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A NOTE ON INFINITELY DIVISIBLE RANDOM VECTORS

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The purpose of this note is to show a characterization of infinitely divisible random vectors whose projection onto a subspace is normally distributed.

In this paper, $\mathbf{x}$ will denote a k-dimensional vector of real numbers, $\mathbf{x}'$ its transpose, $|\mathbf{x}|$ its length, $\boldsymbol{\xi}'\mathbf{x}$ the inner product of the vectors $\boldsymbol{\xi}$, $\mathbf{x}$, R^k the k-dimensional Euclidean space. By a normal distribution we understand either a proper or a singular normal distribution.

A nonnegative measure M defined on R^k is said to be canonical if it has no atom at the origin and $\int_{R^k} [1 + |\mathbf{x}|^2]^{-1}\, dM(\mathbf{x}) < \infty$.

For $\mathbf{X}$ to be an infinitely divisible random vector, it is necessary and sufficient that there exist a canonical measure M, a vector of real numbers $\mathbf{b}$, and a nonnegative definite symmetric matrix $\sum = (\sigma_{ij})$ such that

$$(1) \qquad \log Ee^{i\boldsymbol{\xi}'\mathbf{X}} = \int_{R^k} \frac{e^{i\boldsymbol{\xi}'\mathbf{x}} - 1 - i\boldsymbol{\xi}'\boldsymbol{\tau}(\mathbf{x})}{|\mathbf{x}|^2}\, dM(\mathbf{x}) + i\boldsymbol{\xi}'\mathbf{b} - \tfrac{1}{2}\boldsymbol{\xi}' \textstyle\sum \boldsymbol{\xi}$$

where

$$\begin{aligned} \tau(x) &= -1 & x &\leqq -1 \\ &= x & -1 &\leqq x \leqq +1\,, \qquad [\boldsymbol{\tau}(\mathbf{x})]' = (\tau(x_1), \cdots, \tau(x_k))\,. \\ &= 1 & 1 &\leqq x \end{aligned}$$

This representation is unique (see [1] page 559).

Notice that if $\mathbf{X}$ is normal, $\log Ee^{i\boldsymbol{\xi}'\mathbf{X}} = i\boldsymbol{\xi}'\mathbf{b} - \frac{1}{2}\boldsymbol{\xi}' \sum \boldsymbol{\xi}$, and thus for normally distributed random vectors M is identically zero.

THEOREM. *Let* $\mathbf{X}' = (X_1, \cdots, X_k)$ *be an infinitely divisible random vector, and* $\boldsymbol{\xi}_0 \neq \mathbf{0}$ *an arbitrary vector in k-space. The distribution of* $\boldsymbol{\xi}_0'\mathbf{X}$ *is one-dimensional normal if and only if the canonical measure* M *which corresponds to* $\mathbf{X}$ *in the representation* (1) *is carried by the subspace* $\{\mathbf{x} \mid \boldsymbol{\xi}_0'\mathbf{x} = 0\}$.

PROOF. We need only prove that if $\boldsymbol{\xi}_0'\mathbf{X}$ is normal, then M is carried by $\{\mathbf{x} \mid \boldsymbol{\xi}_0'\mathbf{x} = 0\}$.

Since for every infinitely divisible random vector $\mathbf{X}$ and every matrix A, $A\mathbf{X}$ is infinitely divisible, we may assume without loss of generality that $\boldsymbol{\xi}_0' = (1, 0, \cdots, 0)$. Then:

$$(2) \qquad \log Ee^{it\boldsymbol{\xi}_0'\mathbf{X}} = ib_1 t - \tfrac{1}{2}t^2\sigma_{11} + \int_{R^k} \frac{e^{itx_1} - 1 - it\tau(x_1)}{|\mathbf{x}|^2}\, dM(\mathbf{x})$$

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$$(3)\qquad \int_{R^k} \frac{e^{itx_1} - 1 - it\tau(x_1)}{|\mathbf{x}|^2} dM(\mathbf{x}) = \int_{R^k} \frac{e^{itx_1} - 1 - it\tau(x_1)}{x_1^2} \frac{x_1^2}{|\mathbf{x}|^2} dM(\mathbf{x})$$
$$= \int_{R^1} \frac{e^{itx_1} - 1 - it\tau(x_1)}{x_1^2} dM_1(x_1)$$

where

$$(4)\qquad dM_1(x_1) = \int_{x_1 \times R^{k-1}} \frac{x_1^2}{|\mathbf{x}|^2} dM(\mathbf{x}) .$$

Clearly,

$$\int_{R^1} \frac{1}{1+x_1^2} dM_1(x_1) = \int_{R^k} \frac{x_1^2}{1+x_1^2} \frac{1}{|\mathbf{x}|^2} dM(\mathbf{x})$$
$$\leqq M\{\mathbf{x} \mid \max_{1 \leqq i \leqq k} |x_i| \leqq 1\}$$
$$+ \int_{R^k - \{\mathbf{x} \mid \max_{1 \leqq i \leqq k} |x_i| \leqq 1\}} \frac{1}{|\mathbf{x}|^2} dM(\mathbf{x})$$
$$< \infty .$$

Define:

$$(5)\qquad M_2(S) = M_1(S - \{0\}) , \qquad S \text{ being any measurable subset of } R^1.$$

Thus, M_2 is a canonical measure defined on R^1. From (2) and (3) we have:

$$\log Ee^{it\xi_0'\mathbf{x}} = ib_1 t - \tfrac{1}{2}\sigma_{11} t^2 + \int_{R^1} \frac{e^{itx_1} - 1 - it\tau(x_1)}{x_1^2} dM_1(x_1)$$
$$= ib_1 t - \tfrac{1}{2}(\sigma_{11} + M_1(0))t^2 + \int_{R^1} \frac{e^{itx_1} - 1 - it\tau(x_1)}{x_1^2} dM_2(x_1) .$$

Now, if $\xi_0'\mathbf{X}$ is normal, it follows from the uniqueness of (1) that M_2 is identically zero; and thus from (5) and (4) it follows that for $x_1 \neq 0$, $dM(\mathbf{x}) = 0$. Or: M is carried by the set $\{\mathbf{x} \mid 0 = (1, 0, \cdots, 0)\mathbf{x}\}$. □

COROLLARY 1. *Let* $\mathbf{X}$ *be an infinitely divisible random k-dimensional vector. Then there exists a (trivial or non-trivial) linear subspace L of* R^k, *such that every projection of* $\mathbf{X}$ *onto a linear subspace of L has a normal distribution, and every projection of* $\mathbf{X}$ *onto a linear subspace in* $R^k - L$ *is not normally distributed.*

PROOF. This follows when one takes the orthogonal complement of L to be the minimal subspace which carries the measure M.

Note. It follows from Corollary 1 that an infinitely divisible random vector is normally distributed if and only if its coordinates are normally distributed.

The array $\{\mathbf{X}_{i,n}\}_{i=1,\cdots,n;n=1,2,\cdots}$ is said to be a null triangular array if for each n, $\mathbf{X}_{1,n} \cdots \mathbf{X}_{n,n}$ are independent, and for every $\varepsilon > 0, \max_{1 \leqq i \leqq n} \mathrm{Prob}(|\mathbf{X}_{i,n}| > \varepsilon) \to_{n\to\infty} 0$.

Exactly as in the one-dimensional case, a limit distribution of the sum

$\mathbf{S}_n = \sum_{i=1}^n \mathbf{X}_{i,n}$ is necessarily infinitely divisible. Combining this with the previous remark, we obtain:

COROLLARY 2. *Let* $\{\mathbf{X}_{i,n}\}_{i=1,\cdots,n;n=1,2,\cdots} = \{(X_{i,n}^{(1)}, \cdots, X_{i,n}^{(k)})\}_{i=1,\cdots,n;n=1,2,\cdots}$ *be a null triangular array, such that the distribution of* $\mathbf{S}_n = \sum_{i=1}^n \mathbf{X}_{i,n}$ *tend to the distribution of a random vector* $\mathbf{U}$. *A necessary and sufficient condition for* $\mathbf{U}$ *to be (k-dimensional) normal is that the distributions of the sums* $\sum_{i=1}^n X_{i,n}^{(j)}$, $(j = 1, \cdots, k)$, *tend to a normal (one-dimensional) distribution.*

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REFERENCE

[1] FELLER, WILLIAM (1966). *An Introduction to Probability Theory and its Applications* **2.** Wiley, New York.