The Annals of Mathematical Statistics
1972, Vol. 43, No. 2, 412–427

FUNCTIONS OF ORDER STATISTICS

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Two theorems on the asymptotic normality of linear combinations of functions of order statistics are given. Theorem 1 requires a "smooth" scoring function but the underlying df need not be continuous even and can also depend on the sample size. Theorem 2 allows general scoring functions but places additional restrictions on the df. Applications included.

0. Introduction. Let $\mathscr{G}$ denote the class of left continuous functions on $(0, 1)$ that are of bounded variation on $(\theta, 1 - \theta)$ for all $\theta > 0$. (Associated with any g in $\mathscr{G}$ is a Lebesgue-Stieltjes signed measure g on $(0, 1)$ and its total variation measure $|g|$.) Let $c_{n1}, \cdots, c_{nn}$ for $n \geqq 1$ be known constants. Let $d_{n1}, \cdots, d_{n\kappa}$ for $n \geqq 1$ be known constants of a greater order of magnitude that will be associated with the points $0 < p_1 < \cdots < p_\kappa < 1$. Let [ ] denote the greatest integer function. We will consider the limiting distribution of

$$T_n = n^{-1} \sum_{i=1}^{n} c_{ni}\, g_n(\xi_{ni}) + \sum_{k=1}^{\kappa} d_{nk}\, g_n(\xi_{n,[np_k]+1}) \tag{1}$$

where each g_n is a function in $\mathscr{G}$ and where $0 \leqq \xi_{n1} \leqq \cdots \leqq \xi_{nn} \leqq 1$ are the order statistics of n independent Uniform $(0, 1)$ rv's.

REMARK 1. If $g_n = h(F_n^{-1})$ for some sequence of df's F_n, in the class $\mathscr{F}$ of all df's, then T_n has the same distribution as does $n^{-1} \sum_1^n c_{ni} h(\alpha_{ni}) + \sum_1^\kappa d_{nk} h(\alpha_{n,[np_k]+1})$ where $\alpha_{n1} \leqq \cdots \leqq \alpha_{nn}$ are the order statistics of a random sample of size n from F_n.

For the mean or trimmed mean we have $\kappa = 0$, but for the Winsorized mean $\kappa = 2$ and p_1 and $1 - p_2$ are the Winsorizing percentages.

Theorem 1 and 2 below are both major theorems. Theorem 1 seems to be the first good theorem for general g, though it does require somewhat regular scores. Nonetheless, the rather minor Example 1 is probably sufficient for most applications. Theorem 2 employs a different representation. It allows very general scores, though g must now satisfy smoothness conditions. This second theorem is the one that relates to other theorems for this problem; in particular, it is an improvement on the theorems of [4]. See [4] for a discussion of the literature. The present Theorem 2 and the theorems of [1] and [8] are approximately equally strong; though [1] is probably slightly weaker.

Received January 5, 1971; revised July 1971.

[1] Research for this paper was carried out under NSF Contract, Number GP-13739, while the author was visiting the Mathematisch Centrum in Amsterdam.

Theorem 1 and its Corollary 2 are much cleaner than any of these. Never use Theorem 2 unless Corollary 2 and Theorem 1 fail.

Theorem 1, Example 1 and Corollary 2 are the main results. Also the parallel paper [7] applies the present technique to give the best results yet obtained for the trimmed and Winsorized means.

We stress that the df F need not be continuous.

1. The first main theorem. Define a function J_n on [0, 1] to equal c_{ni} for $(i-1)/n < t \leqq i/n$ and $1 \leqq i \leqq n$ with $J_n(0) = c_{n1}$. Let $d_1, \cdots, d_\kappa$ be finite nonzero constants. We will consider $n^{\frac{1}{2}}(T_n - \mu_n)$ where

$$\mu_n = \int_0^1 g_n J_n \, dI + \sum_{k=1}^{\kappa} d_k g_n(p_k)\,; \tag{2}$$

here I denotes the identity function and $\int \cdot \, dI$ denotes integration with respect to Lebesgue measure.

For fixed b_1, b_2 and M define a "scores bounding function" B by

$$B(t) = Mt^{-b_1}(1-t)^{-b_2} \qquad \text{for} \quad 0 < t < 1\,.$$

For fixed $\delta > 0$ define

$$D(t) = Mt^{-\frac{1}{2}+b_1+\delta}(1-t)^{-\frac{1}{2}+b_2+\delta} \qquad \text{for} \quad 0 < t < 1\,.$$

Let g denote a fixed function in $\mathscr{G}$ and let J denote a fixed measurable function on (0, 1). Define $q(t) = [t(1-t)]^{\frac{1}{2}-\delta/2}$ on [0, 1].

ASSUMPTION 1. (Boundedness) Let $|g| \leqq D$, all $|g_n| \leqq D$, $|J| \leqq B$ and all $|J_n| \leqq B$ on (0, 1).

ASSUMPTION 2. (Smoothness) Except on a set of t's of $|g|$-measure 0 we have both J is continuous at t and $J_n \to J$ uniformly in some small neighborhood of t as $n \to \infty$.

ASSUMPTION 3. (Convergence) $\int_0^1 Bq \, d|g_n - g| \to 0$ as $n \to \infty$.

ASSUMPTION 4. ($\kappa > 0$) For $1 \leqq k \leqq \kappa$ we have $n^{\frac{1}{2}}(d_{nk} - d_k) \to 0$ as $n \to \infty$. In some small neighborhood of each of $p_1, \cdots, p_\kappa$ the functions g_n' for $n \geqq 1$ form an equiuniformly continuous family and $g_n(p_k) \to g(p_k)$ and $g_n'(p_k) \to g'(p_k)$ as $n \to \infty$ for each $1 \leqq k \leqq \kappa$. (If $g_n = g$ for all n we require at each of $p_1, \cdots, p_\kappa$ only that $g'(p_k)$ exist.)

Define

$$\begin{aligned} \sigma^2 = {} & \int_0^1 \int_0^1 (s \wedge t - st) J(s) J(t) \, dg(s) \, dg(t) \\ & + 2 \sum_{k=1}^{\kappa} d_k g'(p_k) \int_0^1 (t \wedge p_k - tp_k) J(t) \, dg(t) \\ & + \sum_{j=1}^{\kappa} \sum_{k=1}^{\kappa} d_j d_k g'(p_j) g'(p_k)(p_j \wedge p_k - p_j p_k)\,. \end{aligned} \tag{3}$$

THEOREM 1. *If Assumptions* 1, 2, 3 *and* 4 *hold*, *then*

$$n^{\frac{1}{2}}(T_n - \mu_n) \to_d N(0, \sigma^2)$$

with μ_n *of* (2) *and* σ^2 of (3) *finite.*

PROOF. We may assume that $0 < \xi_{n1} < \cdots < \xi_{nn} < 1$ are the special Uniform (0, 1) order statistics described in the Appendix. These have empirical df Γ_n, empirical process $U_n = n^{\frac{1}{2}}(\Gamma_n - I)$ on [0, 1] and satisfy $\rho(U_n, U) \to_{\text{a.s.}} 0$ as $n \to \infty$; here U is a special Brownian bridge and ρ denotes the uniform metric. Other properties of these special ξ_{ni}'s and U will be cited from the Appendix as needed.

We give the proof first for $\kappa = 0$. For $0 \leqq t \leqq 1$ define

$$\psi_n(t) = -\int_t^1 J_n\,dI \quad \text{so that} \quad c_{ni}/n = [\psi_n(i/n) - \psi_n((i-1)/n)] \quad \text{for} \quad 1 \leqq i \leqq n\,.$$

Summing by parts gives

$$\begin{aligned} T_n &= \int_0^1 g_n(\Gamma_n^{-1})J_n\,dI = \sum_1^n g_n(\xi_{ni})[\psi_n(i/n) - \psi_n((i-1)/n)] \\ &= -\psi_n(0)g_n(\xi_{n1}) - \sum_1^{n-1} \psi_n(i/n)[g_n(\xi_{ni+1}) - g_n(\xi_{ni})] \\ &=_{\text{a.s.}} -\psi_n(0)g_n(\xi_{n1}) - \int_{\xi_{n1}}^{\xi_{nn}} \psi_n(\Gamma_n)\,dg_n \end{aligned}$$

where the second integral representation uses the fact that it is a.s. true that no ξ_{ni} takes on one of the countable number of values at which g_n is discontinuous. Integration by parts gives

$$\begin{aligned} \mu_n &= \int_{\xi_{n1}}^{\xi_{nn}} g_n\,d\psi_n + \int_{[\xi_{n1},\xi_{nn})^c} g_n J_n\,dI \\ &=_{\text{a.s.}} -\int_{\xi_{n1}}^{\xi_{nn}} \psi_n\,dg_n + \psi_n(\xi_{nn})g_n(\xi_{nn}) - \psi_n(\xi_{n1})g_n(\xi_{n1}) + \int_{[\xi_{n1},\xi_{nn}]^c} g_n\,d\psi_n\,. \end{aligned}$$

Thus

$$n^{\frac{1}{2}}(T_n - \mu_n) = -(S_n + \gamma_{n1} + \gamma_{n2} + \gamma_{n3})$$

where

$$S_n = \int_{\xi_{n1}}^{\xi_{nn}} A_n U_n\,dg_n = \int_0^1 A_n^* U_n\,dg_n \quad \text{with} \quad A_n = [\psi_n(\Gamma_n) - \psi_n]/(\Gamma_n - I)\,,$$

where A_n^* is equal to the difference quotient A_n on $[\xi_{n1}, \xi_{nn})$ and is equal to 0 otherwise, and where

$$\gamma_{n1} = n^{\frac{1}{2}}g(\xi_{n1})[\psi_n(0) - \psi_n(\xi_{n1})]\,, \qquad \gamma_{n2} = n^{\frac{1}{2}}g_n(\xi_{nn})\psi_n(\xi_{nn})\,,$$
$$\gamma_{n3} = n^{\frac{1}{2}} \int_{[\xi_{n1},\xi_{nn})^c} g_n J_n\,dI\,.$$

(For each fixed ω in the probability space, at the at most n points where $\Gamma_n = I$ we define A_n to be 0.)

Define

$$S = \int_0^1 JU\,dg + \sum_{k=1}^{\kappa} d_k g'(p_k)U(p_k) \tag{4}$$

so that S is a $N(0, \sigma^2)$ rv; and σ^2 is finite by Assumption 1. (We remind the reader that we still have $\kappa = 0$ for the time being.)

Let $\chi_{n\varepsilon}$ denote the indicator function of the set $S_{n,\varepsilon}$ of Lemma A. 3; and note also the definition of β_ε made there. Now

$$|\chi_{n\varepsilon}S_n - S| \leqq \gamma_{n4} + \gamma_{n5} \equiv \int_0^1 |\chi_{n\varepsilon}A_n{}^*U_n|d|g_n - g| + \int_0^1 |\chi_{n\varepsilon}A_n{}^*U_n - JU|d|g| \,.$$

Also, by Assumption 1, when $b_1, b_2 > 0$ we have

$$|A_n| = |\int_I^{\Gamma_n} J_n\, dI/(\Gamma_n - I)| \leqq \int_I^{\Gamma_n} B\, dI/(\Gamma_n - I) \leqq B \vee B(\Gamma_n)\,;$$

and since B is a reproducing u-shaped function (see Definition A. 3), Lemma A. 3 gives the key equation of this proof

$$\chi_{n\varepsilon}|A_n{}^*| \leqq B_{\beta_\varepsilon} \leqq M_\varepsilon B \tag{5}$$

for some constant M_ε and for B_{β_ε} as in Definition A. 3. Clearly, (5) also holds for all other values of b_1 and b_2.

For a.e. fixed ω the functions U and U_n for $n \geqq 1$ are uniformly bounded by $M_\omega q$; see Remark A. 5. Thus a.s.

$$|\gamma_{n4}| \leqq M_\varepsilon M_\omega \int_0^1 Bqd|g_n - g| \to 0 \qquad \text{as} \quad n \to \infty$$

using (5) and Assumption 3. Thus $\gamma_{n4} \to_{\text{a.s.}} 0$ as $n \to \infty$.

Integrate by parts and use Assumption 1 to find $\int_0^1 Bqd|g| < \infty$. For fixed ω the integrand of γ_{n5} is dominated by the $|g|$-integrable function $(M_\varepsilon + 1)M_\omega Bq$; and the integrand converges a.e. $|g|$ to 0 since $\rho(U_n, U) \to_{\text{a.s.}} 0$ as $n \to \infty$ and since $A_n(t) \to J(t)$ a.e. $|g|$ as $n \to \infty$ by Assumption 2 and the representation $A_n = \int_I^{\Gamma_n} J_n(s)ds/(\Gamma_n - I)$. Thus $\gamma_{n5} \to_{\text{a.s.}} 0$ as $n \to \infty$ by using the dominated convergence theorem once for each ω.

Thus $\chi_{n\varepsilon}S_n \to_{\text{a.s.}} S$ as $n \to \infty$. Thus $S_n \to_p S$ as $n \to \infty$; since $E\chi_{n\varepsilon} \to 1$ as $n \to \infty$ for each $\varepsilon > 0$ by Lemma A. 3.

In case $\kappa = 0$ it remains only to show that γ_{n1}, γ_{n2} and γ_{n3} are negligible. Case 1: $b_1 < 1$. Then $|\gamma_{n1}| \leqq n^{\frac{1}{2}}D(\xi_{n1}) \int_0^{\xi_{n1}} BdI$ so that on the event $\xi_{n1} < (\frac{1}{2} \wedge p_1)$ we have by Assumption 1 that $|\gamma_{n1}| \leqq M_1 n^{\frac{1}{2}}\xi_{n1}^{\frac{1}{2}+\delta}$ for some constant M_1. Since $n\xi_{n1} = 0_p(1)$ and since $P(\xi_{n1} < (\frac{1}{2} \wedge p_1)) \to 1$ as $n \to \infty$, we have $\gamma_{n1} \to_p 0$ as $n \to \infty$. Case 2: $b_1 \geqq 1$. Then on the event $\xi_{n1} < (\frac{1}{2} \wedge p_1)$ we have

$$\begin{aligned}|\psi_n(0) - \psi_n(\xi_{n1})| &\leqq n^{-1}|\textstyle\sum_1^{[n\xi_{n1}]} c_{ni}| \leqq M_2[n^{b_1-1} + n^{-1}\sum_2^{[n\xi_{n1}]}(i/n)^{-b_1}] \\ &\leqq M_2[n^{b_1-1} + \textstyle\int_{1/n}^{\xi_{n1}} t^{-b_1}\,dt] \leqq M_3[n^{b_1-1} + \xi_{n1}^{1-b_1}]\end{aligned}$$

(replace t^{1-b_1} by $\log t$ when $b_1 = 1$) and also

$$\begin{aligned}|\gamma_{n1}| &\leqq M_4 n^{\frac{1}{2}}\xi_{n1}^{-\frac{1}{2}+b_1+\delta}[n^{b_1-1} + \xi_{n1}^{1-b_1}] \\ &= M_4\xi_{n1}^{\delta}[(n\xi_{n1})^{b_1-\frac{1}{2}} + (n\xi_{n1})^{\frac{1}{2}}]\end{aligned}$$

for constants M_2, M_3, M_4. So again $\gamma_{n1} \to_p 0$ as $n \to \infty$. Analogously $\gamma_{n2} \to_p 0$ as $n \to \infty$ by considering the cases $b_2 < 1$ and $b_2 \geqq 1$. Also $n^{\frac{1}{2}} \int_0^{\xi_{n1}} |g_n J_n| dI \leqq n^{\frac{1}{2}} \int_0^{\xi_{n1}} M^2[t(1-t)]^{-\frac{1}{2}+\delta}\, dt \to_p 0$ as $n \to \infty$; so that $\gamma_{n3} \to_p 0$ as $n \to \infty$ without resort to cases.

This completes the proof for $\kappa = 0$. Assumption 4 was not used.

Suppose now $\kappa > 0$. Now $\gamma_{n1} + \gamma_{n2} + \gamma_{n3} \rightarrow_p 0$ as $n \rightarrow \infty$ by the above proof. Without loss set $\kappa = 1$, $d_{n1} = d_n$, $d_1 = d$ and $p_1 = p$. In addition to the term considered above, $n^{\frac{1}{2}}(T_n - \mu_n)$ contributes $R_n = n^{\frac{1}{2}}[d_n g_n(\xi_{n,[np]+1}) - dg_n(p)]$ and S contributes $R = -dg'(p)U(p)$. But using only Assumption 4 we have

$$|R_n - R| \leqq |n^{\frac{1}{2}}(d_n - d)g_n(\Gamma_n^{-1}(p))| + |dn^{\frac{1}{2}}[g_n(\Gamma_n^{-1}(p)) - g_n(p)] + dg'(p)U(p)| \rightarrow_{\text{a.s.}} 0 \qquad \text{as} \quad n \rightarrow \infty$$

since $\rho(V_n, V) \rightarrow_{\text{a.s.}} 0$ as $n \rightarrow \infty$ with $V_n = n^{\frac{1}{2}}(\Gamma_n^{-1} - I)$ and $V = -U$ by (A. 4). □

REMARK 2. For the special Uniform (0, 1) order statistics and the special Brownian bridge U of the Appendix, we have in fact shown that

$$n^{\frac{1}{2}}(T_n - \mu_n) \rightarrow_p - \textstyle\int_0^1 JUdg - \sum_{k=1}^{\kappa} d_k g'(p_k)U(p_k) \qquad \text{as} \quad n \rightarrow \infty . \tag{6}$$

REMARK 3. A random vector converges in probability if and only if each coordinate does. Thus the obvious vector analog of Theorem 1 follows trivially from Remark 2.

EXAMPLE 1. Let $\alpha_1, \cdots, \alpha_n$ be a random sample from an arbitrary df F for which $E|\alpha|^r < \infty$ for some $r > 0$. Let

$$T_n = n^{-1} \textstyle\sum_1^n J(t_{ni})\alpha_{ni} \tag{7}$$

where $\max_{1 \leqq i \leqq n} |t_{ni} - i/n| \rightarrow 0$ as $n \rightarrow \infty$ and where for some $a > 0$

$$a[(i/n) \wedge (1 - i/n)] \leqq t_{ni} \leqq 1 - a[(i/n) \wedge (1 - i/n)] \qquad \text{for} \quad 1 \leqq i \leqq n .$$

Suppose J is continuous except at a finite number of points at which F^{-1} is continuous, and suppose

$$|J(t)| \leqq M[t(1 - t)]^{-\frac{1}{2}+1/r+\delta} \qquad \text{for} \quad 0 < t < 1$$

for some $\delta > 0$. Then

$$n^{\frac{1}{2}}(T_n - \textstyle\int_0^1 J_n F^{-1}\, dI) \rightarrow_d N(0, \sigma^2) \qquad \text{as} \quad n \rightarrow \infty$$

with finite variance

$$\sigma^2 = \textstyle\int_0^1 \int_0^1 (s \wedge t - st)J(s)J(t)\, dF^{-1}(s)\, dF^{-1}(t) .$$

This result is deficient in that J_n has not been replaced by J in the centering constant; this is true likewise of Theorem 1. But this is a purely deterministic problem, and it seemed advisable to separate it off. We now provide one possible solution. Suppose J' exists and is continuous on (0, 1) with

$$|J'(t)| \leqq M[t(1 - t)]^{-\frac{3}{2}+1/r+\delta} \qquad \text{for} \quad 0 < t < 1 ;$$

and strengthen the "max-condition" to

$$n \max_{1 \leq i \leq n} |t_{ni} - i/n| = O(1) \qquad \text{as} \quad n \to \infty .$$

Then our conclusion may be strengthened to

$$n^{\frac{1}{2}}(T_n - \textstyle\int_0^1 JF^{-1}\, dI) \to_d N(0, \sigma^2) \qquad \text{as} \quad n \to \infty .$$

PROOF. We will apply Theorem 1 with $\kappa = 0$, $g_n = g = F^{-1}$ and $b_1 = b_2 = b \equiv \frac{1}{2} - 1/r - \delta$. Since $E|\alpha|^r < \infty$ we have

$$t|F^{-1}(t)|^r \leqq \textstyle\int_0^t |F^{-1}(s)|^r\, ds \leqq \int_{-\infty}^{F(t)} |s|^r\, dF(s) \to 0 \qquad \text{as} \quad t \to 0$$

using $F^{-1} \circ F(t) \leqq t$ for $-\infty < t < \infty$. Thus $|g| \leqq D$ with the choice $-\frac{1}{2} + b + \delta = -1/r$. By the "$a$-condition" on the t_{ni}'s we have $|J_n| \leqq B \equiv M_a[t(1-t)]^{-\frac{1}{2}+1/r+\delta}$ for some constant M_a; see Definition A. 3. Thus Assumption 1 holds. The continuity of J and the "max-condition" on the t_{ni}'s together imply Assumption 2. The first result now follows from Theorem 1. By the mean value theorem and the "strengthened max-condition" we have

$$\begin{aligned} n^{\frac{1}{2}}|J_n(t) - J(t)||g(t)| &\leqq n^{-\frac{1}{2}} M_1 |J'(t^*)||g(t)| \\ &\leqq n^{-\frac{1}{2}} M_1 M[t^*(1 - t^*)]^{-\frac{3}{2}+1/r+\delta} M[t(1 - t)]^{-1/r} \\ &\leqq n^{-\frac{1}{2}} M_1 M^2 M_a [t(1 - t)]^{-\frac{3}{2}+\delta} \end{aligned}$$

where the final inequality uses Definition A.3 and the "a-condition" bound on t^*. The proof is easily completed using $n^{-\frac{1}{2}} \int_{1/n}^{1-1/n} [t(1 - t)]^{-\frac{3}{2}+\delta}\, dt \to 0$ as $n \to \infty$. □

EXAMPLE 1a. Let $\alpha_1, \cdots, \alpha_n$ be a random sample from a df having $E|\alpha|^r < \infty$ for some $r > 2$. Then

$$n^{\frac{1}{2}}(\bar{\alpha} - E(\alpha)) \to_d N(0, \operatorname{Var}[\alpha]) \qquad \text{as} \quad n \to \infty .$$

This example shows that the ordinary central limit theorem "just fails" to be a corollary to Theorem 1.

EXAMPLE 1b. Let $\alpha_1, \cdots, \alpha_n$ be a random sample from the $N(0, 1)$ df Φ. For integral $r > 0$ let

$$T_n = n^{-1} \textstyle\sum_1^n [\Phi^{-1}(i/(n + 1))]^r \alpha_{ni} .$$

(Or use $(3i - 1)/(3n + 1)$ in place or $i/(n + 1)$ as some would advocate.) Then

$$n^{\frac{1}{2}}(T_n - E(\alpha^{r+1})) \to_d N(0, E(\alpha^{2r+2}) - E^2(\alpha^{r+1})) \qquad \text{as} \quad n \to \infty .$$

EXAMPLE 2 (The linearly trimmed mean). Let $0 < a < \frac{1}{2}$ be fixed, and let $a_n = [na]$. Let $\alpha_1, \cdots, \alpha_n$ be a random sample from $F_\theta = F(\cdot - \theta)$ where F is any df symmetric about 0. For n even we follow Crow and Siddiqui (1967) and define

$$T_n = \textstyle\sum_{a_n+1}^{n/2} [2(i - a_n) - 1](\alpha_{ni} + \alpha_{n,n-i+1})/2(n/2 - a_n)^2 .$$

(We omit n odd.) Then

$$n^{\frac{1}{2}}(T_n - \theta) \to_d [4/(1-2a)^2]2^{\frac{1}{2}} \int_a^{\frac{1}{2}} (t-a)W(t)\,dF^{-1}(t) \qquad \text{as} \quad n \to \infty$$

where W is Brownian motion.

PROOF. Let $c_{ni} = c_{n,n-i+1} = n[2(i-a_n)-1]/2(n/2-a_n)^2$ for $1 \leqq i \leqq n/2$. Then $J(t)$ equals 0 or $4(t-a)/(1-2a)^2$ according as $0 \leqq t \leqq a$ or $a \leqq t \leqq \frac{1}{2}$ while $J(t) = J(1-t)$ for $\frac{1}{2} \leqq t \leqq 1$. Then by Theorem 1 and $n^{\frac{1}{2}} \int_0^1 (J_n - J)F^{-1}\,dI \to 0$ as $n \to \infty$ we have

$$n^{\frac{1}{2}}(T_n - \theta) \to_d -\int_0^1 UJ\,dF^{-1} = 2^{\frac{1}{2}} \int_0^{\frac{1}{2}} WJ\,dF^{-1}$$

using the symmetry of F, where

$$W(t) = -[U(t) + U(1-t)]/2^{\frac{1}{2}} \qquad \text{for} \quad 0 \leqq t \leqq 1 \tag{8}$$

is Brownian motion with $\operatorname{Var}[W(1)] = 1$. □

EXAMPLE 3. Let $\alpha_1, \cdots, \alpha_n$ be independent Bernoulli (θ) rv's. Let $g = F^{-1}$. Thus $g(t)$ equals $-\infty$, 0, 1 for $t = 0$, $0 < t \leqq 1-\theta$, $1-\theta < t \leqq 1$. Let $J(t)$ equal 0, 1 for $0 \leqq t < \frac{1}{2}$, $\frac{1}{2} \leqq t \leqq 1$ and let $c_{ni} = J(i/n)$. Then T_n equals $\frac{1}{2}$ if more than $\frac{1}{2}$ of the α_i's are positive; while T_n equals the proposition of positive α_i's if less than $\frac{1}{2}$ of the α_i's are positive.

(a) Suppose $\theta = \frac{1}{2}$. Then $n^{\frac{1}{2}}(T_n - \mu_n) = n^{\frac{1}{2}}(T_n - \frac{1}{2})$ is asymptotically distributed as a rv having a $N(0, \frac{1}{4})$ density on $(-\infty, 0)$ and having point mass $\frac{1}{2}$ at 0. Note that J is not continuous a.e. $|g|$; and hence the hypotheses of Theorem 1 fail to hold.

(b) Suppose $\theta < \frac{1}{2}$. Then $n^{\frac{1}{2}}(T_n - \mu_n) = n^{\frac{1}{2}}(T_n - \frac{1}{2})$ is asymptotically $N(0, 0)$ by Theorem 1.

(c) Suppose $\theta < \frac{1}{2}$. Then $n^{\frac{1}{2}}(T_n - \mu_n) = n^{\frac{1}{2}}(T_n - \theta)$ is asymptotically $N(0, \theta(1-\theta))$ by Theorem 1.

REMARK 4. The rv T_n of (1) will be asymptotically normal provided there is "not too much weight in the tails." If $|g| \leqq D$, then we only allow $|J| \leqq B$ so that the rv S of (4) will have finite variance. This tradeoff between g and J can be different in each tail (in that b_1 need not equal b_2).

2. Variations on the first main theorem. In this section we consider (Corollary 2) the important weakening of Assumption 3 and (Corollary 1) the rather minor dropping of the existence of a limit g. For these we strengthen Assumption 2.

ASSUMPTION 2′. (Strong smoothness) Let J be continuous on (0, 1) and suppose that $J_n \to J$ uniformly on $[\theta, 1-\theta]$ as $n \to \infty$ for each $\theta > 0$.

We say that g_n *converges weakly* to g if $g_n(t) \to g(t)$ as $n \to \infty$ at each continuity point t of g.

ASSUMPTION 3′. (Weak convergence) Let g_n converge weakly to g as $n \to \infty$.

COROLLARY 1. *Suppose Assumptions* 1, 2′ *and* 4 (*omit the conditions on* g) *hold. Then*

$$n^{\frac{1}{2}}(T_n - \mu_n)/\sigma_n \to_d N(0, 1)$$

provided $\liminf_{n\to\infty} \sigma_n^2 > 0$. (*Replace* g *by* g_n *in* (3) *to obtain* σ_n^2.)

PROOF. In the proof of Theorem 1 let $S_{0n} = +\int_0^1 JU\,dg_n + \sum_1^\kappa d_k g_n'(p_k)U(p_k)$. Then when $\kappa = 0$ we have

$$\begin{aligned} \chi_{n\varepsilon}|S_n - S_{0n}| &= |\textstyle\int_0^1 \chi_{n\varepsilon}[A_n{}^* U_n - JU]\,dg_n| \\ &\leqq \rho_q(U_n, U) \textstyle\int_0^1 qM_\varepsilon B d|g_n| + \rho_q(U, 0) \int_0^1 \chi_{n\varepsilon} q|A_n{}^* - J|d|g_n| \\ &= o_p(1)O(1) + O_p(1) \textstyle\int_0^1 \chi_{n\varepsilon} q|A_n{}^* - J|d|g_n|\,. \end{aligned}$$

Integration by parts and Assumption 1 shows that for any $\eta > 0$ there exists $\theta > 0$ such that $(\int_0^\theta + \int_{1-\theta}^1) qM_\varepsilon B d|g_n| < \eta/2$ for all n. Also on $[\theta, 1-\theta]$ the difference quotient $A_n{}^*$ converges uniformly to J by Assumption 2′. Hence on subsequences n' for which $\int_\theta^{1-\theta} d|g_{n'}|$ remains bounded we have $\int_\theta^{1-\theta} \chi_{n'\varepsilon} q|A_{n'}^* - J|d|g_{n'}| \to_{\text{a.s.}} 0$. Thus $\int_\theta^{1-\theta} \chi_{n\varepsilon} q|A_n{}^* - J|d|g_n| \to_p 0$. Combined with the first remark on the negligible contribution from the tails, this gives that $\int_0^1 \chi_{n\varepsilon} q|A_n{}^* - J|d|g_n| \to_p 0$ for all $\varepsilon > 0$. Hence $\chi_{n\varepsilon}(S_n - S_{0n}) \to_p 0$. We can divide by σ_n without destroying $\to_p$ provided σ_n is bounded away from 0. □

COROLLARY 2. *Suppose Assumptions* 1, 2′, 3′ *and* 4 *hold. Then*

$$n^{\frac{1}{2}}(T_n - \mu_n) \to_d N(0, \sigma^2) \qquad \textit{as} \quad n \to \infty$$

with μ_n *of* (2) *and* σ^2 *of* (3) *finite.*

PROOF. In addition to the proof of Corollary 1 we need only show that $\int_0^1 UJd(g_n - g) \to_p 0$ as $n \to \infty$. But we simply repeat the type of argument used in Corollary 1. That is, truncate off $(\int_0^\theta + \int_{1-\theta}^1)|UJ|d|g_n - g|$ since it makes a small contribution with high probability; and then use weak convergence of g_n to g on $[\theta, 1-\theta]$ and the continuity of J on $[\theta, 1-\theta]$ to show that $\int_\theta^{1-\theta} JUd(g_n - g) \to_{\text{a.s.}} 0$ as $n \to \infty$ for any $\theta > 0$ by the Helly-Bray theorem. □

3. The second main theorem. We will start fresh in regard to notation. The Appendix is rather more heavily relied on.

We now fix g in the class of all left continuous functions and consider

$$T_n = n^{-1} \textstyle\sum_1^n c_{ni} g(\xi_{ni})\,. \tag{9}$$

We suppose throughout that there exist functions C_n on (0, 1) and a signed measure ν on (0, 1) such that

$$c_{ni}/n = \textstyle\int_{(i-1)/n}^{i/n} C_n\,d\nu \qquad \text{for} \quad 1 \leqq i \leqq n \tag{10}$$

where $\int_a^b \cdot\, d\nu = \int_{(a,b]} \cdot\, d\nu$.

EXAMPLE 4. Let ν denote Lebesgue measure and let $C_n(t) = c_{ni}$ for $(i-1)/n < t \leqq i/n$ and $1 \leqq i \leqq n$. Call these "simply generated scores."

EXAMPLE 5. Let ν put mass 1 at $t = \frac{1}{2}$ and zero mass elsewhere. Let $C_n(\frac{1}{2}) = 1$. Then T_n is g evaluated at a sample median (at the sample median for n odd).

In general

$$T_n = \textstyle\int_0^1 g(\Gamma_n^{-1}) C_n \, d\nu \, .$$

Let C denote a fixed measurable function on (0, 1) and let

$$\mu = \textstyle\int_0^1 gC \, d\nu \tag{11}$$

and

$$\sigma^2 = \textstyle\int_0^1 \int_0^1 (s \wedge t - st) g'(s) g'(t) C(s) C(t) \, d\nu(s) \, d\nu(t) \tag{12}$$

provided these exist. Let * now restrict functions on (0, 1) to $[1/n, 1 - 1/n]$. If $C_n{}^*$ replaces C in (11) and (12) the resulting quantities will be called μ_n and $\sigma_n{}^2$.

Now

$$n^{\frac{1}{2}}(T_n - \mu) = T_n{}^* + \gamma_{n1} + \gamma_{n2} + \theta_n \tag{13}$$

where

$$T_n{}^* = \textstyle\int^* V_n A_n C_n \, d\nu \, ,$$
$$A_n = [g(\Gamma_n{}^{-1}) - g]/(\Gamma_n{}^{-1} - I) \, ,$$
$$\gamma_{n1} = n^{-\frac{1}{2}} c_{n1} g(\xi_{n1}) \, , \qquad \gamma_{n2} = n^{-\frac{1}{2}} c_{nn} g(\xi_{nn})$$

and

$$\theta_n = n^{\frac{1}{2}} \textstyle\int_0^1 (C_n{}^* - C) g \, d\nu \, .$$

(Left continuity is used to define A_n at the at most finite number of points, for each fixed ω, at which it might otherwise be undefined.)

(E1) (i) For all large n we have $|C_n| \leqq \psi$ a.e. $|\nu|$ where $\int_0^1 q|g'|\psi d|\nu| < \infty$ for some q in $\mathscr{Q}$.

(ii) $\int_0^1 q|A_n{}^* - g'|\psi d|\nu| \rightarrow_p 0$ as $n \rightarrow \infty$, for this same q.

(E1a) g is absolutely continuous on $(\varepsilon, 1-\varepsilon)$ for all $\varepsilon > 0$. g' exists a.e. $|\nu|$ and $|g'| \leqq R$ a.e. Lebesgue measure where R is a reproducing u-shaped (or increasing) function. For all large n we have $|C_n| \leqq \psi$ a.e. $|\nu|$ where $\int_0^1 qR\psi d|\nu| < \infty$ for some q in $\mathscr{Q}$.

(E2) $\gamma_{n1} + \gamma_{n2} \rightarrow_p 0$ as $n \rightarrow \infty$.

(E2a) $|g| \leqq M[1 - I)]^{-\alpha}$ for some $M > 0$ and $(|c_{n1}| \vee |c_{nn}|) = o(n^{\frac{1}{2}-\alpha})$.

(E2b) $|g| \leqq M(1 - I)^{-\alpha}$ for some $M > 0$ and $|c_{n1}| = o(n^{\frac{1}{2}})$ and $|c_{nn}| = o(n^{\frac{1}{2}-\alpha})$.

(E3) $C_n \to C$ a.e. $|\nu|$ as $n \to \infty$.

(E4) $n^{\frac{1}{2}} \int_0^1 (C_n^* - C)g\,d\nu \to 0$ as $n \to \infty$.

(Roughly speaking, use R increasing in (E1a) and use (E2b) for distributions on $[0, \infty)$.) See below for conditions (E1b) and (E1c).

THEOREM 2. *Conditions* (E1a), (E1b) *and* (E1c) *each imply* (E1). *Conditions* (E2a) *and* (E2b) *each imply* (E2). *If* (E1), (E2), (E3) *and* (E4) *hold, then*

$$n^{\frac{1}{2}}(T_n - \mu) \to_d N(0, \sigma^2)$$

with μ *of* (11) *and* σ^2 *of* (12) *finite. If only* (E1) *and* (E2) *hold, then*

$$n^{\frac{1}{2}}(T_n - \mu_n)/\sigma_n \to_d N(0, 1)$$

provided $\liminf_{n\to\infty} \sigma_n^2 > 0$.

PROOF. We first prove that $n^{\frac{1}{2}}(T_n - \mu) \to_d N(0, \sigma^2)$. Let $T = \int_0^1 Vg'C\,d\nu$, which is a $N(0, \sigma^2)$ rv with σ^2 finite by (E1) (i). Also

$$\begin{aligned}|T_n^* - T| = |\textstyle\int_0^1 [(V_n^* - V)(A_n^* - g' + g')C_n + V(A_n^* - g')C_n \\ + Vg'(C_n - C)]\,d\nu| \leqq \rho_q(V_n^*, V)[\textstyle\int_0^1 q|A_n^* - g'|\psi d|\nu| \\ + \textstyle\int_0^1 q|g'|\psi d|\nu|] + \rho_q(V, 0) \textstyle\int_0^1 q|A_n^* - g'|\psi d|\nu| \\ + \rho_q(V, 0) \textstyle\int_0^1 q|g'||C_n - C|d|\nu| \,.\end{aligned}$$

Now $\rho_q(V_n^*, V) = o_p(1)$, $\rho_q(V, 0) = O_p(1)$ and $\int_0^1 q|g'||C_n - C|d|\nu| \to 0$ by (E1) (i), (E3) and the dominated convergence theorem. Thus $T_n^* \to_p T$ under (E1) and (E3). Referring to (13), (E2) and (E4) show that $n^{\frac{1}{2}}(T_n - \mu) \to_p T$.

The proof for $n^{\frac{1}{2}}(T_n - \mu_n)/\sigma_n$ is even easier. Note that we can divide by σ_n without destroying $\to_p$ since σ_n is bounded away from 0.

We now show that (E2a) implies (E2). Using Stirling's approximation

$$\begin{aligned}E(\gamma_{n1}^2) &= c_{n1}^2 \textstyle\int_0^1 [t(1 - t)]^{-2\alpha} n(1 - t)^{n-1}\,dt/n \\ &= M_1 c_{n1}^2 \Gamma(n - 2\alpha)/\Gamma(n + 1 - 4\alpha) \leqq M_2 c_{n1}^2/n^{1-2\alpha} \to 0 \,.\end{aligned}$$

Likewise (E2b) implies (E2).

We now show that (E1a) implies (E1). Now (E1) (i) is trivial. Since g' exists a.e. $|\nu|$ we have for every ω that the difference quotient A_n converges to g' a.e. $|\nu|$ (since $\rho(\Gamma_n^{-1}, I) \to_{\text{a.e.}} 0$). Also

$$|A_n| = |\textstyle\int_I^{\Gamma_n} g'(s)\,ds/(\Gamma_n - I)| \leqq |\textstyle\int_I^{\Gamma_n} R(s)\,ds/(\Gamma_n - I)| < R \vee R(\Gamma_n) \,.$$

Thus for the set $S_{n,\varepsilon}$ of Lemma A.3

$$\chi_{n\varepsilon}|A_n^*| \leqq R_{\beta_\varepsilon} \leqq M_\varepsilon R \,.$$

Thus we may for each fixed ω apply the dominated convergence theorem to conclude

$$\chi_{n\varepsilon} \textstyle\int_0^1 q|A_n^* - g'|\psi d|\nu| \to_{\text{a.e.}} 0 \,.$$

Thus $\int_0^1 q|A_n^* - g'|\psi d|\nu| \to_p 0$.

To show that (E1b) implies (E1), work Remark A. 4 into the first paragraph of this proof. That (E1c) implies (E1) is also easy. Conditions (E1b) and (E1c) are in the spirit of [1]. They are not stated until after Theorem 2 because they are long, and because I feel that (E1a) is a better condition. □

(E1b) (i) g is absolutely continuous on $(\varepsilon, 1-\varepsilon)$ for all $\varepsilon > 0$. (ii) For all positive β in some neighborhood of 0 there exists $0 < M_\beta < \infty$ such that $|g'(s)/g'(t)| \leqq M_\beta$ whenever $\beta t \leqq s \leqq t + \beta$ and $t \leqq \frac{1}{2}$ and whenever $\beta(1-t) \leqq 1 - s \leqq (1-t) + \beta$ and $t \leqq \frac{1}{2}$. (iii) For all large n we have $|C_n| \leqq \psi$ a.e. $|\nu|$ where $\int_0^1 q|g'|\psi d|\nu| < \infty$ for some q in $\mathscr{Q}$.

(E1c) (i) g satisfies a Lipschitz condition on $(\varepsilon, 1-\varepsilon)$ for all $\varepsilon > 0$ and g' exists a.e. $|\nu|$. (ii) There exists $\delta > 0$ such that for all positive β in some neighborhood of 0 there exists $0 < M_\beta < \infty$ such that $|g'(s)/g'(t)| < M_\beta$ whenever $\beta t \leqq s \leqq t + \beta$ and $t \leqq \delta$ and whenever $\beta(1-t) \leqq 1 - s \leqq (1-t) + \beta$ and $t \leqq \delta$; and $|\nu|([\delta_0, 1-\delta_0]) < \infty$ for some $0 < \delta_0 < \delta$. Or there exists $\delta_0 > 0$ such that $c_{ni} = 0$ for $i \leqq n\delta_0$ and $i \geqq n(1-\delta_0)$ and n exceeding some n_0; and $|\nu|([\delta_0, 1-\delta_0]) < \infty$. (iii) For all large n we have $|C_n| \leqq \psi$ a.e. $|\nu|$ where $\int_0^1 q|g'|\psi d|\nu| < \infty$ for some q in $\mathscr{Q}$.

REMARK 5. A version of Theorem 2 in which g depends on n may be proved with the aid of Corollary 3 of [6].

APPENDIX

A UNIFORMLY CONVERGENT EMPIRICAL PROCESS

A1. Definition of the basic rv's $\xi_1, \cdots, \xi_n$ and processes U_n, U. There is a probability space $(\Omega, \mathscr{A}, P)$ with the following very special random quantities defined on it.

$\xi_1, \cdots, \xi_n$ are independent Uniform (0, 1) rv's on Ω and $0 < \xi_{n1} < \cdots < \xi_{nn} < 1$ for all ω in Ω, where the ξ_{ni}'s denote the ordered ξ_i's. We let Γ_n denote the empirical df of these rv's.

U denotes a Brownian bridge on Ω. That is $\{U(t): 0 \leqq t \leqq 1\}$ is a normal process with all sample paths continuous, $E(U(t)) = 0$ for $0 \leqq t \leqq 1$ and the covariance function of the U process is

$$s \wedge t - st .$$

Note that $V \equiv -U$ is also a Brownian bridge.

For $n \geqq 1$ we define the "uniform empirical process" U_n by

$$U_n(t) = n^{\frac{1}{2}}[\Gamma_n(t) - t] \qquad \text{for} \quad 0 \leqq t \leqq 1 . \tag{A1}$$

We also define the "uniform quantile process" V_n by

$$V_n(t) = n^{\frac{1}{2}}[\Gamma_n^{-1}(t) - t] \qquad \text{for} \quad 0 \leqq t \leqq 1 \tag{A2}$$

(the inverse of a df will always be the left continuous one).

For functions f_1, f_2 on (0, 1) let $\rho(f_1, f_2) = \sup_{0<t<1} |f_1(t) - f_2(t)|$. The special property referred to is

(A3) $$\rho(U_n, U) \to_e 0 \qquad \text{as} \quad n \to \infty$$

and

(A4) $$\rho(V_n, V) \to_e 0 \qquad \text{as} \quad n \to \infty .$$

Equation (A3) says that *every* sample path of the U_n process converges uniformly as $n \to \infty$ to the corresponding sample path of the U process.

That the U_n's exist was shown in Pyke and Shorack (1968). The ξ_i's are introduced here in the hope of making this approach more easily understood. Lemmas A1, A2 and A3 and Theorem A1 were proved or referenced in [3].

Though supressed in the notation, the ξ_i's are actually a triangular array.

A2. The basic convention. Let $Z_1, \cdots, Z_n$ be independent rv's having df F in the class $\mathscr{F}$ of all df's. Let $\mathbb{F}_n$ denote their empirical df. Let

(A5) $$n^{\frac{1}{2}}(\mathbb{F}_n - F) \qquad \text{on} \quad (-\infty, \infty)$$

be the "empirical process"; and let

(A6) $$n^{\frac{1}{2}}(\mathbb{F}_n^{-1} - F^{-1}) \qquad \text{on} \quad (0, 1)$$

be the "quantile process."

PROPOSITION A1 (*The inverse transformation*). *Let ξ be any Uniform* (0, 1) rv. *For an arbitrary F in $\mathscr{F}$ define $X = F^{-1}(\xi)$. Then for all real x we have*

(A7) $$[X \leqq x] = [\xi \leqq F(x)] .$$

Also, *X has* df *F.*

For $1 \leqq i \leqq n$ define $X_i = F^{-1}(\xi_i)$, where the ξ_i's are our special Uniform (0, 1) rv's. Then $X_1, \cdots, X_n$ are independent rv's having df F. Moreover, from (A7) we see that the X_i's have empirical process

(A8) $$U_n(F) \qquad \text{on} \quad (-\infty, \infty)$$

and quantile process

(A9) $$n^{\frac{1}{2}}[F^{-1}(\Gamma_n^{-1}) - F^{-1}] \qquad \text{on} \quad (0, 1) .$$

Thus the processes (A8) and (A9) have the same finite dimensional distributions as do the processes (A5) and (A6).

Also, a statistic $T_n = T_n(Z_1, \cdots, Z_n)$ has the same distribution as does the statistic $T_n = T_n(X_1, \cdots, X_n)$. We can use the second representation in attempts to show T_n is asymptotically normal; and if the statistics we consider can be represented as functionals on certain stochastic processes, (A3) and (A4) will prove fundamental in our proofs.

Because of the above correspondence, *we will simply identify the* X_i's *with the* Z_i's. With this convention we then have

$$n^{\frac{1}{2}}(\mathbb{F}_n - F) = U_n(F) \quad \text{and} \quad \mathbb{F}_n^{-1} = F^{-1}(\Gamma_n^{-1})\,. \tag{A10}$$

Also, the i^{th} order statistic X_{ni} is equal to $F^{-1}(\xi_{ni})$ for $1 \leqq i \leqq n$. Note that all random quantities are now defined on Ω.

Let I denote the identity function and let $\int \cdot\, dI$ denote an integral with respect to Lebesgue measure. We write $\rightarrow_d$ to denote convergence in distribution. Let $\chi(S)$ denote the indicator function of the set S. For function f on $(0, 1)$ and a signed measure ν on $(0, 1)$ with total variation measure $|\nu|$ define

$$||f||_\nu = \int_0^1 |f| d|\nu|\,.$$

A3. The q functions.

DEFINITION A1. Let $\mathscr{Q}(\nearrow)$ denote the class of positive, strictly increasing continuous functions q on $[0, 1]$ for which $\int_0^1 q^{-2}\, dI < \infty$. Let $\mathscr{Q}$ denote the class of all q such that $q(t) = q(1-t) = \bar{q}(t)$ for $0 \leqq t \leqq \frac{1}{2}$ and some $\bar{q}$ in $\mathscr{Q}(\nearrow)$. Let $\mathscr{Q}(\swarrow)$ denote the class of all q such that $q(1-I)$ is in $\mathscr{Q}(\nearrow)$.

For functions f_1, f_2 and q on $(0, 1)$ we let

$$\rho_q(f_1, f_2) = \sup_{0<t<1} |(f_1(t) - f_2(t))/q(t)|\,. \tag{A11}$$

REMARK A1. By far the most important members of $\mathscr{Q}$ are $[I(1-I)]^{\frac{1}{2}-\delta}$ for $\delta > 0$. A function q having $q(t) = q(1-t) = -t^{\frac{1}{2}} \log t$ for $0 \leqq t \leqq e^{-2}$ is also in $\mathscr{Q}$.

REMARK A2. For all q in $\mathscr{Q}$ we have

$$[n^{\frac{1}{2}}q(1/n)]^{-1} \rightarrow 0 \qquad \text{as} \quad n \rightarrow \infty\,,$$

since $q^{-2}(1/n)/n \leqq \int_0^{1/n} q^{-2}\, dI$.

A4. Some properties.

PROPOSITION A2 (Glivenko-Cantelli Lemma). *As* $n \rightarrow \infty$ *we have* $\rho(\mathbb{F}_n, F) \rightarrow_e 0$ *uniformly in all* F *in* $\mathscr{F}$. *Also*

$$\rho(\Gamma_n^{-1}, I) = \rho(\Gamma_n, I) \rightarrow_e 0 \qquad \textit{as} \quad n \rightarrow \infty\,.$$

LEMMA A1. *Let* q *in* $\mathscr{Q}(\nearrow)$. *Then*

$$P(|U_n(t)| \leqq q(t) \quad \text{for} \quad 0 \leqq t \leqq \theta) \geqq 1 - \int_0^\theta q^{-2}\, dI$$

for $n \geqq 0$ *where* $U_0 \equiv U$.

REMARK A3. $\rho_q(U, 0) = O_p(1)$ for any q in $\mathscr{Q}$.

DEFINITION A2. We will say that "* restricts function f on [0, 1] (or (0, 1) etc.) to the interval $[a, b]$ (or $(a, b]$ etc.)" if we mean $f^*(t)$ equals $f(t)$ for $a \leqq t \leqq b$ and equals 0 otherwise. We write $\int^* f\,dI$ for $\int_0^1 f^*\,dI$.

Let N denote an integer and let $n = n(N)$ be a sequence of integers such that $n \to \infty$ as $N \to \infty$. Let $\{K_n(t) : 0 \leqq t \leqq 1\}$ be a process on Ω for which

(i) For any $\varepsilon > 0$ there exists $\beta = \beta_\varepsilon > 0$ such that

$$P(K_N(t) \leqq \beta t \quad \text{for all} \quad t \geqq 1/N) > 1 - \varepsilon \quad \text{and}$$

(ii) $\rho(K_N, K) \to_{\text{a.s.}} 0$ as $N \to \infty$ where K is a positive, increasing continuous function on [0, 1] having $K \leqq MI$ for some $M > 0$.

LEMMA A2. *If* (i) *and* (ii) *hold, then for any* q in $\mathscr{Q}(\nearrow)$ *we have*

$$\rho_q(U_n(K_N)^*, U(K)) \to_p 0 \qquad as \quad N \to \infty\,;$$

where * *restricts functions on* [0, 1] *to* $[1/N, 1]$.

LEMMA A3. *Given* $\varepsilon > 0$ *there exists* $0 < \beta = \beta_\varepsilon < 1$ *and a subset* $S_{n,\varepsilon}$ *of* Ω *having* $P(S_{n,\varepsilon}) > 1 - \varepsilon$ *on which*

$$\begin{aligned}
1 - (1-t)/\beta \leqq \Gamma_n(t) &\leqq t/\beta && \text{for } 0 \leqq t \leqq 1\,,\\
\beta t &\leqq \Gamma_n(t) && \text{for all } t \text{ such that } 0 < \Gamma_n(t)\\
\Gamma_n(t) &\leqq 1 - \beta(1-t) && \text{for all } t \text{ such that } \Gamma_n(t) < 1\,,\\
\beta t \leqq \Gamma_n^{-1}(t) &\leqq 1 - \beta(1-t) && \text{for } 0 \leqq t \leqq 1\,,\\
\Gamma_n^{-1}(t) &\leqq t/\beta && \text{for } t \geqq 1/n \quad \text{and}\\
1 - (1-t)/\beta &\leqq \Gamma_n^{-1}(t) && \text{for } t \leqq 1 - 1/n\,.
\end{aligned}$$

We may also require that for all n exceeding some n_ε we have

$$|\Gamma_n - I| \leqq \beta \quad \text{and} \quad |\Gamma_n^{-1} - I| \leqq \beta \qquad \text{on} \quad [0, 1]$$

provided ω is in $S_{n,\varepsilon}$.

REMARK A4. On the set $S_{n,\varepsilon}$ of the previous lemma we have for $n \geqq n_\varepsilon$ and for all s between t and $\Gamma_n^{-1}(t)$ that $\beta t \leqq s \leqq t + \beta$ for $t \leqq \frac{1}{2}$ and $\beta(1-t) \leqq 1 - s \leqq (1-t) + \beta$ for $t \leqq \frac{1}{2}$.

THEOREM A1 (ρ_q-convergence of U_n and V_n). *For any* q *in* $\mathscr{Q}$ *we have*

$$\rho_q(U_n, U) \to_p 0 \quad and \quad \rho_q(V_n^*, V) \to_p 0 \qquad as \quad n \to \infty$$

where * *restricts functions on* [0, 1] *to* $[1/n, 1 - 1/n]$.

REMARK A5. $\rho_q(U_n, 0) = O_p(1)$ and $\rho_q(V_n^*, 0) = O_p(1)$ for any q in $\mathscr{Q}$.

LEMMA A4. *For all* q *in* $\mathscr{Q}$ *there exists a continuous function* ζ *on* [0, 1] *such that* $\zeta(0) = 0$ *and* ζq *is in* $\mathscr{Q}$.

PROOF. For example, when $q = [I(1-I)]^{\frac{1}{2}-\delta}$, let $\zeta = [I(1-I)]^{\delta/2}$.

It suffices to show that if f is a continuous, strictly positive, strictly decreasing function on (0, 1) such that $\int_0^1 f\,dI < \infty$, then there exists a positive, continuous decreasing function ϕ on (0, 1) such that $\phi(t) \to \infty$ as $t \to 0$ and $\int_0^1 f\phi\,dI < \infty$.

Let $a_n = \int_{1/(n+1)}^{1/n} f\,dI$ and let $\phi_n = \phi(1/n)$ for any fixed positive, continuous, decreasing function ϕ on (0, 1) such that $\phi(t) \to \infty$ as $t \to 0$. If $b_n = \int_{1/(n+1)}^{1/n} f\phi\,dI$, then $b_n \leqq \phi_{n+1} a_n$ for all n. It thus suffices to show that if $\sum_1^\infty a_n$ is any convergent series, then there exists a sequence $c_n \to \infty$ such that $\sum_1^\infty c_n a_n < \infty$.

Let $\varepsilon_m = a^{2m}$ with $0 < a < 1$. Choose a strictly increasing sequence n_m such that $\sum_{n_m}^\infty a_k < \varepsilon_m$ for all m. Let $c_k = \varepsilon_m^{-\frac{1}{2}}$ for $n_m \leqq k < n_{m+1}$ and $m \geqq 1$ with $c_1 = \cdots = c_{n_1 - 1} = 0$. Then

$$\sum_1^\infty c_k a_k = \sum_{m=1}^\infty \sum_{k=n_m}^{n_{m+1}-1} c_k a_k \leqq \sum_{m=1}^\infty \varepsilon_m^{-\frac{1}{2}} \sum_{k=n_m}^\infty a_k$$
$$\leqq \sum_{m=1}^\infty \varepsilon_m^{\frac{1}{2}} = \sum_1^\infty a^m < \infty .$$

This proof was suggested by Jap Fabius. □

A5. Reproducing *u*-shaped function.

DEFINITION A3. A positive function R on (0, 1) will be called "u-shaped" if for some $0 < a < 1$ the function is decreasing on $(0, a]$ and increasing on $(a, 1)$. We introduce the notation R_β for

$$R_\beta = R(\beta t) \qquad \text{for} \quad 0 < t \leqq \tfrac{1}{2}$$
$$= R(1 - \beta(1 - t)) \qquad \text{for} \quad \tfrac{1}{2} < t < 1 .$$

If for all β in a neighborhood of 0 there exists a constant M_β such that $R_\beta \leqq M_\beta R$ on (0, 1), then R will be called a "reproducing u-shaped function."

For any $\theta \geqq 0$ the function $R = [I(1 - I)]^{-\theta}$ is a reproducing u-shaped function.

In certain problems there is a natural asymmetry.

DEFINITION A4. For increasing (decreasing) functions R on (0, 1) we define $R_\beta = R(1 - \beta(1 - t))$ $(= R(\beta t))$. If for all β in a neighborhood of 0 we have $R_\beta \leqq M_\beta R$ on (0, 1) for some M_β, then R will be called a "reproducing increasing (decreasing) function."

Coupled with Lemma A3 this concept yields a powerful tool.

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