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DISTRIBUTION OF THE SUM OF INDEPENDENT DECAPITATED NEGATIVE BINOMIAL VARIABLES

BY J. C. AHUJA

Portland State University

1. Introduction. Let $X_1, X_2, \cdots, X_n$ be n independent and identically distributed random variables having the decapitated negative binomial distribution

$$p(x; k, \theta) = \binom{x+k-1}{x}\theta^x/[(1-\theta)^{-k}-1], \qquad x \in N, \tag{1}$$

where $0 < \theta < 1$, $k > 0$, and N is the set of positive integers. Define their sum as $Z = \sum_{i=1}^n X_i$. Rider [5] has considered the problem of estimating the parameters k and θ in (1), while Govindarajulu [2] and Rider [6] have obtained certain recurrence relations for the inverse moments of a random variable having the distribution (1). In the present note, we derive the exact distribution of Z by applying one of the results established by Patil [3] for the generalized power series distribution (GPSD). The distribution function of Z is also found in an explicit form in terms of a linear combination of the incomplete beta functions.

2. Distribution of sum. The derivation of the distribution of Z is based on a result due to Patil [3] which we state briefly as follows. Let X be a random variable having the GPSD

$$g(x; \theta) = a(x)\theta^x/f(\theta), \qquad x \in T, \tag{2}$$

where T is a subset of the set I of nonnegative integers, $a(x) > 0$, and $f(\theta) = \sum a(x)\theta^x$ is the series function, the summation extending over T. If X_i $(i = 1, 2, \cdots, n)$ is a random sample of size n drawn from the GPSD (2), then $Z = \sum_{i=1}^n X_i$ has also a GPSD with range $n[T]$ and the series function

$$f_n(\theta) = [f(\theta)]^n = \sum b(z, n)\theta^z \tag{3}$$

where the summation extends over $n[T]$, and $b(z, n)$ is the coefficient of θ^z in the expansion of $f_n(\theta)$.

It may now be observed that the decapitated negative binomial distribution (1) is a special case of the GPSD with range N and the series function $f(\theta) = (1-\theta)^{-k} - 1 = \sum_{x=1}^\infty \binom{x+k-1}{x}\theta^x$, so that we have the expansion

$$\begin{aligned}[f(\theta)]^n &= [(1-\theta)^{-k}-1]^n, \\ &= \textstyle\sum_{r=0}^n (-1)^{n-r}\binom{n}{r}(1-\theta)^{-rk}, \\ &= \textstyle\sum_{r=0}^n (-1)^{n-r}\binom{n}{r}\sum_{z=0}^\infty \binom{z+rk-1}{z}\theta^z\end{aligned}$$

which, after changing the order of summation, becomes

$$[f(\theta)]^n = \sum_{z=0}^\infty \left[\sum_{r=0}^n (-1)^{n-r}\binom{n}{r}\binom{z+rk-1}{z}\right]\theta^z. \tag{4}$$

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Using the binomial coefficient identity (12.17) given by Feller ([1] page 65), it can be verified that

$$\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}\binom{z+rk-1}{z}=0$$

for $z=0, 1, \cdots, n-1$, so that (4) reduces to

$$[f(\theta)]^n=\sum_{z=n}^{\infty}\Big[\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}\binom{z+rk-1}{z}\Big]\theta^z. \tag{5}$$

On comparing (3) and (5), we find that Z has a GPSD with the probability function

$$f(z;n,k,\theta)=\sum_{r=1}^{n}(-1)^{n-r}\binom{n}{r}\binom{z+rk-1}{z}\theta^z/[(1-\theta)^{-k}-1]^n \tag{6}$$

for $z=n, n+1, \cdots, \infty$, since the term in the summation is zero for $r=0$. Further, it may be easily seen that the distribution function of Z is obtained as

$$\begin{aligned} F(z;n,k,\theta) &= 1-\sum_{x=z+1}^{\infty}f(x;n,k,\theta), \\ &= 1-[(1-\theta)^{-k}-1]^{-n}\sum_{r=1}^{n}(-1)^{n-r}\binom{n}{r}(1-\theta)^{-rk}I_\theta(z+1,rk) \end{aligned} \tag{7}$$

where $I_\theta(z+1, rk)$ is the incomplete beta function tabulated by Pearson [4].

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