The Annals of Mathematical Statistics
1971, Vol. 42, No. 6, 2029–2035

AN ASYMPTOTIC 0–1 BEHAVIOR OF GAUSSIAN PROCESSES

BY CLIFFORD QUALLS[1] AND HISAO WATANABE[2]

University of North Carolina, Chapel Hill
University of New Mexico and Kyushu University

Let $\{X(t), -\infty < t < \infty\}$ be a stationary Gaussian process with covariance function satisfying: (1) $r(t) = 1 - C|t|^{\alpha} + o(|t|^{\alpha})$ as $t \to 0$: $C > 0$, $0 < \alpha \leqq 2$; and (2) $r(t) = O(t^{-\gamma})$ as $t \to \infty : \gamma > 0$. Then for all positive increasing functions $\phi(t)$ on $[a, \infty)$, $P[X(t) > \phi(t)$ infinitely often$] = 0$ or 1 as $\int_a^\infty \phi(t)^{2/\alpha - 1} \exp\{-\phi^2(t)/2\}\, dt < \infty$ or $= \infty$.

This result generalizes the paper of Watanabe [*Trans. Amer. Math. Soc.* **148** 233–248] by replacing his condition that $r(t) = o(1/t)$ as $t \to \infty$ by condition (2). Our result is extended also to the nonstationary process treated by Watanabe. Our proof treats the problem as a crossing problem using a recent result of Pickands [*Trans. Amer. Math. Soc.* **145** 51–73] and a modification of the Borel lemmas.

0. Introduction. Let $\{X(t), -\infty < t < \infty\}$ be a real separable Gaussian process defined on a probability space $(\Omega, \mathscr{A}, P)$. We assume $EX(t) \equiv 0$ and $v^2(t) = EX^2(t) > 0$. Denote the correlation function by $\rho(s, t) = EX(s)X(t)/(v(s)v(t))$. This paper is concerned with the probability of the event

$$E_\phi = [\exists\, t_0(\omega) : X(t) \leqq v(t)\phi(t) \quad \text{for all} \quad t \geqq t_0(\omega)].$$

One of the authors in [3] gives conditions on the correlation function so that $PE_\phi = 1$ or 0 as $I_\phi < \infty$ or $= \infty$, where the quantity $I_\phi = \int_a^\infty \phi(t)^{2/\alpha - 1} \exp\{-\phi^2(t)/2\} dt$, and α is given below. He considers the problem as a type of the so-called law of the iterated logarithm which appeared in the study of sums of independent random variables. In this paper, we treat the problem from a different point of view as a type of crossing problem. The resulting simpler proof shows the above 0–1 behavior holds for a larger class of processes, and also makes the intuitive content of the result clearer. The Gaussian processes (or rather the corresponding correlation functions) now included satisfy:

I There are positive constants Δ, C_1, C_2, T, and α with $0 < \alpha \leqq 2$, such that $1 - C_1 h^{\alpha} \leqq \rho(t, t+h) \leqq 1 - C_2 h^{\alpha}$ for $0 \leqq h < \Delta$ and all $t \geqq T$; and

II $\rho(t, t+s) = O(s^{-\gamma})$ uniformly in t as $s \to \infty$ for some $\gamma > 0$.

Condition II replaces the condition that $\rho(t, t+s) = o(1/s)$ in Watanabe (1970).

Received February 2, 1971.

[1] Research supported in part by the Office of Naval Research under Contract N00014–67–A–0321–0002.

[2] Research supported in part by the National Science Foundation under Grant GU–2059.

Key words and phrases: Gaussian process, stationary process, crossing problem, a $0-1$ law, a law of the iterated logarithm.

There is also a slight improvement in condition I.

Section 1 gives the proof for stationary processes making use of a recent result due to Pickands [1].

A well-known theorem of Slepian [2] is used in Section 2 to extend the results of Section 1 to nonstationary processes described by conditions I and II above. It has been pointed out in [3] that the result of Section 2 can be made to yield (by a time transformation) the analogous 0–1 behavior for a class of Gaussian processes containing Brownian motion.

1. Stationary case.

THEOREM 1.1. *Let* $\{X(t), -\infty < t < \infty\}$ *be a real separable stationary Gaussian process with* $EX(t) \equiv 0$ *and covariance function r satisfying*

(1) $r(t) = 1-C|t|^\alpha+o(|t|^\alpha)$ *as* $t \to o$ *for some* $C > o$ *and some* α *with* $0 < \alpha \leqq 2$; *and*

(2) $r(t) = O(t^{-\gamma})$ *as* $t \to \infty$ *for some* $\gamma > 0$.

Then for all functions $\phi(t)$ that are positive and nondecreasing on some interval $[a, \infty)$, it follows that

$$PE_\phi \equiv P[\exists\, t_0(\omega) > a : X(t) < \phi(t) \quad \text{for all} \quad t \geqq t_0(\omega)] = 1 \quad \text{or} \quad 0$$

as the integral

$$I_\phi \equiv \int_a^\infty \phi(t)^{2/\alpha-1} \exp\{-\phi^2(t)/2\}\, dt \quad \text{is finite or infinite.}$$

Note that the condition (1) implies the process $X(t)$ has continuous sample functions. The following lemma will be needed for the first half of the proof of this theorem.

LEMMA 1.2. *If condition* (1) *of Theorem* 1.1 *holds and* $A(t) = \inf\{(1-r(s))/|s|^\alpha: 0 < s \leqq t\} > 0$, *then*

$$\lim_{x\to\infty} \frac{P[\max_{0\leqq s\leqq t} X(s) > x]}{tx^{2/\alpha}\Psi(x)} = C^{1/\alpha}H_\alpha, \qquad \textit{where } \Psi(x) = (2\pi)^{-\frac{1}{2}}x^{-1}\, e^{-x^2/2} \textit{ and}$$

$$0 < H_\alpha \equiv \lim_{T\to\infty} T^{-1}\int_0^\infty e^s P[\sup_{0\leqq t\leqq T} Y(t) > s]\, ds < \infty,$$

and $Y(t)$ *is a nonstationary Gaussian process with mean* $EY(t) = -|t|^\alpha$ *and covariance function* $r(s, t) = -|s-t|^\alpha + |s|^\alpha + |t|^\alpha$.

PROOF. This is Lemma 2.9 of Pickands [1]. In addition to condition (1), Pickands required that $A_1(t) = \inf\{(1-r^2(s))/|s|^\alpha: 0 < s \leqq t\} > 0$. However, checking Pickands' proofs, we note that this requirement can be replaced by only the requirement that $A(t) > 0$.

In other words, these proofs permit $r(s) = -1$ but not $r(s) = 1$ in the interval $0 < s \leqq t$.

REMARK. If it should happen that $A(t) = 0$, then there is a smallest $s_0 > 0$ such that $r(s_0) = 1$ and then both $r(t)$ and $X(t)$ are periodic with period s_0. Since $\max_{[o,\tau]} X(s) = \max_{[o,t]} X(s)$ where $\tau = \min(t, s_0)$, the requirement that $A(t) > 0$ can be eliminated from Lemma 1.2 by replacing the denominator $tx^{2/\alpha}\Psi(x)$ by $\tau x^{2/\alpha}\Psi(x)$.

PROOF OF THEOREM 1.1 WHEN $I_\phi < \infty$. Only condition (1) of Theorem 1.1 will be required for this half of the proof. Considering the sequence of intervals $[n, n+1]$ with integer end points, we obtain $\infty > I_\phi \geqq \sum_{n=N}^{\infty} \phi(n+1)^{2/\alpha}\Psi(\phi(n+1))(2\pi)^{\frac{1}{2}}$ where the lower limit $a = N$ of the integral I_ϕ is chosen large enough that the integrand of I_ϕ is a decreasing function in the argument ϕ. Define $F_n = [\max_{n \leqq s \leqq n+1} X(s) \leqq \phi(n)]$.

Since by Lemma 1.2, there is a positive constant K such that

$$PF_n^{\,c} \sim K\phi(n)^{2/\alpha}\Psi(\phi(n))$$

as $\phi(n) \to \infty$, we obtain $\sum_{n=N}^{\infty} PF_n^{\,c} < \infty$. The Borel-Cantelli lemma completes this half of the proof.

Before proceeding to the second half of the proof, we will need the following lemmas.

LEMMA 1.3. *If condition* I *of Theorem* 1.1 *holds and* $A(t) > 0$, *then*

$$\lim_{x\to\infty} \frac{P[\max_{0 \leqq k \leqq m} X(kax^{-2/\alpha}) > x]}{tx^{2/\alpha}\Psi(x)} = C^{1/\alpha}\frac{H_\alpha(a)}{a},$$

where $a > 0$, $m = [t/ax^{-2/\alpha}]$ *and* [ ] *denotes the greatest integer function, and* $H_\alpha(a)$ *is a certain positive constant.*

PROOF. This is Lemma 2.5 of Pickands [1]. All the remarks given for Lemma 1.2 also apply here.

LEMMA 1.4. *If Theorem* 1.1 *for the case* $I_\phi = \infty$ *is true under the additional restriction that for large* t, $2 \log t \leqq \phi^2(t) \leqq 3 \log t$, *then it is true without this restriction.*

PROOF. A similar statement could have been made for the case $I_\phi < \infty$, but it was not needed in the proof. The restriction that $\phi^2(t) \leqq 3 \log t$ is treated in Lemma 4.1 of [3], and the restriction that $\phi^2(t) \geqq 2 \log t$ is only a slight modification of Lemma 4.1 [3] when $\alpha < 2$. Suppose $\alpha = 2$. Since $X(t) > \hat{\phi}(t)$ occurs infinitely often (as $t \to \infty$) implies $X(t) > \phi(t)$ occurs i.o. for any $\hat{\phi} \geqq \phi$, we need only show that $I_\phi = \infty$ implies $I\hat{\phi} = \infty$ for $\hat{\phi} = \max(\phi, u)$ and $u(t) = (2 \log t)^{\frac{1}{2}}$.

Letting $A = \{t > a : \phi(t) \leqq u(t)\}$ and $B = \{t > a : \phi(t) > u(t)\}$ we write $I_\phi = A_\phi + B_\phi = \infty$, $I_u = A_u + B_u = \infty$, and $I\hat{\phi} = A\hat{\phi} + B\hat{\phi} = A_u + B_\phi$, where for example B_ϕ means $\int_B \exp\{-\phi^2(t)/2\}dt$. We may suppose $B_\phi < \infty$ for otherwise the lemma follows immediately. Note that if $t_0 \in B$, then there is a (largest) nonempty interval in B containing t_0. Consequently, B is a union of such (disjoint) intervals I_n whose lengths and left end points will be denoted by Δ_n and t_n. (Unfortunately, it may not be possible to index the t_n's according to their order on the line.) However, we assume $\phi(t)$ crosses $u(t)$ infinitely often as $t \to \infty$, and therefore that the Δ_n's are finite numbers and the sequence $\{t_n\}$ is infinite. If ϕ does not cross $u(t)$ i.o., then either $\phi \leqq u$ and $I\hat{\phi} = I_u = \infty$, or $\phi > u$ and $I\hat{\phi} = I_\phi = \infty$ for some a. Note that $\phi(t_n+\Delta_n) = u(t_n+\Delta_n)$ since the jumps of ϕ are never downward. Now

$$\infty > B_\phi = \sum_n \int_{I_n} \exp\{-\phi^2/2\}\,dt \geqq \sum_n \Delta_n \exp\{-\phi^2(t_n+\Delta_n)/2\} = \sum_n \frac{\Delta_n}{t_n+\Delta_n},$$

and

$$B_u \leqq \sum_n \Delta_n \exp\{-u^2(t_n)/2\} = \sum_n \Delta_n/t_n = \sum_n \frac{\Delta_n}{t_n+\Delta_n}\left\{\frac{1}{1-\Delta_n/(t_n+\Delta_n)}\right\}.$$

Since $\Delta_n/(t_n+\Delta_n) \to 0$, we have $\sum_n \Delta_n/t_n < \infty$. Finally $B_u < \infty$ implies $A_u = \infty$ which in turn implies $I\hat{\phi} = \infty$.

LEMMA 1.5. *Let $X(t)$ be a Gaussian process with zero mean function and covariance function $r(s, t)$ with $r(t, t) \equiv 1$. Let $E_n = [X(t_{n,v}) \leqq x_{n,v} : v = 0, \cdots, m_n]$ with all $t_{n,v}$ distinct. Then*

$$\left|P(\textstyle\bigcap_1^n E_k) - \prod_1^n PE_k\right| \leqq \sum\sum_{1 \leqq i<j \leqq n} \sum_{\mu=0}^{m_j} \sum_{v=0}^{m_i} |r| \int_0^1 \phi(x_{i,v}, x_{j,\mu}; \lambda r)\,d\lambda,$$

where $\phi(x, y; \lambda r)$ is the standard bivariate normal density with correlation coefficient $\lambda r = \lambda r(t_{i,v}, t_{j,\mu})$.

PROOF. This type of lemma now appears in many proofs of asymptotic independence for crossing problems. We include the "standard" proof with the necessary differences.

The event $\bigcap E_k$ involves $N = \prod_1^n (m_k+1)$ random variables $X(t_{n,v})$, and the corresponding covariance matrix will be denoted by $\sum_1 = (r_{kl})$, where the doubly indexed random variables $X(t_{n,v})$ have been renumbered by a single index k (the k in r_{kl}).

Partition $\sum_1 = [\sum_{ij}]$ so that each submatrix $\sum_{ij}$ is the covariances of the random variables of E_i with those of E_j. Now the events E_k would be independent if and only if the corresponding covariance matrix were $\sum_0 = [\sum_{ij}^0]$ with $\sum_{ii}^0 = \sum_{ii}$ but $\sum_{ij}^0 = 0$ matrix for $i \neq j$.

Let $\sum_\lambda = \lambda \sum_1 + (1-\lambda)\sum_0 = (r_{\lambda ij})$ be the covariance matrix for the standardized multivariate normal density $\phi_\lambda(\mathbf{y})$, and

$$F(\lambda) = \int_{-\infty}^{x_{1,0}} \cdots \int_{-\infty}^{x_{1,m_1}} \cdots \int_{-\infty}^{x_{n,0}} \cdots \int_{-\infty}^{x_{n,m_n}} \phi_\lambda(\mathbf{y})\,d\mathbf{y},$$

where $d\mathbf{y} = dy_1\,dy_2 \cdots dy_N$. We now have

$$\left|P(\textstyle\bigcap_1^n E_k) - \prod_1^n PE_k\right| = |F(1)-F(0)| = \left|\int_0^1 F'(\lambda)\,d\lambda\right| \leqq \int_0^1 |F'(\lambda)|\,d\lambda.$$

Since $F'(\lambda) = \int \partial\phi_\lambda/\partial\lambda\,d\mathbf{y}$ and $dr_{\lambda kl}/d\lambda = 0$ or r_{kl} according to whether (k, l) refers to a diagonal $\sum_{kk}$ or not, we obtain by the chain rule for $\partial\phi_\lambda/\partial\lambda$

$$|F'(\lambda)| = \left|\sum_{i<j}\sum{}^* r_{kl} \int \frac{\partial^2\phi_\lambda}{\partial y_k\,\partial y_l}\,d\mathbf{y}\right| \leqq \sum_{i<j}\sum{}^* |\ _{kl}|\phi(x_k, x_l; \lambda r_{kl}).$$

The double sum $\sum^*$ extends over all (k, l) that refer to covariances r_{kl} of $\sum_{ij}$. Integrating this inequality with respect to λ finishes the proof.

PROOF OF THEOREM 1.1 WHEN $I_\phi = \infty$. Note that condition (2) of Theorem 1.1 eliminates the periodic case discussed immediately following Lemma 1.2. Define a sequence of intervals by $I_n = [n\Delta, n\Delta+\beta]$ for $\Delta > 0$ and $0 < \beta < \Delta$. Let $G_k = \{t_{k,v} = k\Delta+(v/n_k); v = 0, \cdots, [\beta n_k]\}$ be a set of points in I_k where n_k shall be chosen later.

Let $E_k = [\max_{s\in G_k} X(s) \leqq \phi(k\Delta+\beta)]$. Now for $a = N\Delta$ sufficiently large $\infty = I_\phi \leqq \sum_{k=N}^{\infty} \Delta\phi(k\Delta)^{2/\alpha}\Psi(\phi(k\Delta))(2\pi)^{\frac{1}{2}}$ implies that $\sum \beta\phi(k\Delta+\beta)^{2/\alpha}\Psi(\phi(k\Delta+\beta)) = \infty$. If we choose $n_k = [\phi(k\Delta+\beta)^{2/\alpha}]$, then Lemma 1.3 implies there is a positive constant K such that $PE_k^{\,c} \sim K\beta\phi(k\Delta+\beta)^{2/\alpha}\Psi(\phi(k\Delta+\beta))$ as $\phi(k\Delta+\beta) \to \infty$. So we have $\sum PE_k^{\,c} = \infty$.

As in the Borel lemma, the main step is

$$1-P[E_k^{\,c}\text{ i.o.}] = \lim_{m\to\infty} \textstyle\prod_m^\infty PE_k + \lim_{m\to\infty}\{P(\bigcap_m^\infty E_k) - \prod_m^\infty PE_k\}.$$

The first limit is zero because $\sum PE_k^{\,c} = \infty$, and the second limit will be zero because of the asymptotic independence of the events E_k. Note the separation between I_k's is $\Delta-\beta$. By Lemma 1.5, we have

$$A_{m,n} = \left|P(\textstyle\bigcap_m^n E_k) - \prod_m^n PE_k\right| \leqq \sum\sum_{m\leqq i<j\leqq n} \sum_{\mu=0}^{[\beta n_j]}\sum_{v=0}^{[\beta n_i]} |r| \int_0^1 g(\phi(i\Delta+\beta), \phi(j\Delta+\beta); \lambda r)\, d\lambda,$$

where $r = r(t_{j,v}-t_{i,\mu})$ and $g(x, y; \lambda p)$ denotes the bivariate normal density. Because $t_{j,v}-t_{i,\mu} \geqq j\Delta-i\Delta-\beta \geqq \Delta-\beta$, and because of condition (2) of Theorem 1.1, Δ can be chosen large enough that $|r(t_{j,v}-t_{i,\mu})| \leqq M_0(j\Delta-i\Delta-\beta)^{-\gamma}$ for all $j > i \geqq m$ and some positive constant M_0 and such that $|r| < \delta \equiv \min(\frac{1}{3}, \gamma/6)$. In fact, for $M = M_0(1-\beta/\Delta)^{-\gamma}$, we have $|r| \leqq M(j\Delta-i\Delta)^{-\gamma}$. Now by Lemma 1.4, we can choose m large enough that $\phi^2(k\Delta+\beta) \geqq u^2(k\Delta+\beta) = 2\log(k\Delta+\beta)$ and $\phi^2(k\Delta+\beta) \leqq w^2(k\Delta+\beta) = 3\log(k\Delta+\beta)$ for all $k \geqq m$. We obtain

$$\begin{aligned}
&g(\phi(i\Delta+\beta), \phi(j\Delta+\beta); \lambda r)\\
&\quad\leqq (2\pi)^{-1}(1-\delta^2)^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}[\phi^2(i\Delta+\beta)-2|r|\phi(i\Delta+\beta)\phi(j\Delta+\beta)+\phi^2(j\Delta+\beta)]\}\\
&\quad\leqq (2\pi)^{-1}(1-\delta^2)^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}[u^2(i\Delta+\beta)-2|r|w(i\Delta+\beta)w(j\Delta+\beta)+u^2(j\Delta+\beta)]\}\\
&\quad\leqq (2\pi)^{-1}(1-\delta^2)^{-\frac{1}{2}}\left(\frac{1}{i\Delta+\beta}\right)\left(\frac{1}{j\Delta+\beta}\right)^{1-3|r|}.
\end{aligned}$$

By Lemma 1.4, $n_i = [\phi(i\Delta+\beta)^{2/\alpha}] \leqq (3\log(i\Delta+\beta))^{1/\alpha}$, and since $w(i\Delta+\beta) \leqq w(j\Delta+\beta)$ and $|r| < \delta$, we have

$$A_{m,\infty} \leqq K\sum_{m\leqq i<j<\infty}\sum \frac{\log(j\Delta+\beta)^{2/\alpha}}{(j\Delta-i\Delta)^\gamma}\left(\frac{1}{i\Delta+\beta}\right)\left(\frac{1}{j\Delta+\beta}\right)^{1-3\delta}.$$

This double series can be seen to be convergent by letting $k = j-i$ and obtaining

$$A_{m,\infty} \leqq K' \sum_{i=m}^{\infty} \sum_{k=1}^{\infty} \frac{(\log(k\Delta + i\Delta + \beta))^{2/\alpha}}{k^{6\delta}} \left(\frac{1}{i}\right)\left(\frac{1}{k+i}\right)^{1-3\delta}$$

$$\leqq K' \sum_{i=m}^{\infty} \sum_{k=1}^{\infty} \frac{(\log(k\Delta + i\Delta + \beta))^{2/\alpha}}{(k+i)^{\delta}} \left(\frac{1}{i}\right)^{1+\delta}\left(\frac{1}{k}\right)^{1+\delta} < \infty.$$

Since this series is convergent, we have $\lim_{m\to\infty} A_{m,\infty} = 0$ where $A_{m,\infty} = |P(\bigcap_m^\infty E_k) - \prod_m^\infty PE_k|$. This completes the proof.

2. Non-stationary case. We now extend the result for the stationary case to the nonstationary case treated in [3]. Here, let $\{X(t), -\infty < t < \infty\}$ be a real separable Gaussian process with zero mean function.

We assume $v^2(t) = EX^2(t) > 0$ and denote the covariance function of $X(t)/v(t)$ by $\rho(s, t) = EX(s)X(t)/(v(s)v(t))$. We shall assume also $\rho(s, t)$ is continuous.

THEOREM 2.1. *Suppose the above process $X(t)$ satisfies*

(3) *there are positive constants δ_1, C_1, T_1 and α with $0 < \alpha \leqq 2$ such that $\rho(t, t+h) \geqq 1 - C_1 h^\alpha$ for $0 < h < \delta_1$ and all $t > T_1$. Then for all functions $\phi(t)$ that are positive and nondecreasing on some interval $[a, \infty)$ such that*

$$I_\phi \equiv \int_a^\infty \phi(t)^{2/\alpha - 1} \exp\{-\phi^2(t)/2\}\, dt < \infty,$$

we have

$$P[\exists\, t_0(\omega) > a : X(t) \leqq v(t)\phi(t) \quad \textit{for all} \quad t \geqq t_0(\omega)] = 1.$$

This theorem is Theorem 1 of [3], but we include the following different proof.

PROOF. Let $Y(t)$ be a separable stationary Gaussian process having a covariance function $q(h)$ satisfying $q(h) = 1 - C_1{}^* \cdot |h|^\alpha + o(|h|^\alpha)$ as $h \to 0$ and $q(h) \leqq 1 - C_1 h^\alpha \leqq \rho(t, t+h)$ for $0 < h < \delta_1{}^*$ and $t \geqq T_1$. This second requirement follows for $C_1{}^* > C_1$. By the nonstationary version of a well-known result due to Slepian (1962) (see Theorem 1), the fact that $q(h) \leqq \rho(t, t+h)$ for all $t \geqq T_1$ implies

$$P[\sup_{(n\Delta, n\Delta+\Delta)} Y(s) \leqq u] \leqq P[\sup_{(n\Delta, n\Delta+\Delta)} X(s) \leqq u]$$

for $\Delta < \delta_1{}^*$ and $n\Delta > T_1$. Now following the "proof of Theorem 1.1 when $I_\phi < \infty$," we have $\sum PG_n{}^c \leqq \sum PF_n{}^c < \infty$, where $F_n = [\sup_{(n\Delta, n\Delta+\Delta)} Y(s) \leqq \phi(n\Delta)]$ and $G_n = [\sup_{(n\Delta, n\Delta+\Delta)} X(s) \leqq \phi(n\Delta)]$. The Borel-Cantelli lemma applied to the G_n's completes the proof.

THEOREM 2.2. *Let the above process $X(t)$ have a correlation function satisfying:*

(3′) *There are positive constants δ_2, C_2, T_2 and α' with $0 < \alpha' \leqq 2$ such that $\rho(t, t+h) \leqq 1 - C_2 h^{\alpha'}$ for $0 < h < \delta_2$ and all $t > T_2$; and*

(4) *$\rho(t, t+s) = O(s^{-\gamma})$ uniformly in t as $s \to \infty$ for some $\gamma > 0$.*

Then for all functions ϕ *as in Theorem* 2.1 *with* $I_\phi = \infty$, *we have*

$$P[X(t) > v(t)\phi(t) \text{ i.o. } \quad in \quad t] = 1.$$

PROOF. Let $Y(t)$ be a separable stationary Gaussian process having a covariance function $q(h)$ satisfying $q(h) = 1 - C_2^*|h|^{\alpha'} + o(|h|^{\alpha'})$ as $h \to 0$ and $\rho(t, t+h) \leqq 1 - C_2 h^{\alpha'} \leqq q(h)$ for $0 < h < \delta_2^*$ $(C_2^* < C_2)$. Applying Slepian's result (see Theorem 1) and the "proof of Theorem 1.1 when $I_\phi = \infty$," we obtain $\infty = \sum PE_n^c \leqq \sum PH_n^c$ for $\beta < \delta_2^*$, where $E_k = [\max_{s \in G_k} Y(s) \leqq \phi(k\Delta + \beta)]$ and $H_k = [\max_{s \in G_k} X(s) \leqq \phi(k\Delta + \beta)]$. Consequently, it only remains to show that the events H_n are asymptotically independent (i.e. in the nonstationary case). Lemma 1.5 yields as in the "proof of Theorem 1.1 when $I_\phi = \infty$"

$$A_{m,n} = \sum\sum_{m \leqq i < j \leqq n} \sum_{\mu=0}^{[\beta n_j]} \sum_{\nu=0}^{[\beta n_i]} |\rho| \int_0^1 g(\phi(i\Delta + \beta), \phi(j\Delta + \beta); \lambda\rho)\, d\lambda,$$

where $\rho = \rho(t_{i,\nu}, t_{j,\mu})$ and $g(x, y; \lambda\rho)$ denotes the bivariate normal density. Again we chose Δ large enough that $|\rho| = |\rho(t_{i,\nu}, t_{j,\mu})| \leqq M(j-i)^{-\gamma}$ and $|\rho| < \delta \equiv \min(1/3, \gamma/6)$ for all $j > i$ and some positive constant M. Consequently we may estimate $A_{m,\infty}$ and complete the proof of Theorem 2.2 exactly as we did for the stationary case, i.e., Theorem 1.1 when $I_\phi = \infty$. □

REMARK. If conditions (3) and (3′) hold simultaneously, then $\alpha \leqq \alpha'$. The nonstationary case theorem exactly analogous to Theorem 1.1 holds if conditions (3) and (3′) with $\alpha = \alpha'$ and condition (4) hold.

Of course, Theorem 6 of [3], which is used by Watanabe as a proof of the asymptotic 0–1 behavior of Brownian motion, can be improved by using a hypothesis analogous to condition (4) of Theorem 2.2.

REFERENCES

[1] PICKANDS, J. (1969). Upcrossing probabilities for stationary Gaussian processes. *Trans. Amer. Math. Soc.* **145** 51–73.

[2] SLEPIAN, D. (1962). The one-sided barrier problem for Gaussian noise. *Bell System Tech. J.* **41** 463–501.

[3] WATANABE, H. (1970). An asymptotic property of Gaussian processes, I. *Trans. Amer. Math. Soc.* **148** 233–248.