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ON THE INDIVIDUAL ERGODIC THEOREM FOR SUBSEQUENCES[1]

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The purpose of this paper is to show that the individual ergodic theorem for subsequences fails to hold for measure preserving (m.p.) transformations of [0, 1] other than the identity.

Ten years ago Blum and Hanson [1] proved the following mean ergodic theorem for subsequences:

THEOREM 1 [1]. *Let T be an invertible* m.p. *transformation of a probability space* $(\Omega, \mathscr{F}, \mu)$. *If T is strongly mixing the averages*

$$f_n = n^{-1} \sum_{i=1}^{n} f \circ T^{k_i} \tag{1}$$

converge in L_1-norm for all $f \in L_1$ and all strictly increasing sequences (k_i) of integers. Conversely, if the limit is required to be the constant $\int f d\mu$, the strong mixing condition is also necessary.

N. Friedman and D. Ornstein [4] gave an example of a strongly mixing T for which there exists an indicator function $f = 1_A$ and a strictly increasing sequence (k_i) such that

$$\liminf_{n\to\infty} f_n = 0 \quad \text{and} \quad \limsup_{n\to\infty} f_n = 1 \tag{2}$$

almost everywhere. Their construction is quite complicated. We show that *every* strongly mixing T could serve as an example. In particular the individual ergodic theorem for subsequences fails for Bernoulli shifts. This answers a question raised in the book of N. Friedman ([3] page 134). Our approach to the problem is quite different from that of [4].

If T is a m.p. transformation of a probability space $(\Omega, \mathscr{F}, \mu)$ we denote by Ω_1 the largest (mod μ) $\mathscr{F}$-measurable set $B \in \mathscr{F}$ such that $T^{-1}A = A$ (mod μ) holds for all $\mathscr{F}$-measurable $A \subseteq B$. Ω_1 is called the identity set of T. If $\mathscr{F}$ is countably generated and separates points we have $\Omega_1 = \{\omega \in \Omega: T\omega = \omega\}$ mod μ. We can now formulate our result as follows:

THEOREM 2. *There exists a universal strictly increasing sequence (k_i) of nonnegative integers such that for every* m.p. *transformation T of a probability space* $(\Omega, \mathscr{F}, \mu)$ *there exists an indicator function* $f = 1_A (A \in \mathscr{F})$ *with*

$$\liminf_{n\to\infty} f_n = 0 \quad \textit{and} \quad \limsup_{n\to\infty} f_n = 1 \quad \text{a.e. } \textit{on} \quad \Omega \backslash \Omega_1. \tag{3}$$

A m.p. transformation T in $(\Omega, \mathscr{F}, \mu)$ is called aperiodic if for every $n \geqq 1$ the identity set Ω_n of T^n is a nullset. A modification of the proof of Theorem 2 yields the following theorem, the proof of which we leave to the reader.

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THEOREM 3. *There exists a universal strictly increasing sequence* (k_i) *of nonnegative integers such that for every aperiodic* m.p. *transformation* T *of a probability space* $(\Omega, \mathscr{F}, \mu)$ *and for every* p *with* $1 \leqq p < \infty$ *the set of functions* $f \in L_p$ *with*

$$\liminf_{n\to\infty} f_n = -\infty \quad \textit{and} \quad \limsup_{n\to\infty} f_n + = \infty \quad \text{a.e.} \tag{4}$$

is a dense G_δ *in* L_p, *and the system of sets* $A \in \mathscr{F}$ *for which* $f = 1_A$ *satisfies* (2) a.e. *is a dense* G_δ *in* $\mathscr{F}$, (*with metric* $d(A, B) = \mu(A \triangle B)$).

PROOF OF THEOREM 2. As Ω_1 is T-invariant we may assume $\Omega_1 = \varnothing$. For $n \geqq 2$ let $\Omega^{(n)} = \Omega_n \backslash \bigcup_{i=1}^{n-1} \Omega_i$ be the set where T is periodic with period n. We shall make frequent use of the following results of Rohlin (see [5], [6] Lemmas 2.1–2.3):

(i) $\Omega^{(n)}$ is a disjoint union of n sets $E_{n,1}, \cdots, E_{n,n} \in \mathscr{F}$ such that $E_{n,k+1} = T^{-1}E_{n,k}$ $(k = 1, \cdots, n-1)$ and $E_{n,1} = T^{-1}E_{n,n}$.

(ii) Let $\Omega^{(0)} = \Omega \backslash \bigcup_{m=1}^{\infty} \Omega_m$. For every $\varepsilon > 0$ and $n \in \mathbb{N} = \{1, 2, 3, \cdots\}$ there exists a set $E \in \mathscr{F}$ such that $E \subseteq \Omega^{(0)}$, $E, T^{-1}E, \cdots, T^{-(n-1)}E$ are disjoint, and $\mu(\Omega^{(0)} \backslash \bigcup_{k=0}^{n-1} T^{-k}E) < \varepsilon$.

We shall assume $\Omega = \Omega^{(0)}$ in our construction. The set $A \in \mathscr{F}$ and the sequence (k_i) will be defined inductively. In the tth step we determine $k_{m_{t-1}+1}, k_{m_{t-1}+2}, \cdots, k_{m_t}$ and an approximation A_t of A so as to produce divergence of f_n on $\Omega^{(0)}$. To prove the theorem in full generality a subsequence of the sequence of steps of the construction must be reserved to the definition of some k_i in such a way as to produce divergence on $\Omega^{(k)}$ $(k \geqq 2)$. It will be clear from the present proof, how to proceed. The assumption $\Omega = \Omega^{(0)}$ is only made in order to keep notation and technicalities down.

We start the construction (step 1) by defining $m_1 = 1, k_1 = 0, A_1 = \varnothing$.

At the end of step $t-1$ $(t \geqq 2)$ we have defined a strictly increasing finite sequence $1 = m_1 < m_2 < \cdots < m_{t-1}$ of integers, a strictly increasing finite sequence $0 = k_1 < k_2 < \cdots < k_{m_{t-1}}$ of integers and $t-1$ sets $A_\tau \in \mathscr{F}$ $(\tau = 1, \cdots, t-1)$. Let

$$S_n 1_{A_\tau} = n^{-1} \textstyle\sum_{\nu=1}^{n} 1_{A_\tau} \circ T^{k_\nu},$$

$$G_{i,\tau} = \{0 \leqq \inf_{m_i < n \leqq m_{i+1}} S_n 1_{A_\tau} < 2^{-i}\} \quad \text{and}$$

$$H_{i,\tau} = \{1 - 2^{-i} < \sup_{m_i < n \leqq m_{i+1}} S_n 1_{A_\tau} \leqq 1\}.$$

The sets A_τ $(1 < \tau \leqq t-1)$ and the numbers k_ν and m_i have been chosen in such a way that the inequalities

$$\mu(G_{i,\tau}) > 1 - 2^{i}(1 - 2^{-\tau}) \quad \text{and} \tag{5}$$

$$\mu(H_{i,\tau}) > 1 - 2^{-i}(1 - 2^{-\tau}) \tag{6}$$

are satisfied for $1 \leqq i < t-1$ and $i < \tau < $ t. (Note that nothing has to be proved for $t = 2$, since there is no i with $1 \leqq i < t-1$ in that case.)

Step t. Let $\alpha_t = m_{t-1}^{-1} 2^{-2(t+2)}$. If $A_t \in \mathscr{F}$ is such that $\mu(A_{t-1} \triangle A_t) \leqq \alpha_t$ the inequalities (5) and (6) will hold for $1 \leqq i < t-1$ and $\tau = t$. To see this observe that $G_{i,t-1} \triangle G_{i,t}$ and $H_{i,t-1} \triangle H_{i,t}$ are contained in the set $\bigcup_{v=1}^{m_t-1} T^{-k_v}(A_{t-1} \triangle A_t)$, so that

$$\mu(G_{i,t}) \geqq \mu(G_{i,t-1}) - \mu(G_{i,t-1} \triangle G_{i,t})$$
$$\geqq 1-2^{-i}(1-2^{-t+1})-2^{-2(t+2)} \geqq 1-2^{-i}(1-2^{-t}).$$

The same argument applies to (6).

Let p_t be an integer with $p_t > 2\alpha_t^{-1}$. The idea is to choose q_t very large and to apply Rohlin's result (ii) with $n = p_t q_t$. If q_t is large T behaves for a long time just like a periodic transformation with period p_t.

Let $r_0 = r_0(t)$ be the smallest multiple of p_t which is larger than $k_{m_{t-1}}$. Pick $l_1 = l_1(t) \in \mathbb{N}$ such that

$$l_1(l_1+m_{t-1})^{-1} > 1-2^{-t}. \tag{7}$$

Let $k_{m_{t-1}+j} = r_0 + jp_t$ $(j = 1, 2, \cdots, l_1)$. If l_s $(s \geqq 1)$ has been determined find $l_{s+1} = l_{s+1}(t) \in \mathbb{N}$ with

$$l_{s+1}(l_1+l_2+\cdots+l_{s+1}+m_{t-1})^{-1} > 1-2^{-t}. \tag{8}$$

For j with $l_1+l_2+\cdots+l_s < j \leqq l_1+l_2+\cdots+l_{s+1}$ $(s \leqq p_t-1)$ define $k_{m_{t-1}+j}$ by

$$k_{m_{t-1}+j} = r_0 + jp_t + s. \tag{9}$$

Let $m_t = m_{t-1} + \sum_{s=1}^{p_t} l_s$. We have now completely specified $k_1 < k_2 < \cdots < k_{m_t}$. Let $q_t \in \mathbb{N}$ be such that $q_t > k_{m_t} \cdot 2^{t+3}$. We apply Rohlin's theorem (ii) with $n = n_t = p_t q_t$ and $\varepsilon = \varepsilon_t = 2^{-(t+3)}$. We obtain the existence of a set $E_t \in \mathscr{F}$ such that the sets $E_t, T^{-1}E_t, \cdots, T^{-(n_t-1)}E_t$ and disjoint and

$$\mu(\Omega \backslash \bigcup_{v=0}^{n_t-1} T^{-v}E_t) < 2^{-(t+3)}. \tag{10}$$

Let $D_t = \bigcup_{j=0}^{q_t-1} T^{-jp_t}E_t$. We complete step t of the construction by defining $A_t = D_t \cup (A_{t-1} \backslash T^{-1}D_t)$.

The set D_t has measure at most equal to $p_t^{-1} < 2^{-1}\alpha_t$. It follows that $\mu(A_t \triangle A_{t-1}) < \alpha_t < 2^{-(t+1)}$. This implies that the sequence A_t converges to a set $A \in \mathscr{F}$. It remains to prove that (3) holds with $f = 1_A$ and with the inductively defined sequence (k_i).

Let $\omega \in \bigcup_{v=k_{m_t}}^{n_t-1} T^{-v}E_t$. For some integer ρ with $0 \leqq \rho < p_t$ we have $\omega \in T^{-\rho}D_t$. It follows that

$$T^{k_{m_{t-1}+j}}\omega \in D_t \subseteq A \qquad \text{for all } j \text{ with} \tag{11}$$

$\sum_{u=1}^{\rho} l_u < j \leqq \sum_{u=1}^{\rho+1} l_u$, because $T^{\rho}\omega \in D_t$ and then the point ω revisits D_t periodically with period p_t until it reaches E_t. This does not happen before time $k_{m_t} \geqq k_{m_{t-1}+j}$. It follows from (11) that the last $l_{\rho+1}$ terms in the sequence $1_A \circ T^{k_v}(\omega)(1 \leqq v \leqq m_{t-1} + \sum_{u=1}^{\rho+1} l_u)$ are equal to 1. By (7) or (8) we obtain $\omega \in H_{t-1,t}$. From $H_{t-1,t} \supseteq \bigcup_{v=k_{m_t}}^{n_t-1} T^{-v}E_t$ we get

$$\mu(H_{t-1,t}) \geqq 1 - \mu(\Omega \backslash \bigcup_{v=0}^{n_t-1} T^{-v}E_t) - \mu(E_t) \cdot k_{m_t}.$$

The inequality (10) and the inequalities

$$\mu(E_t)k_{m_t} \leqq n_t^{-1}k_{m_t} = p_t^{-1}q_t^{-1}k_{m_t} < 2^{-(t+3)}$$

now imply $\mu(H_{t-1,t}) > 1-2^{-(t+2)} > 1-2^{-(t-1)}(1-2^{-t})$.

We have proved (6) for $i = t-1$, $\tau = t$.

The proof of (5) is similar: Let $\omega \in \bigcup_{\nu=k_{m_t}}^{n_t-1} T^{-\nu}E_t \backslash D_t$. In this case there is an integer ρ with $1 \leqq \rho < p_t$ and $\omega \in T^{-\rho}D_t$. It follows that

$$T^{k_{m_{t-1}}+j}\omega = T^{r_0+jp_t+\rho-1}\omega \in T^{-1}D_t \subseteq A_t^c \tag{12}$$

for all j with $\sum_{u=1}^{\rho-1} l_u < j \leqq \sum_{u=1}^{\rho} l_u$. Using (7) or (8) we obtain $\omega \in G_{t-1,t}$. Hence

$$\mu(G_{t-1,t}) \geqq 1-\mu(\Omega\backslash\bigcup_{\nu=0}^{n_t-1} T^{-\nu}E_t)-\mu(E_t)k_{m_t}-\mu(D_t).$$

Using the previous estimates and $\mu(D_t) \leqq p_t^{-1} < 2^{-(t+2)}$ we get (5) for $i = t-1$, $\tau = t$.

By our choice of α_t the inequalities (5) and (6) remain valid for each larger τ. Passing for fixed i with τ to infinity we get for $i \geqq 1$;

$$\mu\{0 \leqq \inf_{m_i<n\leqq m_{i+1}} S_n 1_A \leqq 2^{-i}\} \geqq 1-2^{-i}$$

and

$$\mu\{1-2^{-i} \leqq \sup_{m_i<n\leqq m_{i+1}} S_n 1_A\} \geqq 1-2^{-i},$$

where $S_n 1_A = f_n$. Clearly this implies (3). □

In [7] Professor Sucheston and the author proved a mean ergodic theorem for subsequences for m.p. "mixing" transformations fo an infinite σ-finite measure space. Using stacking constructions (see [3] page 85) it is possible to see that the corresponding individual ergodic theorem for subsequences fails for certain m.p. "mixing" transformations T. We have made no attempt to find out whether it fails for all conservative m.p. transformations.

In [2] Brunel and Keane have proved the following individual ergodic theorem with weighted averages: A m.p. transformation T of a probability space $(\Omega, \mathscr{F}, \mu)$ is strongly mixing if and only if for each strictly increasing sequence (k_i) and each $f \in L_1$ there exists a decreasing sequence (c_i) of positive real numbers with divergent sum such that $(\sum_{i=1}^{n} c_i)^{-1}(\sum_{i=1}^{n} c_i f \circ T^{k_i}) \to \int f\,d\mu$ a.e. It is easy to observe that (c_i) can be chosen independent of (k_i). Using the methods of the present paper it is possible to see that (c_i) cannot be chosen in such a way that it depends on T only.

It is also shown in [2] that f_n converges a.e., if T is weakly mixing and (k_i) is a sequence of a special type, called *uniform* in [2]. Professor Brunel has pointed out to the author that the weak mixing condition in the Corollary on page 236 [2] is also necessary. If T is not weakly mixing a uniform sequence (k_i) for which f_n diverges for some f is obtained by considering a rotation of the unit circle by an eigenvalue $\neq 1$.

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