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# THE HARTMAN–WINTNER LAW OF THE ITERATED LOGARITHM FOR MARTINGALES[1]

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According to the Hartman–Wintner law of the iterated logarithm [4], $\{Y_i, i \geqq 1\}$ independent identically distributed with $EY_1 = 0$ and $EY_1^2 = 1$ implies that $\limsup \sum_{i=1}^n Y_i/(2n \log_2 n)^{\frac{1}{2}} = 1$ almost surely (a.s.). We generalize this result to stationary ergodic martingale difference sequences.

THEOREM. *Let* $(Y_i, i \geqq 1)$ *be a stationary ergodic stochastic sequence with* $E[Y_i \mid Y_1, Y_2, \cdots, Y_{i-1}] = 0$ a.s. *for all* $i \geqq 2$ *and* $EY_1^2 = 1$. *Then* $\limsup \sum_{i=1}^n Y_i/(2n \log_2 n)^{\frac{1}{2}} = 1$ a.s.

PROOF. In order to prove the result, we assume a particular representation for the stochastic sequence, namely that the sample space $\Omega$ is the Cartesian product $\prod_{i=-\infty}^{\infty} R_i$ of copies of the real line, that the $Y_i$'s are the coordinate variables of $\Omega$, that $\mathscr{F}$ is the $\sigma$-field of events generated by the $Y_i$'s, and that $P$ is the probability induced on $(\Omega, \mathscr{F})$ by the finite dimensional distributions of the $Y_i$'s. Letting $\mathscr{F}_n$ be the $\sigma$-field generated by $\{Y_m, -\infty < m \leqq n\}$, we note that $E[Y_i \mid \mathscr{F}_{i-1}] = 0$ a.s. for all integers $i$. Let $K_i$ be a sequence of positive constants to be specified later and let $b_i = K_i(i/\log_2 i)^{\frac{1}{2}}$. Let $Y_i' = Y_i I(|Y_i| \leqq b_i) + b_i I(Y_i > b_i) - b_i I(Y_i < -b_i)$ and $Z_i' = Y_i' - E[Y_i' \mid \mathscr{F}_{i-1}]$. Likewise let $Y_i'' = Y_i I(|Y_i| \leqq M) + MI(Y_i > M) - MI(Y_i < -M)$ and $Z_i'' = Y_i'' - E[Y_i'' \mid \mathscr{F}_{i-1}]$ for $M > 0$. This modification of the usual truncation procedure enables us to conclude that $E[Y_i^2 \mid \mathscr{F}_{i-1}] \geqq E[(Z_i')^2 \mid \mathscr{F}_{i-1}] \geqq E[(Z_i'')^2 \mid \mathscr{F}_{i-1}]$ a.s. when $b_i \geqq M$. (See Corollary 4 of [3] for a proof of this.) By the Birkoff ergodic theorem, $\sum_{i=1}^n E[(Z_i'')^2 \mid \mathscr{F}_{i-1}]/n \to E(Z_0'')^2$ a.s. and $\sum_{i=1}^n E[Y_i^2 \mid \mathscr{F}_{i-1}]/n \to 1$ a.s. $K_i$ will be chosen such that $K_i \to 0$ and $b_i \to \infty$. We then obtain $1 = \limsup \sum_{i=1}^n E[Y_i^2 \mid \mathscr{F}_{i-1}]/n \geqq \limsup \sum_{i=1}^n E[(Z_i')^2 \mid \mathscr{F}_{i-1}]/n \geqq \limsup \sum_{i=1}^n E[(Z_i'')^2 \mid \mathscr{F}_{i-1}]/n = E(Z_0'')^2$. Since $E(Z_0'')^2 \to 1$ as $M \to \infty$, it follows that

$$\sum_{i=1}^n E[(Z_i')^2 \mid \mathscr{F}_{i-1}]/n \to 1 \text{ a.s.} \tag{1}$$

and hence that

$$\sum_{i=1}^n E[(Z_i')^2 \mid \mathscr{F}_{i-1}] \to \infty \text{ a.s.} \tag{2}$$

According to [5], if $(Z_i, \mathscr{F}_i, i \geqq 1)$ is a martingale difference sequence with $s_n^2 = \sum_{i=1}^n E[Z_i^2 \mid \mathscr{F}_{i-1}] \to \infty$ a.s., $u_n = (2 \log_2 s_n^2)^{\frac{1}{2}}$, $\mathscr{F}_{i-1}$ measurable random variables $L_i \to 0$ a.s., and $|Z_i| \leqq L_i s_i/u_i$ a.s. for all $i \geqq 1$, then $\limsup \sum_{i=1}^n Z_i/(s_n u_n) = 1$ a.s.

Recalling (1) and (2), $Z_i'$ satisfies the hypotheses of this theorem with $L_i = 2K_i u_i (i/\log_2 i)^{\frac{1}{2}}/s_i$ since $|Z_i'| \leqq 2K_i(i/\log_2 i)^{\frac{1}{2}}$ a.s. Thus, using (1), $\limsup \sum_{i=1}^n Z_i'/(2n \log_2 n)^{\frac{1}{2}} = \limsup \sum_{i=1}^n Z_i'/(s_n u_n) = 1$ a.s.

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To complete the proof it suffices to show that

(3) $$\sum_{i=1}^{n} E[Y_i' \mid \mathscr{F}_{i-1}]/(2n\log_2 n)^{\frac{1}{2}} \to 0 \quad \text{a.s.}$$ and that

(4) $$\sum_{i=1}^{n} \mid Y_i - Y_i' \mid /(2n\log_2 n)^{\frac{1}{2}} \to 0 \quad \text{a.s.}$$

Noting that $E[Y_i' \mid \mathscr{F}_{i-1}] = -E[Y_i - Y_i' \mid \mathscr{F}_{i-1}]$, it suffices by the Kronecker Lemma to establish $\sum_{i=16}^{\infty} E \mid Y_i - Y_i' \mid /(2i\log_2 i)^{\frac{1}{2}} < \infty$ in order to prove (3) and (4).

To this end (following the approach of [2]) the sequence $K_i$ is chosen to depend on the distribution of $Y_0$ in a rather involved manner. Let $c_i = i^2 P[i-1 < |Y_0| \leqq i]$, noting that $\sum_{i=1}^{\infty} c_i < \infty$ follows from $EY_0^2 < \infty$. Let $n_k \geqq 2$ be an increasing sequence of integers such that $n_{k+1} > 2^{n_k}$ and $\sum_{i=n_k}^{\infty} c_i < 2^{-k}$ for all $k \geqq 1$. For each $i \geqq n_1$ letting $K_i = (k)^{-\frac{1}{2}}$ when $n_{k-1} \leqq i < n_k$ it follows that

(5) $$\sum_{i=n_1}^{\infty} c_i/K_i = \sum_{k=2}^{\infty} k^{\frac{1}{2}} \sum_{i=n_{k-1}}^{n_k - 1} c_i \leqq \sum_{k=2}^{\infty} k^{\frac{1}{2}} 2^{-k+1} < \infty.$$

Note that the above choice of $K_i$ is consistent with prior requirements that $K_i \to 0$ and $b_i \to \infty$. With $b_i = K_i(i/\log_2 i)^{\frac{1}{2}}$ let $N(m)$ be the largest integer $n$ such that $[b_n] \leqq m$ where $[\cdot]$ is the greatest integer function, noting that $b_i \to \infty$. Since $K_i \downarrow$ and $K_i/K_{i^3} \to 1$, it follows that there exists an integer $m_0$ such that $m \geqq m_0$ implies

$$\{(4m^2 \log_2 m/K_m^2)K^2_{[4m^2 \log_2 m/K_m^2]}/\log_2 (4m^2 \log_2 m/K_m^2)\}^{\frac{1}{2}}$$
$$\geqq \{(4m^2 \log_2 m/K_m^2)K_{m^3}^2/\log_2 m^3\}^{\frac{1}{2}} > m.$$

Thus for $m \geqq m_0$

(6) $$N(m) \leqq 4m^2 \log_2 m/K_m^2.$$

$$\sum_{i=16}^{\infty} E|Y_i - Y_i'|/(2_i \log_2 i)^{\frac{1}{2}}$$
$$\leqq \sum_{i=16}^{\infty} E|Y_i| I(|Y_i| > b_i)/(2i\log_2 i)^{\frac{1}{2}}$$
$$\leqq \sum_{i=16}^{\infty} \sum_{m=[b_i]}^{\infty} (m+1)P[m < |Y_0| \leqq m+1]/(2i\log_2 i)^{\frac{1}{2}}$$
$$= \sum_{m=[b_{16}]}^{\infty} \sum_{i=16}^{N(m)} (m+1)P[m < |Y_0| \leqq m+1]/(2i\log_2 i)^{\frac{1}{2}}.$$

By elementary integration and (6),

$$\sum_{i=16}^{N(m)} (2i\log_2 i)^{-\frac{1}{2}} < c(N(m)/\log_2 N(m))^{\frac{1}{2}}$$
$$\leqq c((4m^2 \log_2 m/K_m^2)/\log_2 (4m^2 \log_2 m/K_m))^{\frac{1}{2}} \approx 2c\, m/K_m.$$

Using (5) and combining, it follows that $\sum_{i=16}^{\infty} E|Y_i - Y_i'|/(2i\log_2 i)^{\frac{1}{2}} < \infty$, thus completing the proof.

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