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MARTINGALE CENTRAL LIMIT THEOREMS

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1. Introduction, summary and acknowledgments. The classical Lindeberg–Feller CLT for sums of independent random variables (rv's) provides more than the convergence in distribution of the sum to a normal law. The independence of summands also guarantees the weak convergence of all finite dimensional distributions of an a.e. sample continuous stochastic process (suitably defined in terms of the partial sums) to those of a Gaussian process with independent increments, namely, the Wiener process. Moreover, the distributions of said process converge weakly to Wiener measure on $C[0, 1]$, the latter result being known as an *invariance principle*, or *functional* CLT, an idea originating with Erdos and Kac [10] and Donsker [5], then developed by Billingsley, Prohorov, Skorohod and others.

The present work contains an invariance principle for a certain class of martingales, under a martingale version of the classical Lindeberg condition. In the case of sums of independent rv's, our results reduce to the conventional invariance principle (see, for example, Parthasarathy [16]) in the setting of the classical Lindeberg–Feller CLT.

Theorem 1 contains a type of martingale characteristic function convergence which is strictly analogous to the classical CLT, while Theorem 2 provides weak convergence of finite dimensional distributions to those of a Wiener process, followed by (Theorem 3) the weak convergence of corresponding induced measures on $C[0, 1]$ to Wiener measure, thus entailing an invariance principle for martingales.

Notation and results are listed in Section 2. Section 3 defines the Lindeberg condition for martingales and gives it several equivalent forms. Sections 4 and 5 contain the proofs of Theorems 1 and 2, respectively, while Theorem 3 is proved in Section 6 by use of a martingale inequality derived from an upcrossing inequality of Doob [7]. Section 7 contains brief remarks.

Among the large literature on CLT's for sums of dependent rv's, mention of a martingale CLT is first made by Lévy [12], [13], followed by Doob [6] page 383. Billingsley [2] and Ibragimov [11] gave a version for stationary ergodic martingales, and Csorgo [4] considered related problems. The author also knows of Dvoretsky [8]. Invariance principles for various dependent rv's were found by Billingsley [1], and in [3] for stationary ergodic martingales, the latter result being given by Rosén [17] for bounded summands. The present Theorem 3 relaxes the stationarity and ergodicity requirements of Billingsley's Theorem (23.1) of [3].

Since preparing the original version of this paper, the author's attention has been drawn to Dvoretsky [9], which announces a result strongly resembling the present Theorem 2, and to the CLT [14] and invariance principle [15] for reversed martingales, due to Loynes.

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The methods of Section 5 are owed to Billingsley [2] (who in turn acknowledges a "debt to Lévy") and to ideas given by Dvoretsky [8]. Finally, I wish to thank Dr. C. C. Heyde, for making a vital remark, and Mr. David Scott, for helpful criticism.

2. Notation and results. Let $\{S_n, \mathscr{F}_n, n = 1, 2, \cdots\}$ be a martingale on the probability space $\{\Omega, \mathscr{F}, P\}$, with $S_0 = 0$, and $X_n = S_n - S_{n-1}$, $n = 1, 2, \cdots$. $\mathscr{F}_0$ need not be the trivial σ-field $\{\phi, \Omega\}$. Let

$$\phi_j(t) = E(e^{itX_j} \mid \mathscr{F}_{j-1}) = E_{j-1}(e^{itX_j}),$$

where E_{j-1} denotes $E(\cdot \mid \mathscr{F}_{j-1})$, and let

$$\sigma_n^2 = E_{n-1}(X_n^2),$$
$$V_n^2 = \textstyle\sum_{j=1}^n \sigma_j^2,$$
$$s_n^2 = EV_n^2 = ES_n^2,$$
$$f_n(t) = \textstyle\prod_{j=1}^n \phi_j(t/s_n) \qquad \text{and}$$
$$b_n = s_n^{-2} \max_{j \leqq n} \sigma_j^2$$

for $n = 1, 2, \cdots$. Following Parthasarathy [16] page 220, define

$$\xi_n(t) = s_n^{-1}(S_k + X_{k+1}(ts_n^2 - s_k^2)/(s_{k+1}^2 - s_k^2))$$

for $0 \leqq t \leqq 1$ and $s_k^2 \leqq ts_n^2 \leqq s_{k+1}^2$, $k = 0, 1, \cdots, n-1$. $\xi_n(t)$ is a.e. continuous on $0 \leqq t \leqq 1$ for all n, being composed of straight line segments joining the points $(s_k^2/s_n^2, S_k/s_n)$, $k = 0, 1, 2, \cdots, n$.

Throughout, we consider martingales for which

$$V_n^2 s_n^{-2} \to_p 1 \qquad \text{as } n \to \infty. \tag{1}$$

For this class of martingales, the Lindeberg condition is said to hold if

$$s_n^{-2} \textstyle\sum_{j=1}^n EX_j^2 I(|X_j| \geqq \varepsilon s_n) \to_p 0 \qquad \text{as } n \to \infty \tag{2}$$

for all $\varepsilon > 0$, where $I(A)$ denotes the indicator function of the set A.

THEOREM 1. *Assume that equation* (1) *holds. Then*

$$f_n(t) \to_p e^{-\frac{1}{2}t^2} \qquad \textit{as } n \to \infty, \textit{ for all } t, \textit{ and} \tag{3}$$

$$b_n \to_p 0 \qquad \textit{as } n \to \infty \tag{4}$$

if and only if the Lindeberg condition (2) *holds.*

THEOREM 2. *If* (1) *and the Lindeberg condition hold, then in addition to* (3) *and* (4),

$$\lim_{n\to\infty} P[S_n/s_n \leqq x] = \Phi(x) = (2\pi)^{-\frac{1}{2}} \textstyle\int_{-\infty}^x e^{-\frac{1}{2}y^2}\, dy$$

for all x. Furthermore, all finite dimensional distributions of $\xi_n(t)$ converge weakly, as $n \to \infty$, to those of a Wiener process $W(t)$ on $0 \leqq t \leqq 1$, where $W(0) = 0$ a.e. *and $EW^2(1) = 1$.*

THEOREM 3. *Let* $\{C, \mathscr{B}, P_W\}$ *be the probability space where* $C = C[0, 1]$ *with the* sup *norm topology,* $\mathscr{B}$ *being the Borel σ-field generated by open sets in C, and* P_W, *Wiener measure. Let* $\{P_n\}$ *be the sequence of probability measures on* $\{C, \mathscr{B}\}$ *determined by the distribution of* $\{\xi_n(t), 0 \leqq t \leqq 1\}$. *Then if* (1) *and the Lindeberg condition hold,* $P_n \to P_W$ *weakly as* $n \to \infty$.

Throughout, we use the notations $X_+ = \max(0, X)$ and $X_- = \max(0, -X)$, while $\mathscr{R}lz$ is used to denote the real part of z.

3. The Lindeberg condition for martingales. For the class of martingales for which (1) holds, the Lindeberg condition is defined by (2). The definition corresponds exactly to that of the classical Lindeberg condition which is NSC for (3) and (4) when $X_1, X_2, \cdots$ are independent rv's. However, in the martingale case, and under (1), it is equivalent to several alternative conditions listed in Lemma 2 below. To introduce such alternatives, consider

$$\begin{aligned} g(n,\varepsilon) &= V_n^{-2}\textstyle\sum_{j=1}^n E_{j-1}X_j^2 I(|X_j| \geqq \varepsilon s_n),\\ G(n,\varepsilon) &= V_n^2 s_n^{-2}\, g(n,\varepsilon),\\ h(n,\varepsilon) &= V_n^{-2}\textstyle\sum_{j=1}^n E_{j-1}X_j^2 U(|X_j|\varepsilon^{-1}s_n^{-1}), \qquad \text{and}\\ H(n,\varepsilon) &= V_n^2 s_n^{-2} h(n,\varepsilon), \end{aligned}$$

where $U(x)$ is any continuous nonnegative function of bounded variation on $[0, \infty)$ for which $U(0) = 0$ and $U(x) \to \text{const} \ (>0)$ as $x \to \infty$. G, H denote rv's with a divisor of s_n^2, while g, h denote corresponding rv's with a divisor of V_n^2 instead.

LEMMA 1. (1) *is equivalent to*

$$\lim_{n\to\infty} E\,|V_n^2 s_n^{-2} - 1| = 0. \tag{5}$$

PROOF. It suffices to show that $(1) \Rightarrow (5)$. But $(1) \Rightarrow \lim_{n\to\infty} E(V_n^2 s_n^{-2} - 1)_- = 0$, which $\Rightarrow (5)$, since

$$E(V_n^2 s_n^{-2} - 1)_- = E(V_n^2 s_n^{-2} - 1)_+.$$

LEMMA 2. *Under the condition* (1), *or alternatively* (5), *the Lindeberg condition is equivalent to the convergence to zero as* $n \to \infty$ *of* $g(n, \varepsilon)$, $G(n, \varepsilon)$, $h(n, \varepsilon)$ *or* $H(n, \varepsilon)$, *for all* $\varepsilon > 0$; *either in probability or in the mean or order* 1.

PROOF. Convergence in mean being a priori stronger than convergence in probability, it will suffice to show firstly the mutual equivalence of convergences in probability of g, G, h and H, then secondly to show that each such convergence in probability implies a corresponding convergence in mean. (Noting of course that the Lindeberg condition is defined by $EG(n, \varepsilon) \to 0$ as $n \to \infty$ for all $\varepsilon > 0$.)

Firstly, concerning convergences in probability, those of the pairs g and G; and h, H are equivalent because of (1), while $h \to_p 0$ implies $g \to_p 0$ by noting that there exist constants $a, b > 0$ for which

$$bI(x \geqq a) \leqq U(x) \qquad \text{for all } x \geqq 0. \tag{6}$$

Finally it is readily verified that

$$h(n,\varepsilon) = \int_0^\infty g(n,\varepsilon y)\,dU(y) \qquad \text{pointwise a.e.,}$$
$$\leqq \int_0^\delta dU(y) + K\cdot g(n,\varepsilon\delta) \qquad \text{a.e.}$$

since $g(n, y) \leqq 1$ a.e. and $g(n, y)\downarrow$ as $y\downarrow$, for each n; where K is the total variation of U. Thus, by choosing δ small, it is apparent that the condition

(7) $$g(n,\varepsilon)\to_p 0 \qquad \text{as } n\to\infty \text{ for all } \varepsilon>0 \text{ implies that}$$

(8) $$h(n,\varepsilon)\to_p 0 \qquad \text{as } n\to\infty \text{ for all } \varepsilon>0.$$

Secondly, we must show that convergences in probability of g, G, h and H imply corresponding convergences in mean. In the case of g and h this is trivial since the rv's in question are uniformly bounded; thus we can assume that

(9) $$\lim_{n\to\infty} Eh(n,\varepsilon) = 0, \qquad \text{for all } \varepsilon>0.$$

It will suffice now to show that (8) and hence (9) imply

$$\lim_{n\to\infty} EH(n,\varepsilon) = 0 \qquad \text{for all } \varepsilon>0,$$

for the mean convergence of G will then follow by applying equation (6). Write $V_n^2 s_n^{-2} = 1+\delta_n$. Therefore

$$EH(n,\varepsilon) = E(1+\delta_n)h(n,\varepsilon)$$
$$= Eh(n,\varepsilon) + E\delta_n\, h(n,\varepsilon)$$

which $\to 0$ as $n\to\infty$ from (9) and since $E|\delta_n h(n,\varepsilon)| \leqq K\cdot E|\delta_n|$, $\to 0$ as $n\to\infty$ by Lemma 1. The proof is completed.

4. Proof of Theorem 1. Define functions Q and M by writing $e^{ix} = 1+ix-\frac{1}{2}x^2+\frac{1}{2}x^2 Q(x)$, and $M(x) = \min(x/3, 2)$. Thus

(10) $$|1-Q(x)| \leqq 1 \qquad \text{and}$$

(11) $$|Q(x)| \leqq M(x) \qquad \text{for all } x.$$

By using (10) and the inequality $|\log z - z + 1| \leqq |z-1|^2/(1-|z-1|)$ (for $|z-1|<1$) we can write, after some computation,

(12) $$\log f_n(t) = \sum_{j=1}^n \log\phi_j(ts_n^{-1})$$
$$= -\tfrac{1}{2}t^2 V_n^2 s_n^{-2} + \tfrac{1}{2}t^2 s_n^{-2}\sum_{j=1}^n E_{j-1}(X_j^2 Q(tX_j s_n^{-1})) + A_n(t)$$

where for each fixed t, $A_n(t)\to_p 0$ if $b_n\to_p 0$ as $n\to\infty$. But it is easy to show that the Lindeberg condition implies that $b_n\to_p 0$ as $n\to\infty$. Theorem 1 will then follow if the convergence in probability to zero as $n\to\infty$ of the second term of (12) is equivalent to the Lindeberg condition. But this equivalence can be shown by firstly applying the inequality (11) and noting that the function $M(\cdot)$ has the properties required of the function $U(\cdot)$ in Lemma 2; and then conversely by noting that the convergence in probability to zero of the second term of (12) implies the corresponding convergence of its real part, and that the function $\mathscr{R}l Q(\cdot)$ has the properties required of $U(\cdot)$ in Lemma 2.

5. Proof of Theorem 2. It is assumed throughout that (1) and the Lindeberg condition both hold. Adapting an idea used in Billingsley [2], we can choose any constant $C > 1$ and for each fixed n set

$$X_j^*(n) = X_j I(V_j^2 \leqq Cs_n^2), \qquad j = 1, 2, \cdots, n.$$

$\{X_1^*(n), X_2^*(n), \cdots, X_n^*(n)\}$ will form a martingale difference sequence and $\lim_{n\to\infty} P \bigcap_{j=1}^n [X_j = X_j^*(n)] = 1$ because of (1), so that to prove convergence in distribution properties for $\sum_j X_j/s_n$ as $n \to \infty$ (where the summation of j can be over any subset of the integers from 1 to n), it suffices to prove them for $\sum_j X_j^*(n)/s_n$ as $n \to \infty$. Moreover, the Lindeberg condition holds with X_j replaced by $X_j^*(n)$, $j = 1, 2, \cdots, n$, and

$$P[\textstyle\sum_{j=1}^n E_{j-1}(X_j^*(n))^2 \leqq Cs_n^2] = 1, \tag{13}$$

so instead of introducing the $\{X_j^*(n)\}$ rv's, we can work with the original $\{X_j\}$ rv's and assume, in addition, that

$$P[V_n^2 \leqq Cs_n^2] = 1, \qquad \text{all } n = 1, 2, \cdots. \tag{14}$$

For fixed k; $t_1, t_2, \cdots, t_k$ not all zero; and $0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{k-1} < \alpha_k = 1$, set

$$\begin{aligned} m_j &= \max\{m \geqq 0; s_m^2 \leqq \alpha_j s_n^2\}, \\ \theta_r &= t_j \qquad \text{for } m_{j-1} < r \leqq m_j, \\ \theta &= \max_{r\leqq n} |\theta_r| = \max_j |t_j|. \end{aligned} \tag{15}$$

For each fixed n, let $Y_r = \theta_r X_r/s_n$, $T_r = \sum_{j=1}^r Y_j$, and $U_r^2 = \sum_{j=1}^r E_{j-1}(Y_j^2)$. We wish to show that

$$\lim_{n\to\infty} Ee^{iT_n} = e^{-\frac{1}{2}\sigma^2}, \tag{16}$$

where $\sigma^2 = \sum_{j=1}^k t_j^2(\alpha_j - \alpha_{j-1})$. We will do this by showing that

$$\lim_{n\to\infty} E(\exp(iT_n + \tfrac{1}{2}U_n^2) - 1) = 0, \qquad \text{and} \tag{17}$$

$$\lim_{n\to\infty} E|\exp(\tfrac{1}{2}U_n^2) - e^{\frac{1}{2}\sigma^2}| = 0, \tag{18}$$

for routine computations from (17) and (18) then establish that (16) holds.

Proof of Equation (17). For n fixed let

$$\begin{aligned} Z_j &= (\exp(iT_{j-1} + \tfrac{1}{2}U_j^2))(e^{iY_j} - \exp(-\tfrac{1}{2}\theta_j^2\sigma_j^2/s_n^2)) \\ &= (\exp(iT_{j-1} + \tfrac{1}{2}U_j^2))(iY_j - \tfrac{1}{2}Y_j^2(1 - Q(Y_j)) \\ &\qquad + \tfrac{1}{2}\theta_j^2\sigma_j^2/s_n^2 - Z(\tfrac{1}{2}\theta_j^2\sigma_j^2/s_n^2)), \end{aligned} \tag{19}$$

where $Z(x) = e^{-x}-1+x$ for $x \geqq 0$. Therefore

$$|E_{j-1}Z_j| \leqq e^{\frac{1}{2}\theta^2 C}|\tfrac{1}{2}E_{j-1}Y_j^2Q(Y_j)-Z(\tfrac{1}{2}\theta_j^2\sigma_j^2/s_n^2)|$$
$$\leqq \tfrac{1}{2}e^{\frac{1}{2}\theta^2 C}(E_{j-1}Y_j^2M(|Y_j|)+\tfrac{1}{4}\theta^4\sigma_j^4 s_n^{-4})$$

(using (11), (14), $|\theta_j| \leqq \theta$, and the inequality $Z(x) \leqq \frac{1}{2}x^2$),

$$\leqq \tfrac{1}{2}\theta^2 e^{\frac{1}{2}\theta^2 C}s_n^{-2}(E_{j-1}X_j^2M(|\theta X_j|/s_n)+\theta^2 b_n\sigma_j^2/4).$$

But from (19) we have

$$|E(\exp(iT_n+\tfrac{1}{2}U_n^2)-1)| = |E\textstyle\sum_{j=1}^n Z_j|$$
(20) $$\leqq E\textstyle\sum_{j=1}^n |E_{j-1}Z_j|$$
$$\leqq \tfrac{1}{2}\theta^2 e^{\frac{1}{2}\theta^2 C}E(h(n,\theta^{-1})+\theta^2 b_n V_n^2/4s_n^2)$$

where (see Section 3) the function $M(\cdot)$ has the properties required by the function $U(\cdot)$ for Lemma 2 to be applicable. The right-hand side of (20) tends to zero as $n\to\infty$ by Lemma 2 and (4) and (14), thus proving (17).

PROOF OF EQUATION (18). For any $j \leqq n$,

$$s_n^{-2}EX_j^2 \leqq \varepsilon + s_n^{-2}EX_j^2 I(|X_j| \geqq \varepsilon s_n)$$
$$\leqq \varepsilon + EG(n,\varepsilon) \qquad \text{(see Section 3)}$$
$$\to \varepsilon \qquad \text{as } n\to\infty$$

by Lemma 2, since the Lindeberg condition holds. Therefore

(21) $$\lim_{n\to\infty} s_n^{-2}\max_{j\leqq n}EX_j^2 = 0,$$

from which it follows, with (15), that

$$\lim_{n\to\infty} s_n^{-2}(s_{m_j}^2 - s_{m_{j-1}}^2) = \alpha_j - \alpha_{j-1}$$

for $1 \leqq j \leqq k$; thus permitting an induction argument to be constructed, starting from (1), to show that

(21a) $$(V_{m_j}^2 - V_{m_{j-1}}^2)(s_{m_j}^2 - s_{m_{j-1}}^2)^{-1} \to 1$$

in probability as $n\to\infty$ for $1 \leqq j \leqq k$. It follows that $U_n^2 \to \sigma^2$ in probability as $n\to\infty$, and also, since $U_n^2 \leqq \theta^2 C$, that equation (18) holds.

REMAINDER OF PROOF OF THEOREM 2. Having proved equation (16), the proof of Theorem 2 will be completed by noting that, for each $j = 1, 2, \cdots k$,

$$|\xi_n(\alpha_j)-\xi_n(s_{m_j}^2/s_n^2)| \leqq |\xi_n(s_{m_{j+1}}^2/s_n^2)-\xi_n(s_{m_j}^2/s_n^2)|$$
$$= |X_{m_{j+1}}/s_n|$$
$$\to 0 \quad \text{in probability} \qquad \text{as } n\to\infty,$$

by (21).

6. Convergence to Wiener measure. In order to prove Theorem 3 from the weak convergence of finite dimensional distributions obtained in Theorem 2, the property of *relative compactness*, or *tightness* (see Billingsley [3]) must be verified; namely ([16] page 222) in the case of $C[0, 1]$ that

$$\lim_{h\to 0} \limsup_{n\to\infty} P[\sup_{|t-y|\leqq h} |\xi_n(t)-\xi_n(y)| > \varepsilon] = 0 \tag{22}$$

for all $\varepsilon > 0$. To establish (22), we employ a martingale inequality (Lemma 4) which is derived from Doob's upcrossing inequality (Doob [7]), namely

LEMMA 3. *Let β be the number of upcrossings of the interval $[a, b]$ by the submartingale $U_0, U_1, \cdots, U_n$. Then $(b-a)E\beta + E(U_0-a)_+ \leqq E(U_n-a)_+$.*

LEMMA 4. *Let $U_0 = 0, U_1, U_2, \cdots, U_n$ be martingale. Then, for all constants $c > 0$,*

$$P[\max_{j\leqq n} |U_j| > 2c] \leqq P[|U_n| > c] + \int_{[|U_n|\geqq 2c]} (c^{-1}|U_n|-2) \leqq \int_{[|U_n|\geqq c]} c^{-1}|U_n|.$$

PROOF. Let $A_n = [\min_{k\leqq n} U_k < -2c]$, and let β_1 be the number of upcrossings of $[-2c, -c]$ by $U_0, U_1, \cdots, U_n$. Then

$$PA_n = PA_n[U_n \geqq -c] + PA_n[U_n < -c] \leqq P[\beta_1 > 0] + P[U_n < -c], \tag{23}$$

and similarly, if $B_n = [\max_{k\geqq n} U_k > 2c]$ and if β_2 is the number of upcrossings of $[-2c, -c]$ by $U_0, -U_1, \cdots, -U_n$, then

$$PB_n \leqq P[\beta_2 > 0] + P[U_n > c]. \tag{24}$$

Therefore $P(A_n \cup B_n) \leqq PA_n + PB_n$, and the Lemma follows by adding equations (23) and (24), noting that $P[\beta_i > 0] \leqq E\beta_i$, $i = 1, 2$, and then applying Lemma 4 to $E\beta_1$ and $E\beta_2$.

PROOF OF THEOREM 3. To establish (22), we follow the proof of Parthasarathy [16] page 222, and hence we omit most details. We have

$$\begin{aligned} &P[\sup_{|t-y|\leqq h} |\xi_n(t)-\xi_n(y)| > \varepsilon] \\ &\qquad \leqq \sum_{kh<1} P[\sup_{kh<t\leqq(k+1)h} |\xi_n(t)-\xi_n(kh)| > \varepsilon/4] \end{aligned} \tag{25}$$

and

$$\sup_{kh<t\leqq(k+1)h} |\xi_n(t)-\xi_n(kh)| \leqq 2\max_{q_k\leqq r-1\leqq q_{k+1}} |\textstyle\sum_{j=q_k+1}^{r} X_j s_n^{-1}| \tag{26}$$

where $q_k = \max\{j \geqq 1 : s_j^2 \leqq khs_n^2\}$, $k = 0, 1, 2, \cdots$. Apply (26) in (25) and then apply Lemma 4. (22) then follows by the convergence of finite dimensional distributions (Theorem 2) and (21).

7. Remarks. (i) The Lindeberg condition, which by Theorem 2 and Theorem 3 is sufficient for the CLT and the invariance principle, is also necessary and sufficient for a type of convergence of conditional characteristic functions specified by (3) and (4). The question can then be asked: are (3) and (4) also necessary conditions for the CLT and invariance principle, as they are in the classical case of independent rv's?

(ii) The results of Theorem 1, Theorem 2 and Theorem 3 are obtained under the assumption of (1). Though this condition might be dispensable for the purposes of the CLT, it seems to be a natural (and perhaps necessary) condition in the case of the invariance principle. This is because it leads, with other conditions, to (21a), which specifies the fact that the (random) function of conditional variance converges in some sense to that of a Wiener process; namely a constant, additive function.

(iii) Conditions of the type $\sigma_j^2 \leqq$ constant, a.e., or $\sigma_j^2 =$ constant a.e., $j = 1, 2, \cdots$ are used in [4], [6], [8], [12], and [13]. The CLTS of Billingsley and Ibragimov ([2], [11]) for stationary ergodic martingales assume no extra conditions, but in fact (1) and (2) follow from ergodicity and stationarity, the convergences produced in (1), (2) being a.e., compared with convergence in probability here.

Loynes [14] uses some conditions very similar to ours, with a.e. convergence replacing our convergence in probability. Dvoretsky [9] mentions conditions corresponding closely to ours.

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