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CONVERGENCE OF SUMS TO A CONVOLUTION OF STABLE LAWS

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1. Statement of the problem. Let $\{X_n\}$ be a sequence of independent random variables such that the distribution function of X_n is one of $F_1, \cdots, F_r$, where $F_1, \cdots, F_r$ are r distribution functions and $F_i \in \mathscr{D}(\lambda_i)$, $i = 1, \cdots, r$. Here, $\mathscr{D}(\lambda)$ denotes the domain of attraction of the stable type with characteristic exponent λ. Assume $0 < \lambda_1 < \cdots < \lambda_r \leqq 2$. Let $n_i(n)$ be the number of random variables among $X_1, \cdots, X_n$ which have F_i as their distribution function, $i = 1, \cdots, r$. Assume $n_i(n) \to \infty$ as $n \to \infty$, for each i.

If we assume that there are constants $\{A_n\}$ and $\{B_n\}$, with $0 < B_n \to \infty$ as $n \to \infty$, such that $B_n^{-1}(X_1 + \cdots + X_n) - A_n$ converges in law to a nondegenerate distribution function G, then Theorem 2 of [1] gives a necessary and sufficient condition (without the assumption that $F_i \in \mathscr{D}(\lambda_i)$) that G be a convolution of r distinct stable laws. The purpose of this paper is to obtain a necessary and sufficient conditon (with the assumption that $F_i \in \mathscr{D}(\lambda_i)$ but without the assumption of the existence of $\{A_n\}$ and $\{B_n\}$) that G be a convolution of $l \leqq r$ distinct stable laws.

2. Statement of theorem. Let $X(i, m)$ be the mth random variable in the sequence $\{X_n\}$ whose distribution function is F_i, $i = 1, \cdots, r$. Then, for each i, there are constants $\{A(i, n)\}$ and $\{B(i, n)\}$, with $0 < B(i, n) \to \infty$ as $n \to \infty$, such that $(B(i, n))^{-1}(X(i, 1) + \cdots + X(i, n)) - A(i, n)$ converges in law to a stable distribution with characteristic exponent λ_i. By Lemma 5 of [2], for each i, there exists a measurable slowly varying function L_i defined over $(0, \infty)$ such that $B(i, n) \sim n^{\lambda_i^{-1}} L_i(n)$. By Karamata's representation theorem,

$$L_i(x) = c_i(x) \exp\{\int_0^x (\theta_i(t)/t)\, dt\},$$

where $c_i(\cdot)$ is a measurable function such that $c_i(x) > 0$ for all x and $c_i(x) \to c_i > 0$ as $x \to \infty$, and $\theta_i(\cdot)$ is Lebesgue-integrable over every finite interval $(0, x)$ and $\theta_i(t) \to 0$ as $t \to \infty$.

Theorem. *A necessary and sufficient condition that there exist constants $\{A_n\}$ and $\{B_n\}$, with $0 < B_n \to \infty$ as $n \to \infty$, such that $B_n^{-1}(X_1 + \cdots + X_n) - A_n$ converges in law to a nondegenerate distribution function G is that for some set of indices $\{i_1, \cdots, i_l\}$, with $1 \leqq i_1 < \cdots < i_l \leqq r$,*

$$B(j, n_j(n))/B(i_l, n_{i_l}(n)) \to p_j \qquad \text{as} \quad n \to \infty,$$

where $p_j = 0$ for $j \notin \{i_1, \cdots, i_l\}$ and $0 < p_j < \infty$ for $j \in \{i_1, \cdots, i_l\}$.

Furthermore, G is a convolution of l stable laws with characteristic exponents $\lambda_{i_1}, \cdots, \lambda_{i_l}$, and there is a constant $b > 0$ such that $B_n = bB(i_l, n_{i_l}(n))$ for all n.

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Also, the condition of the theorem may be stated as follows:

$$(1/\lambda_j)\log n_j(n)-(1/\lambda_{i_l})\log n_{i_l}(n)+\int_0^{n_j(n)}(\theta_j(t)/t)\,dt-\int_0^{n_{i_l}(n)}(\theta_{i_l}(t)/t)\,dt\to t_j \quad \text{as} \quad n\to\infty,$$

where $t_j=-\infty$ *for* $j\notin\{i_1,\cdots,i_l\}$ *and* $|t_j|<\infty$ *for* $j\in\{i_1,\cdots,i_l\}$.

Hence, the rate that $n_i(n)$ goes to infinite and the rate that $\theta_i(t)$ goes to zero determine convergence.

3. Proof of theorem. The sufficiency is obvious.

Necessity. Let k_1 be the smallest integer such that $\limsup n_{k_1}(n)/n>0$ and $\lim n_i(n)/n=0$ for $1\leqq i<k_1$.

In the proof of Theorem 1 of [1], it was shown that there exist a strictly increasing sequence $\{m(n)\}$ of natural numbers and an integer $k\leqq k_1$ such that

$$B(j,n_j(m(n)))/B(k,n_k(m(n)))\to p_j \qquad \text{as} \quad n\to\infty, \quad \text{for} \quad j=1,\cdots,r$$

with $p_j=0$ for $j>k$, and $0\leqq p_j<\infty$ for $1\leqq j\leqq k$.

Let $\{i_1,\cdots,i_l\}$ be the set of indices of all the nonzero p_j's. Assume $i_1<\cdots<i_l$.

For each i, let Y_i be a random variable such that

$$(B(i,n))^{-1}(X(i,1)+\cdots+X(i,n))-A(i,n)$$

converges in law to Y_i. Also, let

$$Z_n=(B(i_l,n_{i_l}(n)))^{-1}\{X_1+\cdots+X_n-\textstyle\sum_{j=1}^r B(j,n_j(n))A(j,n_j(n))\}.$$

We see that $Z_{m(n)}$ converges in law to $p_{i_1}Y_{i_1}+\cdots+p_{i_l}Y_{i_l}$. Since $B_{m(n)}^{-1}(X_1+\cdots+X_{m(n)})-A_{m(n)}$ converges in law to G, we see that G must be a convolution of l stable laws with characteristic exponents $\lambda_{i_1},\cdots,\lambda_{i_l}$.

What remains to be shown is that the sequence $\{m(n)\}$ may be taken to be $\{n\}$. Let f_n and f be the characteristic functions of Z_n and $p_{i_1}Y_{i_1}+\cdots+p_{i_l}Y_{i_l}$, respectively.

Suppose f_n does not converge to f. Then there exist $u\in(-\infty,\infty)$, $\varepsilon>0$ and a sequence $\{q(n)\}$ of natural numbers such that

$$|f_{q(n)}(u)-f(u)|\geqq\varepsilon \quad \text{for all} \quad n. \tag{1}$$

Then there exist a subsequence $\{s(n)\}$ of $\{q(n)\}$ and $s\in\{1,\cdots,r\}$ such that

$$B(j,n_j(s(n)))/B(s,n_s(s(n)))\to\rho_j \qquad \text{as} \quad n\to\infty, \quad \text{for} \quad j=1,\cdots,r,$$

with $\rho_j=0$ for $j>s$ and $0\leqq\rho_j<\infty$ for $1\leqq j\leqq s$.

Therefore, $(B(s,n_s(s(n))))^{-1}\{X_1+\cdots+X_{s(n)}-\sum_{j=1}^r B(j,n_j(s(n)))A(j,n_j(s(n)))\}$ converges in law to $\rho_1Y_1+\cdots+\rho_sY_s$.

Since $B_{s(n)}^{-1}(X_1+\cdots+X_{s(n)})-A_{s(n)}$ converges in law to G, there are positive constants b_1 and b_2 such that $b_1(p_{i_1}Y_{i_1}+\cdots+p_{i_l}Y_{i_l})$ and $b_2(\rho_1Y_1+\cdots+\rho_sY_s)$ have the same distribution function. Since $\rho_s=p_{i_l}=1$, an easy symmetrization argument shows that $i_l=s$, $b_1=b_2$, $p_j=\rho_j$ for $j\in\{i_1,\cdots,i_l\}$, and $\rho_j=0$ for $j\notin\{i_1,\cdots,i_l\}$.

Therefore, $f_{s(n)} \to f$, which contradicts (1). Hence, every subsequence of f_n converges to f which shows that $\{m(n)\}$ may be taken to be $\{n\}$. $\square$

Since G is of the same type as $p_{i_1} Y_{i_1} + \cdots + p_{i_l} Y_{i_l}$, given any normalizing coefficients $\{B_n\}$, there is a constant $b > 0$ such that $B_n = bB(i_l, n_{i_l}(n))$ for all n.

The alternate statement of the condition follows from the fact that

$$B(j, n_j(n))/B(i_l, n_{i_l}(n)) \sim c_j c_{i_l}^{-1} \exp\{(1/\lambda_j)\log n_j(n) - (1/\lambda_{i_l})\log n_{i_l}(n) + \int_0^{n_j(n)} (\theta_j(t)/t)\,dt - \int_0^{n_{i_l}(n)} (\theta_{i_l}(t)/t)\,dt\}.$$

Hence, $p_j = 0$ if and only if $t_j = -\infty$, and $0 < p_j < \infty$ if and only if $|t_j| < \infty$.

4. Remarks. If each F_k is in the domain of *normal* attraction, then, since $L_j(x) = 1$ for all x and all j, the condition of the theorem becomes

$$(n_j(n))^{1/\lambda_j}/(n_{i_l}(n))^{1/\lambda_{i_l}} \to p_j \qquad \text{as} \quad n \to \infty.$$

There are examples (e.g. [3]) that show that G being a convolution of stable laws does not imply that $F_i \in \mathscr{D}(\lambda_i)$ nor even that F_i be in the domain of partial attraction of any law.

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