The Annals of Mathematical Statistics
1971, Vol. 42, No. 6, 2126–2130

SHORT COMMUNICATIONS

INEQUALITIES FOR MODES OF L FUNCTIONS[1]

BY STEPHEN JAMES WOLFE

University of California, Riverside and University of Delaware

Inequalities are obtained for the modes of L functions whose Lévy spectral functions have support on the positive axis. An application of these inequalities is made to certain stochastic processes.

1. Introduction. For a definition of the class of L functions, i.e., distribution functions in class L, and some of their elementary properties, see [3, Chapter 6].

N. L. Johnson and C. A. Rogers have shown [4, Theorem 2, page 434] that if a unimodal distribution function has mean m, mode $\mathscr{M}$, and standard deviation σ, then $(\mathscr{M}-m)^2 \leqq 3\sigma^2$. It has been shown in [6, Theorem 1] that every L function that has a Lévy spectral function with support on the positive axis is unimodal. In this paper other inequalities are derived for the modes of these L functions. An application is made to certain stochastic processes.

2. A lemma.

LEMMA 1. *Let $F(x)$ be an infinitely divisible distribution function with a centering constant γ and a Lévy–Khintchine function $G(u)$. The distribution function $F(x)$ has a finite mean m if and only if $G(u)$ has a finite mean m^* in which case $m = \gamma + m^*$.*

PROOF. It follows from [5, Theorem 2,] that $F(x)$ has a finite mean if and only if $G(u)$ has a finite mean. It is easy to see that $m = \gamma + m^*$.

3. Another lemma. Let $0 < p_1 < \cdots < p_k < \infty$. Let $\lambda_1, \cdots, \lambda_k$ be positive constants. Let

$$\begin{aligned} \lambda_0(u) &= \lambda_1 + \cdots + \lambda_k && \text{if} \quad 0 < u \leqq p_1 \\ &= \lambda_2 + \cdots + \lambda_k && \text{if} \quad p_1 < u \leqq p_2 \\ &\vdots \\ &= 0 && \text{if} \quad u > p_k. \end{aligned}$$

Let

$$\hat{f}_0(t) = \exp\{\textstyle\int_{+0}^{+\infty} (e^{iut}-1)(\lambda_0(u)/u)du\}. \tag{1}$$

Received August 25, 1970.

[1] This research is from the author's doctoral dissertation submitted to the Department of Mathematics, University of California, Riverside. It was supported in part by the Air Force Office of Scientific Research, Grant No. AF–AFOSR 851–66.

The function $\hat{f}_0(t)$ is the characteristic function of an L function $F_0(x)$. It has been shown in the proof of [6, Lemma 1] that $F_0(x)$ is unimodal with a unique mode $\mathscr{M}$ and that

$$f(x) = 0 \qquad \text{if} \quad x < 0, \tag{2}$$

$$f(x) = (\lambda_1/x)[F_0(x)-F_0(x-p_1)]+ \cdots + (\lambda_k/x)[F_0(x)-F_0(x-p_k)] \qquad \text{if} \quad x > 0, \tag{3}$$

(4) $f(x)$ is strictly increasing on the interval $(0, \mathscr{M}]$, and

(5) $f(x)$ is strictly decreasing on the interval $[\mathscr{M}, \infty)$.

LEMMA 2. *Let $F_0(x)$ be the distribution function defined above. If $\lambda_0(+0) > 1$ then $F_0(x)$ has a finite mean m and a unique mode $\mathscr{M}$ such that $m-p_k < \mathscr{M} < m$.*

PROOF. Let $G_0(u)$ be the Lévy–Khintchine function of $F_0(x)$. It follows from Lemma 1 that $F_0(x)$ has a finite mean m and that

$$m = \gamma+\textstyle\int_0^\infty u dG(u) = \int_{+0}^\infty \lambda_0(u)du = \sum_{i=1}^k \lambda_i p_i.$$

It follows from (3) that if $x > 0$,

$$f(x) = (\lambda_1/x) \textstyle\int_{x-p_1}^x f(y)dy+ \cdots +(\lambda_k/x) \int_{x-p_k}^x f(y)dy. \tag{6}$$

It follows from (6) that

$$f(m) = (\lambda_1/m) \textstyle\int_{m-p_1}^m f(y)dy+ \cdots +(\lambda_k/m) \int_{m-p_k}^m f(y)dy. \tag{7}$$

It follows from (4) and (5) that $f(x)$ cannot be constant on any interval of $(0, \infty)$. Thus $f(x)$ is not constant on $[m-p_k, m]$. If $f(x)$ were increasing on $[m-p_k, m]$ it would follow from (7) that

$$f(m) < (\lambda_1 p_1/m+ \cdots +\lambda_k p_k/m)f(m) = f(m)$$

and this is an obvious contradiction. Similarly if $f(x)$ were decreasing on $[m-p_k, m]$ a contradiction could be derived in a similar manner. Thus it follows that $m-p_k < \mathscr{M} < m$. □

4. The main theorem.

THEOREM 1. *Let $F(x)$ be an L function with a Lévy spectral function $M(u)$ such that $M(u) = 0$ for $u < 0$. If there exists a constant $p > 0$ such that $M(u) = 0$ for $u > p$ then $F(x)$ has a finite mean m and a mode $\mathscr{M}$ such that $m-p \leqq \mathscr{M} \leqq m$. If $F(x)$ has a finite mean m then $F(x)$ has a mode $\mathscr{M}$ such that $\mathscr{M} \leqq m$.*

PROOF. Let $G(u)$ be the Lévy–Khintchine function of $F(x)$. Then $G(u) = 0$ for $u < 0$ and $F(x)$ has characteristic function

$$\hat{f}(t) = \exp \{i\gamma t+\textstyle\int_0^\infty (e^{iut}-1-iut(1+u^2)^{-1})((1+u^2)/u^2)dG(u)\}. \tag{8}$$

The theorem will first be proved when $F(x)$ does not have a normal component. In this case $G(u)$ is continuous at 0 and (8) is equivalent to

$$\hat{f}(t) = \exp\{i\gamma t + \int_{+0}^{\infty}(e^{iut}-1-iut(1+u^2)^{-1})(\lambda(u)/u)du\} \tag{9}$$

where $\lambda(u) = uM'(u)$. If $M(u) = 0$ for $u > p$ then $G(u)$ has support on $[0, p]$ and it follows from Lemma 1 that $F(x)$ has a finite mean m and that

$$m = \gamma + \int_0^{\infty} u dG(u) = \gamma + \int_{+0}^{\infty} u^2\lambda(u)(1+u^2)^{-1}du. \tag{10}$$

It has been shown in the proof of [6, Theorem 1] that it is possible to construct a sequence of L functions $\{F_n\}$ such that $F_n \to_c F$ and such that each L function F_n has a characteristic function

$$\hat{f}_n(t) = \exp\{i\gamma t + \int_{+0}^{\infty}(e^{iut}-1-iut(1+u^2)^{-1})(\lambda_n(u)/u)du\}$$

where $\lambda_n(u)$ is a nonnegative step function for each value of n of the type described at the beginning of Section 3 and $\lambda_n(u)$ converges to $\lambda(u)$ from below. From Lemma 1 it follows that each $F_n(x)$ has a finite mean m_n and that

$$m_n = \gamma + \int_{+0}^{\infty} u^2\lambda_n(u)(1+u^2)^{-1}du. \tag{11}$$

It follows from (10), (11), and the monotone convergence theorem that $m = \lim_{n\to\infty} m_n$.

If $\lambda(+0) \leqq 1$ then $\lambda_n(+0) \leqq 1$ for each value of n. It follows from the proof of [6, Lemma 1] that $F_n(x)$ has a mode at

$$\mathscr{M}_n = \gamma - \int_0^{\infty} \lambda_n(u)(1+u^2)^{-1}du.$$

By a theorem of A. L. Lapin [3, Theorem 4, page 160] $\mathscr{M} = \limsup_{n\to\infty} \mathscr{M}_n$ is a mode of $F(x)$. It follows from the monotone convergence theorem that

$$\mathscr{M} = \gamma - \int_0^{\infty} \lambda(u)(1+u^2)^{-1}du.$$

Thus $\mathscr{M} \leqq m$. If $M(u) = 0$ for $u > p$ then it follows from (10) and the fact that $\lambda(u)$ is non-increasing on $(0, \infty)$ that $m-p \leqq \mathscr{M}$.

If $\lambda(+0) > 1$ then it can be assumed without loss of generality that $\lambda_n(+0) > 1$ for all values of n. It follows from Lemma 2 that each L function $F_n(x)$ has a mode $\mathscr{M}_n$ such that $\mathscr{M}_n < m_n$. By Lapin's theorem $\mathscr{M} = \limsup_{n\to\infty} \mathscr{M}_n$ is a mode of $F(x)$. It is easily seen that $\mathscr{M} \leqq m$. If $M(u) = 0$ for $u > p$ then it follows from Lemma 2 that $m-p \leqq \mathscr{M}$.

Finally, assume that $F(x)$ has a normal component variance $\sigma^2 > 0$. In this case $G(u)$ has a discontinuity at 0 and (8) is equivalent to

$$\hat{f}(t) = \exp\{i\gamma t - \sigma^2 t^2/2 + \int_{+0}^{\infty}(e^{iut}-1-iut(1+u^2)^{-1})(\lambda(u)/u)du\} \tag{12}$$

where $\lambda(u) = uM'(u)$. Let $p^* > 0$ and let $\{a_n\}$ be a sequence of constants such that $0 < a_n < 2$ for each value of n and $\lim_{n\to\infty} a_n = 2$. For each value of n let

$$\begin{aligned} \lambda_n{}^*(x) &= \lambda(x) + (2-a_n)\sigma^2 x^{-a_n} && \text{if } 0 < x \leqq p^* \\ &= \lambda(x) && \text{if } x > p^*. \end{aligned}$$

and let

$$M_n(u) = 0 \qquad \text{if} \quad u < 0$$
$$= -\int_u^\infty \lambda_n^*(x)/x\, dx \qquad \text{if} \quad u > 0.$$

It is easily seen that for each value of n, $M_n(u)$ is a Lévy spectral function. Let $H_n(x)$ be the distribution function with characteristic function

$$\hat{h}_n(t) = \exp\{i\gamma t + \int_{+0}^\infty (e^{iut} - 1 - iut(1+u^2)^{-1}) dM_n(u)\}.$$

Then $H_n(x)$ is an L function without a normal component. By simple computations it can easily be seen that $\lim_{n\to\infty} M_n(u) = M(u)$ for $u \neq 0$ and that

$$\lim_{\varepsilon\to 0+} \limsup_{n\to\infty} \int_{+0}^{+\varepsilon} u^2 dM_n(u) = \lim_{\varepsilon\to 0+} \liminf_{n\to\infty} \int_{+0}^{+\varepsilon} u^2 dM_n(u) = \sigma^2.$$

It follows from [3, Theorem 2, page 88] that $H_n(x) \to_c F(x)$. Let $G_n(u)$ be the Lévy–Khintchine function of $H_n(x)$. It is easily seen that $G_n(u)$ has a finite mean for each value of n and

$$\lim_{n\to\infty} \int_0^\infty u dG_n(u) = \int_0^\infty u dG(u).$$

It follows from Lemma 1 that $H_n(x)$ has a finite mean m_n^* for each value of n and $\lim_{n\to\infty} m_n^* = m$ where m is the mean of $F(x)$. It has been shown that for each value of n, $H_n(x)$ has a mode $\mathscr{M}_n^*$ such that $\mathscr{M}_n^* \leqq m_n^*$. Thus $F(x)$ has a mode $\mathscr{M}$ such that $\mathscr{M} \leqq m$. If $M(u) = 0$ for $u > p$ then it can be assumed that $p^* < p$. It follows that $m_n^* - p \leqq \mathscr{M}_n^*$ for each value of n and therefore $m - p \leqq \mathscr{M}$. □

5. An application. This paper will be concluded with an application of Theorem 1. Let a and b be finite constants. Let $\{X(v), a \leqq v \leqq b\}$ be a real, centered stochastic process with independent increments and with no fixed points of discontinuity. Let $F(v, x)$ be the distribution function of $X(v) - X(a)$. Then $F(v, x)$ is infinitely divisible for $a \leqq v \leqq b$ and has a characteristic function

$$\hat{f}(v, t) = \exp\{i\gamma(v)t + \int_{-\infty}^{\infty} (e^{iut} - 1 - iut(1+u^2)^{-1})((1+u^2)/u^2) dG(v, u)\}.$$

Also $\gamma(v)$ is continuous on $[a, b]$ and $G(v_2, u) - G(v_1, u)$ is a Lévy–Khintchine function for $a \leqq v_1 < v_2 \leqq b$ (see [1, Chapter VIII, Section 7]). It follows that if $G(b, u)$ has a finite mean then $G(v, u)$ has a finite mean for $a \leqq v \leqq b$. The stochastic process $\{X(v), a \leqq v \leqq b\}$ with independent increments will be called an L process [2, page 186] if the distribution function $F(v, x)$ of $X(v) - X(a)$ is an L function for $a \leqq v \leqq b$. The following theorem follows immediately from Lemma 1 and Theorem 1.

THEOREM 2. *Let $\{X(v), a \leqq v \leqq b\}$ be a real centered L process with no fixed points of discontinuity. For $a \leqq v \leqq b$ let $F(v, x)$ be the distribution function of $X(v) - X(a)$ and let $G(v, u)$ be the Lévy–Khintchine function of $F(v, x)$. Assume that $G(v, u) = 0$ for $a \leqq v \leqq b$ and $u < 0$. If there exists a constant $p > 0$ such that $G(v, u) = G(v, p)$ for $a \leqq v \leqq b$ and $u > p$ then for $a \leqq v \leqq b$, $F(v, x)$ has a finite mean $m(v)$ and a mode $\mathscr{M}(v)$ such that $m(v) - p \leqq \mathscr{M}(v) \leqq m(v)$. If $F(b, x)$ has a* *finite mean then for* $a \leqq v \leqq b$, $F(v, x)$ *has a finite mean* $m(v)$ *and a mode* $\mathscr{M}(v)$ *such that* $\mathscr{M}(v) \leqq m(v)$.

Acknowledgment. I wish to express my gratitude to my dissertation adviser, Professor Howard G. Tucker, for the guidance and encouragement he has given to me.

REFERENCES

[1] DOOB, J. L. (1953). *Stochastic Processes.* Wiley, New York.

[2] FISZ, M. (1963). On the orthogonality of measures induced by L functions. *Trans. Amer. Math. Soc.* **106** 185–192.

[3] GNEDENKO, B. V. and KOLMOGOROV, A. N. (1954). *Limit Distributions for Sums of Independent Random Variables.* Addison-Wesley, Reading.

[4] JOHNSON, N. L. and ROGERS, C. A. (1951). The moment problem for unimodal distributions. *Ann. Math. Statist.* **22** 433–439.

[5] WOLFE, S. J. (1971). On moments of infinitely divisible distribution functions. *Ann. Math. Statist.* **42** 2036–2043.

[6] WOLFE, S. J. (1971). On the unimodality of L functions. *Ann. Math. Statist.* **42** 912–918.