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INADMISSIBILITY OF A CLASS OF ESTIMATORS OF A NORMAL QUANTILE[1]

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0. Summary. Suppose that independent normally distributed random vectors $W^{n\times 1}$ and $T^{k\times 1}$ are observed with $E(W) = 0$, $E(T) = \mu$, $\text{Cov}(W) = \sigma^2 I$ and $\text{Cov}(T) = \sigma^2 I$. In this paper it is shown that each member of a certain class of estimators of $\mu + \eta\sigma$ for a given vector η is inadmissible if loss is dimension-free quadratic loss. This class includes the best invariant estimator. The proof is carried out by exhibiting, for each member, $\hat{\theta}$, of the class, an estimator depending on $\hat{\theta}$ whose risk is uniformly smaller than that of $\hat{\theta}$.

1. Introduction. Let X be a normally distributed random variable with mean μ and variance σ^2, where both μ and σ are unknown ($-\infty < \mu < \infty$, $\sigma > 0$). Let $v = \mu + \eta\sigma$ for a given constant η. Then v is a quantile of the distribution of X. Suppose $X_1, \cdots, X_n$ ($n \geqq 2$) are independent copies of X on the basis of which v is to be estimated. If an estimate, t, is selected, a loss $\sigma^{-2}(t-v)^2$ is incurred.

A minimax estimator of v is $\hat{\theta} = \bar{X} + \eta C_n S^{\frac{1}{2}}$ where $\bar{X} = n^{-1}\Sigma X_i$, $S = \Sigma(X_i - \bar{X})^2$ and

$$C_n = \Gamma\left(\frac{n}{2}\right)\left(2^{\frac{1}{2}}\Gamma\left(\frac{n+1}{2}\right)\right)^{-1}, \qquad n = 2, 3, \cdots. \tag{1.1}$$

In this paper we prove a conjecture of Stein [1] that if $\eta \neq 0$, $\hat{\theta}$ is an inadmissible estimator of v. We do so by exhibiting a second estimator $\hat{\theta}_1$ for which $E_{\mu,\sigma}(\hat{\theta}_1 - v)^2 < E_{\mu,\sigma}(\hat{\theta} - v)^2$. To be precise, $\hat{\theta}_1 = \min\{1, 1 + H(\bar{X}S^{-\frac{1}{2}})S\bar{X}^{-2}\}\bar{X} + \eta C_n S^{\frac{1}{2}}$ where $H(t) = (\frac{1}{4})n(\eta t)^2 - nC_n(\eta t) + 1$, $-\infty < t < \infty$. It is easy to see that $\hat{\theta}_1 = \hat{\theta}$ unless $|\frac{1}{2}\eta\bar{X}S^{-\frac{1}{2}} - C_n| < (C_n^{\,2} - n^{-1})^{\frac{1}{2}}$.

The result described above is a corollary of Theorem 1 which concerns a more general problem than that described above. Suppose $S, T_1, \cdots, T_k (k \geqq 1)$ are independent random variables, that $S\sigma^{-2}$ has the chi-squared distribution with n degrees of freedom, and that T_i is normally distributed with mean μ_i and variance σ^2, $i = 1, 2, \cdots, k$. We assume that $\sigma^2(\sigma > 0)$ and the $\mu_i(-\infty < \mu_i < \infty$, $i = 1, 2, \cdots, k)$ are unknown constants.

Let R^k ($k = 1, 2, \cdots$) denote k dimensional Euclidean space. If $x \in R^k$, x' will denote the transpose of x. Moreover for $x = (x_1, \cdots, x_k)' \in R^k$ and $y = (y_1, \cdots, y_k)' \in R^k$, we let $x'y = \Sigma x_i y_i$ and $||x|| = +(x'x)^{\frac{1}{2}}$. Let $\mu = (\mu_1, \cdots, \mu_k)'$ and $\eta \in R^k$ be a given vector.

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The result of Theorem 1 pertains to the estimation of $v = \mu+\eta\sigma$ when loss is measured by $\sigma^{-2}||t-v||^2$ if an estimate, t, of v is chosen. For this problem a minimax estimator is $\hat{\theta} = \hat{\theta}(T, S) = T+\eta C_{n+k}S^{-\frac{1}{2}}$, where C_{n+k} is defined as in equation (1.1) and $T = (T_1, \cdots, T_k)'$. Also, $\hat{\theta}$ is "best" among all estimators having the property, $\hat{\theta}(at+b, a^2s) = a\hat{\theta}(t, s)+b$, for all a, b, t, s ($a > 0$, $s > 0$, $b \in R^k$, $t \in R^k$).

Observe that $\hat{\theta}$ is a member of the class of all estimators of the form $S^{\frac{1}{2}}\psi(Y)$ where $Y = S^{-\frac{1}{2}}T$ and ψ is a measurable function. Define ψ^* by $\psi^*(y) = \psi(y)+\min\{0, \Delta(y)\}y$, where

$$\Delta(y) = (||y||^2+1+(\tfrac{1}{4})(\eta'y)^2(n+k)^{-1}-y'\psi(y))||y||^{-2}$$

if $\eta'y > 0$ and $\Delta(y) = 0$ if $n'y \leqq 0$. In Section 2 we prove the following result.

THEOREM 1. *If ψ^* is defined as above*

$$E_{\mu,\sigma}||\psi^*(Y)S^{\frac{1}{2}}-v||^2 \leqq E_{\mu,\sigma}||\psi(Y)S^{\frac{1}{2}}-v||^2. \tag{1.2}$$

Moreover, if $P_{\mu,\sigma}(\psi^*(Y) \neq \psi(Y)) > 0$ then strict inequality holds in (1.2).

If we set $\psi(y) = y+\eta C_{n+k}$, then $\psi^*(y) = \psi(y)$ unless $(\frac{1}{4})(\eta'y)^2(n+k)^{-1}-(\eta'y)C_{n+k}+1 < 0$, that is, unless

$$|\tfrac{1}{2}\eta'y(n+k)^{-1}-C_{n+k}| < [C_{n+k}^2-(n+k)^{-1}]^{\frac{1}{2}}. \tag{1.3}$$

This last assertion depends on the observation that $(n+k)C_{n+k}^2-1 > 0$. To see this note that

$$(n+k)C_{n+k}^2-1 = \Gamma^2\left(\frac{n+k}{2}\right)\Gamma^{-2}\left(\frac{n+k+1}{2}\right)K\left(\frac{n+k}{2}\right)$$

where $K(\alpha) = \Gamma(\alpha+1)/\Gamma(\alpha)-(\Gamma(\alpha+\frac{1}{2})/\Gamma(\alpha))^2$, $\alpha > 0$. But $K(\alpha)$ is just the variance of $W^{\frac{1}{2}}$ when W has the gamma density which is $(\Gamma(\alpha))^{-1}w^{\alpha-1}e^{-w}$ or zero according as w is greater than zero or not. It follows for this choice of ψ that $P_{\mu,\sigma}(\psi(Y) \neq \psi^*(Y)) > 0$ for all μ and σ and consequently, by Theorem 1, that $T+\eta C_{n+k}S^{\frac{1}{2}}$ is inadmissible.

In [2] the more general problem of estimating $A\mu+\eta\sigma$ under dimension-free quadratic loss is considered where A is a known $m\times k$ matrix and $\eta \in R^m$. It is shown there that $\hat{t} = AT+\eta C_{n+k}S^{\frac{1}{2}}$ is inadmissible if $||\eta|| > C$, C being a sufficiently large constant. If $||\eta|| > C$, an estimator with uniformly smaller risk is

$$\hat{t}_1 = [AY+\eta\min\{C_{n+k}, C_{n+k+1}(1+||Y||^2)^{\frac{1}{2}}\}]S^{\frac{1}{2}}.$$

Set $m = k$ and A equal to the identity matrix. We conclude that $\hat{\theta}$ is inadmissible if $||\eta|| > C$. The result of the present paper generalizes this last result in two directions. First, the restriction, $||\eta|| > C$, is replaced by $||\eta|| \neq 0$. Second, the result is obtained not only for $\hat{\theta}$ but for each member of a family of possible estimators of v. The generalization is achieved using a different approach. In [2], the improvement (for $||\eta|| > C$) is achieved by simply replacing $C_{n+k}S^{\frac{1}{2}}$, in $\hat{\theta}$, by an improved estimator of σ. Here we use a conditional expectation argument which is analogous, in form, to that involved in the Blackwell–Rao method for improving unbiased estimators.

2. Proof of Theorem 1. We adopt the following notation:

$$\Delta_r(a) = \int_0^\infty g^r \exp(-\tfrac{1}{2}g^2 + ga)\,dg,$$
$$T_r(a) = \Delta_r(a)[\Delta_{r+1}(a)]^{-1}$$
$$G_r(a) = \tfrac{1}{2}r^{-1}(+(a^2+4r)^{\frac{1}{2}} - a),$$
$$H_r(a, b) = (a+b)G_r(a)$$
$$-\infty < a, b < \infty, r = 1, 2, \cdots.$$

LEMMA 2. *If* $b > 0$, $\sup_a (a+b)T_r(a) \leqq 1 + (\frac{1}{4})b^2 r^{-1}$.

PROOF. Jensen's inequality implies $T_r(a) > T_{r+1}(a)$ for all a and r. After integrating by parts we find that $\Delta_r(a) = (r+1)^{-1}(\Delta_{r+2}(a) - a\Delta_{r+1}(a))$. Thus

$$a = (T_{r+1}(a))^{-1} - (r+1)T_r(a)$$
$$< (T_{r+1}(a))^{-1} - (r+1)T_{r+1}(a),$$
$$-\infty < a < \infty, r = 0, 1, 2, \cdots.$$

It follows that $rT_r^2(a) + aT_r(a) - 1 < 0$, that is, $0 < T_r(a) < G_r(a)$, for all a and r.

Now $\sup_a (a+b)T_r(a) = \sup_{a>-b}(a+b)T_r(a) \leqq \sup_{a>-b} H_r(a, b)$. Consider $(\partial/\partial a)H_r(a, b) = \frac{1}{2}r^{-1}[+(a^2+4r)^{\frac{1}{2}} - a + (a+b)(a(a^2+4r)^{-\frac{1}{2}} - 1)]$. It is readily seen that $(\partial/\partial a)H_r(a, b) = 0$ when $a = a_0 = \frac{1}{2}(4r - b^2)b^{-1}$. Furthermore $a_0 > -b$ and $H_r(a_0, b) = 1 + (\frac{1}{4})b^2 r^{-1} > \lim_{a\to\infty} H_r(a, b) = 1 > H_r(-b, b) = 0$. Thus the maximum of $H_r(\cdot, b)$ is achieved at a_0 and the conclusion of the lemma is immediate.

PROOF OF THEOREM 1. Observe that if we let $\lambda = \mu\sigma^{-1}$ and take ψ to be any measurable function,

$$E_{\mu,\sigma}||S^{\frac{1}{2}}\psi(Y) - v||^2\sigma^{-2} = E_{\lambda,1}||S^{\frac{1}{2}}\psi(Y) - \lambda - \eta||^2$$
$$= E_{\lambda,1}c(Y, \lambda)||\psi(Y) - \omega(Y, \lambda)||^2 + b(\lambda)$$

where

$$c(Y, \lambda) = E_{\lambda,1}(S \mid Y),$$
$$\omega(Y, \lambda) = (\lambda + \eta)E_{\lambda,1}(S^{\frac{1}{2}} \mid Y)\cdot E_{\lambda,1}(S \mid Y)^{-1},$$

and b is a function whose value does not depend on ψ and plays no role in the argument. It is easy to show that

$$\omega(Y, \lambda) = (\lambda + \eta)T_{n+k}(\lambda' Y(1 + ||Y||^2)^{-\frac{1}{2}})(1 + ||Y||^2)^{\frac{1}{2}}.$$

For simplicity, let $v = \omega(y, \lambda)$, $x = y'\psi(y) - 1 - ||y||^2 - (\frac{1}{4})(\eta' y)^2(n+k)^{-1}$, and $w = y'\psi(y) - y'v$. Observe that $y'(\psi(y) - w||y||^{-2}y - v) \equiv 0$ so that

$$||\psi(y) - v||^2 = w^2||y||^{-2} + ||\psi(y) - w||y||^{-2}y - v||^2.$$

Also

$$||\psi(y)-x||y||^{-2}y-v||^2 = (w-x)^2||y||^{-2}+||\psi(y)-w||y||^{-2}y-v||^2.$$

Thus

$$||\psi(y)-v||^2-||\psi(y)-x||y||^{-2}y-v||^2 = ||y||^{-2}(w^2-(w-x)^2). \tag{2.1}$$

For convenience let $z = y(1+||y||^2)^{-\frac{1}{2}}$. It follows that

$$\begin{aligned} w &= y'\psi(y)-(1+||y||^2)(\lambda'z+\eta'z)T_{n+k}(\lambda'z) \\ &\geqq y'\psi(y)-(1+||y||^2)(1+(\tfrac{1}{4})(\eta'z)^2(n+k)^{-1}) \\ &= x \end{aligned}$$

if $\eta'y > 0$. Furthermore, if y is such that $x > 0$ and $\eta'y > 0$, the quantity on the left-hand side of (2.1) is positive. Theorem 1 is an immediate consequence of this observation.

REFERENCES

[1] STEIN, C. (1961). Summary of Wald lectures (unpublished).
[2] ZIDEK, J. V. (1969). Inadmissibility of the best invariant estimator of extreme quantiles of the normal law under squared error loss, *Ann. Math. Statist.* **40** 1801–1808.