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A NOTE ON THE ZERO-ONE LAW[1]

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Let $\mathscr{M} = \{\mu_n : n \geqq 1\}$ be a sequence of probability measures defined on the measurable space $(\mathscr{R}_n, \mathscr{B}_n)$ and suppose that the measures $\{\mu_n : n \geqq 1\}$ satisfy the following condition (B): $\forall \varepsilon > 0, k \geqq 1$ and $m \geqq 1$, there exists an $n \geqq m$ such that $\|\mu_k - \mu_n\| < \varepsilon$. We show that if $A \in \times_1^\infty \mathscr{B}_n$ and if A is permutation invariant then $\mu(A) = 0$ or 1. The zero-one laws of Hewitt and Savage [*Trans. Amer. Math. Soc.* **80** (1955) 470–501] and Horn and Schach [*Ann. Math. Statist.* **41** (1970) 2130–2131] follow as special cases of our result.

1. Let $\mathscr{M} = \{\mu_n : n \geqq 1\}$ be a sequence of probability measures defined on the measurable space $(\mathscr{R}_n, \mathscr{B}_n)$. Consider the product space $\{\times_1^\infty \mathscr{R}_n, \times_1^\infty \mathscr{B}_n, \mu = \times^\infty \mu_n\}$ and suppose that the measures $\{\mu_n : n \geqq 1\}$ satisfy the following condition.

(B) For each $\varepsilon > 0, k \geqq 1$, and $m \geqq 1$, there exists an $n \geqq m$ such that $||\mu_k - \mu_n|| < \varepsilon$.

The main object of this note is to establish the following zero-one law.

(1.1) Theorem. *Consider the probability space* $\{\times_1^\infty \mathscr{R}_n, \times_1^\infty \mathscr{B}_n, \mu = \times_1^\infty \mu_n\}$ *and suppose that the probability measures* $\{\mu_n : n \geqq 1\}$ *satisfy condition* (B). *Let* $A \in \times_1^\infty \mathscr{B}_n$ *and suppose that* A *is invariant under all permutations of finitely many coordinates. Then* $\mu(A) = 0$ *or* 1.

This theorem is an extension of the zero-one laws in Hewitt-Savage [2] and Horn and Schach [3]. The substitution $\mu_1 = \mu_2 = \cdots$ in the above theorem yields the Hewitt-Savage zero-one law while the assumption that $\forall k \geqq 1$ and $m \geqq 1$ there is an $n \geqq m$ such that $\mu_k = \mu_m$ yields the zero-one law due to Horn-Schach [3].
To prove the theorem we need the following preliminary results.

(1.2) Lemma. *Let* $(\Omega, \mathscr{A}, \mu)$ *be a probability space and let* $\{\mathscr{A}_n : n \geqq 1\}$ *be a decreasing sequence of sub-*σ*-algebras of* $\mathscr{A}$. *Let* $A \in \mathscr{A}$. *Suppose that* $\forall \varepsilon > 0$ *and* $n \geqq 1$ *there exists a* $B_n \in \mathscr{A}_n$ *such that* $\mu(A \bigtriangleup B_n) < \varepsilon$. *Then there is a set* $B \in \mathscr{A}_\infty = \cap \mathscr{A}_n$ *such that* $\mu(A \bigtriangleup B) = 0$.

Proof. For each $n \geqq 1$ choose a $B_n \in \mathscr{A}_n$ such that $\mu(A \bigtriangleup B_n) < 1/2^n$. Now let $B = \lim\sup B_n$.
We say that a set $A \in \mathscr{A}_1$ is "tail-approximable" if $\forall \varepsilon > 0$ and $\forall n \geqq 1$ there exists a B_n such that $\mu(A \bigtriangleup B_n) < \varepsilon$. The preceding lemma shows that if $\mathscr{A}_\infty$ is trivial (i.e. $\mu(B) = 0$ or $1 \forall B \in \mathscr{A}_\infty$) then every "tail-approximable" set is also

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trivial. This result is stronger than some of the well-known zero-one laws and enables us to prove a somewhat stronger version of the zero-one law given above in Theorem 1.1.

(1.3) LEMMA. *Let* $\{\mu_k : k \geqq 1\}$ *and* $\{\nu_k : k \geqq 1\}$ *be probability measures and suppose for some* $\varepsilon > 0$ *that* $||\mu_k - \nu_k|| < \varepsilon$. *Then* $||\times_1^n \mu_k - \times_1^n \nu_k|| < n\varepsilon$.

PROOF. This is based on induction and the observation that if μ is a probability measure and ν a signed measure with $||\nu|| < \varepsilon$ then $||\mu \times \nu|| < \varepsilon$.

(1.4) PROOF OF THEOREM 1.1. Let $\mathscr{B}^n = \times_{n+1}^\infty \mathscr{B}_k$. Then $\mathscr{B}^1 \supset \mathscr{B}^2 \supset \cdots$ is a decreasing sequence of σ-algebras. From the classical zero-one law it follows that $\mathscr{B}^\infty = \lim \mathscr{B}^n$ is a trivial σ-algebra. Consequently it suffices to show that every permutation invariant set (i.e. invariant under all permutations of *finitely many* coordinates) is tail-approximable. Now let A be permutation invariant. Let $\varepsilon > 0$ and $n \geqq 1$. Let A_m be a cylinder set based on $\mathscr{B}_1 \times \mathscr{B}_2 \times \cdots \times \mathscr{B}_m$ such that $\mu(A \triangle A_m) < \varepsilon/2m$. For each k, $1 \leqq k \leqq m$, choose $k(m) \geqq \max\{m+1, n\}$ such that $||\mu_k - \mu_{k(m)}|| < \varepsilon/2m$ and such that $1(m), 2(m), \cdots, m(m)$ are all different. For each $(x_1, x_2, \cdots) \in \times_1^\infty \mathscr{R}_k$ define the permutation, T, which interchanges every x_k, $1 \leqq k \leqq m$, with $x_{k(m)}$. It is easily seen, as a consequence of Lemma 1.3, that $||\mu - \mu T^{-1}|| < \varepsilon/2$. So

$$\mu(A \triangle T(A_m)) \leqq \mu T^{-1}(A \triangle T(A_m)) + ||\mu - \mu T^{-1}|| < \varepsilon/2 + \mu T^{-1}(A \triangle T(A_m)) .$$

Since A is permutation invariant,

$$\mu(A \triangle T(A_m)) < (\varepsilon/2) + \mu(A \triangle A_m) < \varepsilon .$$

Since $T(A_m)$ is a $\mathscr{B}^n$-measurable cylinder set, it follows that A is tail-approximable. Therefore $\mu(A) = 0$ or 1.

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