A* **25** 91–100.