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THE OCCUPATION TIME OF A SET BY COUNTABLY MANY RECURRENT RANDOM WALKS

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Let $A_0(x)$, $x \in Z$ be independent nonnegative integer-valued random variables with $\mu_j(x) = E(A_0(x)(A_0(x) - 1) \cdots (A_0(x) - j + 1))$. Assume that $\{\mu_j(x)\}_x$ has limits for $j = 1, 2$ and that it is bounded for $3 \leqq j \leqq 6$. Suppose at time zero there are $A_0(x)$ particles at $x \in Z$ and subsequently the particles move independently according to the transition function $P(x, y)$ of a recurrent random walk. For a finite nonempty subset B of Z denote by $A_n(B)$ the number of particles in B at time n. Then $S_n(B) = \sum_{k=1}^n A_k(B)$ is the total occupation time of B by time n of all particles.

Assuming that the n step transition function $P_n(x, y)$ is such that there is an α, with $1 < \alpha \leqq 2$, so that $P_n(0, x) \sim cn^{-1/\alpha}$ for all x, it is proved that the strong law of large numbers and the central limit theorem hold for the sequence $\{S_n(B)\}$.

1. Introduction. Suppose at time zero that $A_0(x)$ particles are placed at $x \in Z$ and then the particles move independently according to some transition law. Let $A_n(x)$ denote the number of particles in x at time n and $A_n(B) = \sum_{x \in B} A_n(x)$ for a finite nonempty set B. Also let $S_n(B) = \sum_{k=1}^n A_k(B)$—the total occupation time of B by time n.

Assuming that the random variables $A_0(x)$, $x \in Z$ are *independent Poisson variables* with means $\mu(x)$, $x \in Z$ and that the particles move independently according to the transition function $P(x, y)$ of a Markov chain which has $\mu(x)$ as an invariant measure (*i.e.* $\sum_x \mu(x)P(x, y) = \mu(y)$ for all y). Derman, in [1], proved that the system maintains equilibrium in the sense that at any time n, $A_n(x)$, $x \in Z$ are independent Poisson variables with means $\mu(x)$, $x \in Z$. In [4] Port showed that if $P(x, y)$ is the transition function of a transient chain then

$$P(\lim_{n\to\infty} S_n(B)/n = \mu(B)) = 1\,, \tag{1.1}$$

$$[S_n(B) - n\mu(B)]/[\operatorname{Var} S_n(B)]^{\frac{1}{2}} \to_D \Phi \tag{1.2}$$

where Φ is the standard normal distribution. (1.1) shows that the number of particles per unit time in B is $\mu(B) = \sum_{x \in B} \mu(x)$ and (1.2) reveals the fact that the total occupation time of B is asymptotically normally distributed. If the Markov chain is null recurrent (e.g. a recurrent random walk) then results in [5] show that (1.1) and (1.2) also hold in this case.

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In [8] the author showed that the Poisson assumption can be dropped and (1.1) and (1.2) will still hold if it is assumed that $P(x, y)$ is the transition function of a *transient aperiodic random walk*. In this case we need only assume that $A_0(x)$, $x \in Z$ are independent and identically distributed random variables with finite fourth moments (here we take $\mu(x) \equiv$ constant). It should be noted that the proofs in this case differ greatly from those in the Poisson case.

The purpose of this paper is to establish (1.1) and (1.2) for the infinite particle system for *recurrent random walks* without the Poisson hypothesis. We assume that $A_0(x)$, $x \in Z$ are independent nonnegative integer-valued random variables with finite sixth moments and that there are constants $\lambda > 0$, ν, and M such that

$$(1.3)\qquad \begin{aligned} &\text{i)}\ \mu_1(x) \to \lambda \quad \text{as} \quad |x| \to \infty\,, \\ &\text{ii)}\ \mu_2(x) \to \nu \quad \text{as} \quad |x| \to \infty\,, \\ &\text{iii)}\ \mu_j(x) \leqq M\,, \qquad 1 \leqq j \leqq 6\,, \qquad x \in Z \end{aligned}$$

where $\mu_j(x) = E(A_0(x)(A_0(x) - 1) \cdots (A_0(x) - j + 1))$. Note that (1.3) is satisfied if $A_0(x)$, $x \in Z$ are independent and identically distributed random variables with finite sixth moments.

2. Notation and preliminaries. Let X_n, $n \geqq 0$ be independent integer-valued random variables and assume that for $n \geqq 1$ the variables are identically distributed. Then the process $\{Y_n; n \geqq 0\}$ defined by $Y_n = X_0 + X_1 + \cdots + X_n$ is called a random walk. This process is a Markov chain with n-step transition function given by $P_n(x, y) = P(X_1 + \cdots + X_n = y - x)$. $P(x, y) \equiv P_1(x, y)$ is called the transition function of the random walk. Let $F_n(x, y) = P(Y_n = y;\ Y_\nu \neq y,\ 1 \leqq \nu \leqq n - 1 \mid Y_0 = x)$. Then the random walk is said to be recurrent if $\sum_{n=1}^{\infty} F_n(0, 0) = 1$ and is said to be transient otherwise. The random walk is said to be aperiodic if the group generated by the set $\Theta = \{x\colon P(0, x) > 0\}$ is the group of all integers. It is said to be strongly aperiodic if for all $x \in Z$, the group generated by $x + \Theta$ is the group of all integers. If B is a finite nonempty subset of the integers and $P_n(x, y)$ is as above, we will use the following notation:

$$\begin{aligned} P_n(x, B) &= \textstyle\sum_{y \in B} P_n(x, y)\,, \\ G_n(x, B) &= \textstyle\sum_{k=1}^{n} P_k(x, B)\,, \\ H_n(x, B) &= 1 + G_n(x, B)\,. \end{aligned}$$

Also define $1_B(x) = 1$ if $x \in B$ and $= 0$ if $x \notin B$. Finally, let Z denote the integers, N denote the positive integers, and let $|B|$ be the cardinality of B.

3. Statement of results. Suppose that $A_0(x)$, $x \in Z$ are independent random variables satisfying (i)—(iii) of Section 1 and that at time zero there are $A_0(x)$ particles at $x \in Z$. Assume the particles then move independently according to the same transition function $P(x, y)$ of a recurrent random walk. More precisely, suppose the random variables $X_{nx}^{(k)}$, for n, $k \in N$, $x \in Z$ are independent and identically distributed with $P(X_{nx}^{(k)} = y) = P(0, y)$. Also suppose that the random variables $A_0(x)$ and $X_{nx}^{(k)}$ are independent. Then the process $\{Y_{nz}^{(k)}\}_{n=0}^{\infty}$ defined by $Y_{0z}^{(k)} = z$ and $Y_{nz}^{(k)} = z + X_{1z}^{(k)} + \cdots + X_{nz}^{(k)}$ for $n \geqq 1$ is a random walk with transition function $P(x, y)$ and represents the position of the kth particle starting at z at time n.

The following assumption is made on the n-step transition function $P_n(x, y)$ which regulates the movement of the particles: There is an α with $1 < \alpha \leqq 2$ such that for all $x \in Z$

$$P_n(0, x) \sim cn^{-1/\alpha} \tag{3.1}$$

where c is a positive constant.

REMARKS. Let $\{\xi_n\}_{n=1}^{\infty}$ be a sequence of independent and identically distributed random variables and let $F(t) = P(\xi_1 \leqq t)$. The law $F(t)$ is said to belong to the domain of normal attraction of the stable law $V(t)$, if for some $a > 0$ and some A_n,

$$\lim_{n\to\infty} P\left(\frac{1}{an^{1/\alpha}} \sum_{k=1}^{n} \xi_k - A_n \leqq x\right) = V(x) .$$

Here α is the exponent of the stable law V.

Now if the recurrent random walk with transition function $P(x, y)$ is strongly aperiodic and if $F(t) = \sum_{x \leq t} P(0, x)$ is in the domain of normal attraction of a stable law with exponent α, $1 < \alpha < 2$ then it follows (see [3], page 236) that (3.1) holds. Also, for any strongly aperiodic recurrent random walk with $\sigma^2 = \sum_x x^2 P(0, x) < \infty$ we have (see [6], page 75) $P_n(0, x) \sim cn^{-\frac{1}{2}}$ where $c = 1/\sigma(2\pi)^{\frac{1}{2}}$. Hence (3.1) is also valid in this case with $\alpha = 2$.

Finally, we mention that results in [5] along with some theorems in [7] make it plausible that (1.1) and (1.2) hold for arbitrary recurrent random walk.

Throughout, B will denote a finite nonempty subset of Z. Our first result is the strong law of large numbers for the quantity $S_n(B)$.

THEOREM 1. *Let $S_n(B)$ be the total occupation time of B by time n of all particles. Then with probability one*

$$\lim_{n\to\infty} S_n(B)/n = \lambda|B| . \tag{3.2}$$

The next theorem shows that the total occupation time of B is asymptotically normally distributed.

THEOREM 2. *Let the notation be as above and suppose that*

$$\liminf_{n\to\infty} \operatorname{Var} S_n(B)/n^{2-1/\alpha} > 0 . \tag{3.3}$$

Then for any $u \in R$,

$$\lim_{n\to\infty} P\left[\frac{S_n(B) - ES_n(B)}{[\text{Var } S_n(B)]^{\frac{1}{2}}} \leqq u\right] = \Phi(u) \tag{3.4}$$

where Φ is the normal distribution function with mean zero and variance one.

We conclude by showing that (3.3) will hold whenever the initial distribution of particles is a deterministic constant.

THEOREM 3. *Suppose that the initial distribution of particles is $A_0(x) = \lambda > 0$ with probability one for all $x \in Z$. Then*

$$\liminf_{n\to\infty} \text{Var } S_n(B)/n^{2-1/\alpha} \geqq \rho\lambda|B|^2 \tag{3.5}$$

where $\rho = c[\alpha(2 - \alpha^{-1})^{-1}(1 - \alpha^{-1})^{-1}]^{-1}$.

Now, for each $x \in Z$, let $\{Y_{nx}\}$ be a random walk with $Y_{0x} = x$ and n-step transition function $P(x, y)$. Let $N_{nx}(B) = \sum_{k=1}^{n} 1_B(Y_{kx})$. Then $EN_{nx}(B) = \sum_{k=1}^{n} P_k(x, B) = G_n(x, B)$. Also, denote by $Z_{nx}^{(k)}(B)$ the total occupation time of B by time n of the kth particle starting at x at time zero. Then $Z_{nx}^{(k)}(B)$ $k \in N$, $x \in Z$ are independent for each $n \in N$ and are distributed as $N_{nx}(B)$. If $S_{nx}(B)$ is the occupation time by time n of the particles starting at x then $S_{nx}(B)$, $x \in Z$ are independent random variables for each $n \in N$ and

$$S_{nx}(B) = \sum_{k=1}^{A_0(x)} Z_{nx}^{(k)}(B) \qquad (= 0 \quad \text{if} \quad A_0(x) = 0)\,, \tag{3.6}$$

$$S_n(B) = \sum_x S_{nx}(B)\,. \tag{3.7}$$

Next we exhibit the first four moments of $S_{nx}(B)$. From the assumptions it is clear that $E(S_{nx}(B) \mid A_0(x) = m) = mEN_{nx}(B)$ and consequently

$$ES_{nx}(B) = \mu_1(x)EN_{nx}(B)\,. \tag{3.8}$$

Similar arguments give

$$ES_{nx}(B)^2 = \mu_1(x)EN_{nx}(B)^2 + \mu_2(x)[EN_{nx}(B)]^2\,, \tag{3.9}$$

$$\begin{aligned} ES_{nx}(B)^3 = {} & \mu_1(x)EN_{nx}(B)^3 + 3\mu_2(x)EN_{nx}(B)^2EN_{nx}(B) \\ & + \mu_3(x)[EN_{nx}(B)]^3\,, \end{aligned} \tag{3.10}$$

$$\begin{aligned} ES_{nx}(B)^4 = {} & \mu_1(x)EN_{nx}(B)^4 + 3\mu_2(x)[EN_{nx}(B)^2]^2 \\ & + 4\mu_2(x)EN_{nx}(B)^3EN_{nx}(B) \\ & + 6\mu_3(x)EN_{nx}(B)^2[EN_{nx}(B)]^2 \\ & + \mu_4(x)[EN_{nx}(B)]^4\,. \end{aligned} \tag{3.11}$$

Using (3.7)—(3.9) and the independence of $\{S_{nx}(B)\}_x$ we obtain for each $n \in N$

$$\begin{aligned} \text{Var } S_n(B) = {} & \sum_x \mu_1(x)EN_{nx}(B)^2 \\ & + \sum_x (\mu_2(x) - \mu_1(x)^2)[EN_{nx}(B)]^2\,. \end{aligned} \tag{3.12}$$

Note that in the Poisson case $\mu_2(x) = \mu_1(x)^2$ so that the second term on the right in (3.12) does not arise in that case.

4. Proof of Theorem 1. The proof of the strong law of large numbers for $S_n(B)$ is obtained by using a sixth moment argument. In order to apply this argument it is necessary to establish several results concerning the asymptotic behavior of such quantities as $\sum_x E|S_{nx}(B) - ES_{nx}(B)|^3$. Many of these results require lengthy calculations and computations for verification and so most of the details will be omitted. In order to indicate the general ideas involved, however, we will prove the following lemma in some amount of detail. Before beginning, note that it follows from (3.1) that

$$H_n(0, 0) = O(n^{1-1/\alpha}) . \tag{4.1}$$

For brevity write $H_n = H_n(0, 0)$ and recall that for any random walk we have for all $x, y \in Z$ and $n \in N$ that $G_n(x, y) \leqq H_n$. Hence for all $n \in N$, $x \in Z$

$$EN_{nx}(B) = G_n(x, B) \leqq |B|H_n . \tag{4.2}$$

LEMMA 1. *Let the notation be as above. Then*

$$\operatorname{Var} S_n(B) = O(n^{2-1/\alpha}) , \tag{4.3}$$

$$\textstyle\sum_x E|S_{nx}(B) - ES_{nx}(B)|^3 = O(n^{3-2/\alpha}) . \tag{4.4}$$

PROOF. To prove (4.3) first note that the independence of the random variables $\{S_{nx}(B)\}_x$ implies $\operatorname{Var} S_n(B) = \sum_x \operatorname{Var} S_{nx}(B)$. Also we get from (1.3), (3.7), and (3.8) that $\sum_x \operatorname{Var} S_{nx}(B) = O(\sum_x EN_{nx}(B)^2 + \sum_x [EN_{nx}(B)]^2)$. Since $N_{nx}(B)^2 = N_{nx}(B) + 2\sum_{i<j} 1_B(Y_{ix})1_B(Y_{jx})$ we have $EN_{nx}(B)^2 = EN_{nx}(B) + 2\sum_{y\in B}\sum_{i<j} P_i(x, y)P_{j-i}(y, B)$ and so

$$\textstyle\sum_x EN_{nx}(B)^2 = n|B| + 2\sum_{y\in B}\sum_{i<j} P_{j-i}(y, B) . \tag{4.5}$$

Since $\alpha > 1$ it follows that $n = O(n^{2-1/\alpha})$. Noting that $\sum_{i<j} P_{j-i}(y, B) = \sum_{i=1}^{n-1} G_i(y, B) \leqq |B| \sum_{i=1}^{n-1} H_n$ and applying (4.1) we see that $\sum_{i<j} P_{j-i}(y, B) = O(n^{2-1/\alpha})$. Hence we conclude that

$$\textstyle\sum_x EN_{nx}(B)^2 = O(n^{2-1/\alpha}) . \tag{4.6}$$

By (4.2), $[EN_{nx}(B)]^2 \leqq |B|H_n EN_{nx}(B)$ and thus by (4.1) and the fact that $\sum_x EN_{nx}(B) = n|B|$ it follows that

$$\textstyle\sum_x [EN_{nx}(B)]^2 = O(n^{2-1/\alpha}) . \tag{4.7}$$

The validity of (4.3) now follows from (4.6) and (4.7). To establish (4.4) first notice that $\sum_x E|S_{nx}(B) - ES_{nx}(B)|^3 = O(\sum_x EN_{nx}(B)^3 + \sum_x EN_{nx}(B)^2 EN_{nx}(B) + \sum_x [EN_{nx}(B)]^3)$. We will show that each of the three terms above is $O(n^{3-2/\alpha})$. Arguments as above give $\sum_x EN_{nx}(B)^3 = O(n + \sum_{y\in B}\sum_{i<j} P_{j-i}(y, B) + \sum_{y,z\in B}\sum_{i<j<k} P_{j-i}(y, z)P_{k-j}(z, B))$, and some routine calculations show that the last term is dominated by $|B| \sum_{k=1}^{n-2} H_n{}^2 = O(n^{3-2/\alpha})$. This and previous results give

$$\textstyle\sum_x EN_{nx}(B)^3 = O(n^{3-2/\alpha}) . \tag{4.8}$$

To continue, note that $\sum_x EN_{nx}(B)^2 EN_{nx}(B) \leqq |B| H_n \sum_x EN_{nx}(B)^2$ and so (4.1) and (4.6) imply that $\sum_x EN_{nx}(B)^2 EN_{nx}(B) = O(n^{3-2/\alpha})$. Finally, note that $\sum_x [EN_{nx}(B)]^3 \leqq |B|^2 H_n^2 \sum_x EN_{nx}(B) = O(n^{3-2/\alpha})$ by (4.1) and the fact that $\sum_x EN_{nx}(B) = n|B|$. This completes the proof of the lemma.

The proof of Theorem 1 requires two more estimates similar to the ones in the previous lemma. The general idea of the proofs of the validity of these estimates is quite similar to that of the above proof. Consequently, we only sketch the verification of the following lemma.

LEMMA 2. *With the same notation as above we have*

$$\sum_x E[S_{nx}(B) - ES_{nx}(B)]^4 = O(n^{4-3/\alpha}) \tag{4.9}$$

and

$$\sum_x E[S_{nx}(B) - ES_{nx}(B)]^6 = O(n^{6-5/\alpha}) . \tag{4.10}$$

PROOF. Using (1.3) along with (3.8)—(3.11) it is not too hard to show that

$$\begin{aligned} \sum_x E[S_{nx}(B) - ES_{nx}(B)]^4 \\ = O(\sum_x \{EN_{nx}(B)^4 + [EN_{nx}(B)^2]^2 + EN_{nx}(B)^3 EN_{nx}(B) \\ + EN_{nx}(B)^2 [EN_{nx}(B)]^2 + [EN_{nx}(B)]^4\}) . \end{aligned} \tag{4.11}$$

Arguments similar to the ones above (see the previous lemma) show that for any positive integer m

$$\sum_{x_0, \cdots, x_{m-1} \in B} \sum_{i_0 < \cdots < i_{m-1}} \prod_{j=1}^{m-1} P_{i_j - i_{j-1}}(x_{j-1}, x_j) = O(n^{m-(m-1)/\alpha}) \tag{4.12}$$

and from this fact it follows that for any positive integer m and any finite non-empty subset B of Z

$$\sum_x EN_{nx}(B)^m = O(n^{m-(m-1)/\alpha}) . \tag{4.13}$$

Applying this fact along with the fact that $\sup_x EN_{nx}(B)^2 = O(n^{2-2/\alpha})$ to (4.11) yields (4.9).

To establish (4.10) it is only necessary to obtain analogous formulas to (3.8)–(3.11) for the fifth and sixth moments of $S_{nx}(B)$ and use (4.13) in a manner similar to the one above.

The next lemma gives the asymptotic behavior of $ES_n(B)$. In order to obtain this we need the following fact: Suppose for each $x \in Z$, $\{a_n(x)\}_n$ is a non-negative sequence of real numbers such that $\sum_x a_n(x) \to a$ as $n \to \infty$ and $a_n(x) \to 0$ as $n \to \infty$ for each $x \in Z$. Also suppose $\{b(x)\}_x$ is such that $b(x) \to b$ as $|x| \to \infty$. Then

$$\lim_{n \to \infty} \sum_x b(x) a_n(x) = ba . \tag{4.14}$$

With this fact in mind we prove the following:

LEMMA 3. *Let $S_n(B)$ be as before. Then*

(4.15) $$ES_n(B) \sim n\lambda|B| .$$

PROOF. We have $ES_n(B) = \sum_x ES_{nx}(B) = \sum_x \mu_1(x)EN_{nx}(B)$ by (3.8) But it is easy to see that $\sum_x \mu_1(x)EN_{nx}(B) = \sum_x \mu_1(x) \sum_{y\in B} \sum_{k=1}^n P_k(x, y)$. Thus

(4.16) $$ES_n(B) = \sum_{y\in B} \sum_{k=1}^n (\sum_x \mu_1(x)P_k(x, y)) .$$

Now for fixed $y \in B$ let $b(x) = \mu_1(x)$ and $a_n(x) = P_n(x, y)$. Then $b(x) \to \lambda$ as $|x| \to \infty$ by (3.1). Also, $a_n(x) \to 0$ as $n \to \infty$ for each x and $\sum_n a_n(x) = 1$ for all n by the spatial homogeneity of random walks. Thus (4.14) shows that $\sum_x \mu_1(x)P_k(x, y) \to \lambda$ as $n \to \infty$ for all $y \in Z$. This fact along with (4.16) establishes the lemma.

We are now in a position to prove the strong law of large numbers for $S_n(B)$:

For convenience let $A_{nx} = S_{nx}(B) - ES_{nx}(B)$. Then the random variables $\{A_{nx}\}_x$ are independent for each $n \in N$, $EA_{nx} = 0$, and $EA_{nx}^2 = \mathrm{Var}\, S_{nx}(B)$. Moreover, it is clear that $S_n(B) - ES_n(B) = \sum_x A_{nx}$. The idea of the proof rests on establishing the fact that

(4.17) $$E[S_n(B) - ES_n(B)]^6 = O(n^{6-3/\alpha}) .$$

To obtain this fact, first note that we can write

$$[\sum_x A_{nx}]^6 = \sum_x A_{nx}^6 + c_1 \sum_{x\neq y} A_{nx}^4 A_{nx}^2 + c_2 \sum_{x\neq y} A_{nx}^3 A_{ny}^3 + c_3 \sum_{x\neq y\neq z} A_{nx}^2 A_{ny}^2 A_{nz}^2 + R_n$$

where R_n consists of a linear combination of those terms whose sum contains a first power of A_{nx}. For example, $\sum_{x\neq y\neq z} A_{nx}A_{ny}^2A_{nz}^3$ and $\sum_{x\neq y} A_{nx}A_{ny}^5$. Using the independence of the $\{A_{nx}\}_x$ and the fact that $EA_{nx} = 0$ we see that $ER_n = 0$ and

(4.18) $$E[\sum_x A_{nx}]^6 = \sum_x EA_{nx}^6 + c_1 \sum_{x\neq y} EA_{nx}^4 EA_{ny}^2 + c_2 \sum_{x\neq y\neq z} EA_{nx}^2 EA_{ny}^2 EA_{nz}^2 + c_3 \sum_{x\neq y} EA_{nx}^3 EA_{ny}^3 .$$

From (4.10) we know that $\sum_x EA_{nx}^6 = O(n^{6-5/\alpha})$. Using (4.3) and (4.9) we see that $\sum_x EA_{nx}^4 \sum_x EA_{nx}^2 = O(n^{6-4/\alpha})$. Also, by (4.3) we get $[\sum_x EA_{nx}^2]^3 = O(n^{6-3/\alpha})$. Finally, (4.4) implies that $[\sum_x E|A_{nx}|^3]^2 = O(n^{6-4/\alpha})$. From these facts and (4.18) we get (4.17).

Now by Chebychev's inequality we have for any $\varepsilon > 0$

(4.19) $$P[|S_n(B) - ES_n(B)| > n\varepsilon] \leqq \frac{E[S_n(B) - ES_n(B)]^6}{n^6\varepsilon^6} .$$

From (4.7) we conclude that the term on the right of the above inequality is $O(n^{-3/\alpha})$. Since $\alpha \leqq 2$ this fact implies that for any $\varepsilon > 0$

(4.20) $$\sum_{n=1}^\infty P[|S_n(B) - ES_n(B)| > n\varepsilon] < \infty .$$

Using the Borel-Cantelli lemma we conclude that $P[|S_n(B) - ES_n(B)| > n\varepsilon \text{ i.o.}] = 0$ and from this it follows easily that with probability one $[S_n(B) - ES_n(B)]/n \to 0$. Lemma 3 and this last fact establish the theorem.

5. Proofs of Theorems 2 and 3. In order to prove the central limit theorem for $\{S_n(B)\}$ it is first necessary to establish one more lemma. We will also use the fact that if $a_n, b_n \geqq 0$, $a_n \sim b_n$, and $\sum b_n = \infty$ then $\sum_{k=1}^n a_k \sim \sum_{k=1}^n b_k$. Since $\alpha > 1$, $\sum n^{-1/\alpha} = \infty$ and thus (3.1) along with the above fact show that

$$G_n(0, x) \sim \frac{c}{1 - 1/\alpha} n^{1-1/\alpha}, \tag{5.1}$$

$$H_n \sim \frac{c}{1 - 1/\alpha} n^{1-1/\alpha}, \tag{5.2}$$

$$\sum_{k=1}^{n-1} G_k(0, x) \sim \frac{c}{(2 - 1/\alpha)(1 - 1/\alpha)} n^{2-1/\alpha}. \tag{5.3}$$

LEMMA. 4. *Let $S_{nx}(B)$ be as above. Then*

$$\sup_x \operatorname{Var} S_{nx}(B) = o(n^{2-1/\alpha}) \tag{5.4}$$

and

$$\sup_x E|S_{nx}(B) - ES_{nx}(B)|^3 = o(n^{3-3/2\alpha}). \tag{5.5}$$

PROOF. Using (1.3), (3.8), and (3.9) it is not difficult to see that $\operatorname{Var} S_{nx}(B) = O(EN_{nx}(B)^2 + [EN_{nx}(B)]^2)$. In the proof of Lemma 1 we saw that $EN_{nx}(B)^2 = G_n(x, B) + 2\sum_{y \in B}\sum_{i<j} P_i(x, y)P_{j-i}(y, B)$. Similar arguments to those in the above lemmas yield $\sup_x EN_{nx}(B)^2 = O(n^{2-2/\alpha})$ and $\sup_x [EN_{nx}(B)]^2 = O(n^{2-2/\alpha})$. These facts establish (5.4). To establish (5.5) first note that $E|S_{nx}(B) - ES_{nx}(B)|^3 = O(EN_{nx}(B)^3 + EN_{nx}(B)^2 EN_{nx}(B) + [EN_{nx}(B)]^3)$. Then some estimates show that each of the terms in the last expression are $O(n^{3-3/\alpha})$ uniformly in x. This gives the desired result.

We now show that $S_n(B)$ is asymptotically normally distributed:

Let ϕ_n and ψ_{nx} be the characteristic functions of the random variables $[S_n(B) - ES_n(B)]/[\operatorname{Var} S_n(B)]^{1/2}$ and $[S_{nx}(B) - ES_{nx}(B)]/[\operatorname{Var} S_n(B)]^{1/2}$, respectively. Since $S_n(B) = \sum_x S_{nx}(B)$ and the random variables $\{S_{nx}(B)\}_x$ are independent for each n

$$\phi_n(\theta) = \prod_x \psi_{nx}(\theta). \tag{5.6}$$

If X is a random variable with finite third moment and if $f(\theta)$ is the characteristic function of X then (see Feller [2], page 487) $f(\theta) = 1 + i\theta EX - \theta^2 EX^2/2 + \varepsilon(\theta)$ where $|\varepsilon(\theta)| \leqq |\theta|^3 E|X|^3/3!$. Applying this to ψ_{nx} we get

$$\psi_{nx}(\theta) = 1 - \theta^2 \operatorname{Var} S_{nx}(B)/2 \operatorname{Var} S_n(B) + R_{nx}(\theta) \tag{5.7}$$

where

(5.8) $$|R_{nx}(\theta)| \leqq |\theta|^3 E|S_{nx}(B) - ES_{nx}(B)|^3/3\,!\,[\mathrm{Var}\, S_n(B)]^{\frac{3}{2}} .$$

For convenience set $B_{nx}(\theta) = \psi_{nx}(\theta) - 1$. By assumption ($\liminf_{n\to\infty} [\mathrm{Var}\, S_n(B)]^{3/2}/n^{3-3/2\alpha}) > 0$. It now follows from the above facts that $\sup_x |B_{nx}(\theta)| \to 0$ (See Lemma 4 and (3.3)). Using the Taylor expansion of $\log(1+z)$ about $z = 0$ one finds that for $|z| \leqq 1/2$, $\log(1+z) = z + \varepsilon(z)|z|^2$ where $|\varepsilon(z)| \leqq 1$. Consequently, for large n we have $\log(1 + B_{nx}(\theta)) = B_{nx}(\theta) + \Lambda_{nx}(\theta)|B_{nx}(\theta)|^2$ for all $x \in Z$, where $|\Lambda_{nx}(\theta)| \leqq 1$. From Lemma 1 and (3.3) we are able to deduce that $\sum_x |R_{nx}(\theta)| \to 0$ as $n \to \infty$. Using this fact along with ($\sum_x \mathrm{Var}\, S_{nx}(B)) = \mathrm{Var}\, S_n(B)$ it follows immediately that

(5.9) $$\lim_{n\to\infty} \sum_x B_{nx}(\theta) = -\theta^2/2 .$$

and

(5.10) $$\limsup_{n\to\infty} \sum_x |B_{nx}(\theta)| \leqq \theta^2/2 .$$

Since $|\sum_x \Lambda_{nx}(\theta)|B_{nx}(\theta)|^2| \leqq [\sup_x |B_{nx}(\theta)|] \sum_x |B_{nx}(\theta)|$ and $\sup_x |B_{nx}(\theta)| \to 0$ it is clear that (5.10) imples $\sum_x \Lambda_{nx}(\theta)|B_{nx}(\theta)|^2 \to 0$ as $n \to \infty$. Writing (5.6) in the form $\phi_n(\theta) = \exp[\sum_x \log \psi_{nx}(\theta)]$ and noting that the above results show $\sum_x \log(1 + B_{nx}(\theta)) \to -\theta^2/2$ as $n \to \infty$ we conclude that $\phi_n(\theta) \to e^{-\theta^2/2}$. The continuity theorem now yields the desired result.

Finally we prove Theorem 3: Using (3.12) we can conclude that $\mathrm{Var}\, S_n(B) = \lambda[\sum_x EN_{nx}(B)^2 - \sum_x [EN_{nx}(B)]^2]$. In the proof of Lemma 1 it was established that $\sum_x EN_{nx}(B)^2 = n|B| + 2 \sum_{y\in B} \sum_{k=1}^{n-1} G_k(y, B)$. For convenience let $\gamma = c/(1 - \alpha^{-1})$ and $\delta = c/(2 - \alpha^{-1})(1 - \alpha^{-1})$. From (5.3) and the fact that $\alpha > 1$ it is clear that

(5.11) $$\sum_x EN_{nx}(B)^2 \sim 2|B|^2 \delta n^{2-1/\alpha} .$$

Also using (5.2) we have $\sum_x [EN_{nx}(B)]^2 \leqq |B| H_n \sum_x EN_{nx}(B) \sim |B|^2 \gamma n^{2-1/\alpha}$. This fact along with (5.11) implies

$$\liminf_{n\to\infty} \mathrm{Var}\, S_n(B)/n^{2-1/\alpha} \geqq \lambda |B|^2 (2\delta - \gamma) .$$

But $(2\delta - \gamma) = c[\alpha(2 - \alpha^{-1})^{-1}(1 - \alpha^{-1})^{-1}]^{-1}$. This completes the proof of Theorem 3.

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