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A MARTINGALE DECOMPOSITION THEOREM[1]

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Let Z be a random variable with $E|Z| < \infty$ and define recursively

$$Z_0 = EZ, \qquad Z_n = E^{\mathscr{F}_n}Z, \tag{1}$$

where

$$\mathscr{F}_n = \mathscr{B}(Z_{n-1}, I(Z \geqq Z_{n-1})) \quad \text{for} \quad n = 1, 2, \cdots.^2 \tag{2}$$

The Z_n sequence constitutes a martingale decomposition of Z in the sense of the following

THEOREM.

(i) $Z_0, Z_1, \cdots, Z_n, \cdots, Z$ *is a martingale.*

(ii) *The conditional distribution of* Z_n *given* Z_{n-1} *is a one or two point distribution* a.s. *for* $n = 1, 2, \cdots$.

(iii) $Z_n \to Z$ a.s. *as* $n \to \infty$.

PROOF. It is useful to define a closely related sequence by

$$Y_0 = EZ, \qquad Y_n = E^{\mathscr{G}_n}Z, \tag{3}$$

where

$$\mathscr{G}_n = \mathscr{B}(Y_i, I(Z \geqq Y_i); i = 0, \cdots, n-1) \quad \text{for} \quad n = 1, 2, \cdots. \tag{4}$$

We shall show that

$$\bar{\mathscr{F}}_n = \bar{\mathscr{G}}_n \tag{5}$$

from which we may conclude (i) (cf., [1] page 293) and

$$Y_n = Z_n \text{ a.s.} \quad \text{for} \quad n = 0, 1, \cdots. \tag{6}$$

To show (5), it suffices to show for $0 \leqq j < k$ that

$$Z \geqq Y_j \quad \text{if, and only if,} \quad Y_k \geqq Y_j \quad \text{a.s.} \tag{7}$$

and

$$Y_j \text{ is measurable with respect to } \bar{\mathscr{B}}(Y_k). \tag{8}$$

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[2] We shall assume that everything is defined on a basic probability space $(\Omega, \mathscr{F}, P)$. For an arbitrary event $A \in \mathscr{F}$ and arbitrary random vector W, we denote $I(A)$ and $\mathscr{B}(W)$ as the indicator function (taking the value 1 on A and 0 off A) and the σ-field generated by W respectively. $\bar{\mathscr{B}}(W)$ will refer to the smallest σ-field containing $\mathscr{B}(W)$ and the null sets of $\mathscr{F}$.

For then

$$\begin{aligned}\bar{\mathscr{G}}_n &= \bar{\mathscr{B}}(Y_i, I(Z \geqq Y_i); i = 0, \cdots, n-1) \\ &= \bar{\mathscr{B}}(Y_{n-1}, I(Z \geqq Y_{n-1}), Y_i, I(Y_{n-1} \geqq Y_i); i = 0, \cdots, n-2) \quad (\text{cf., } (7)) \\ &= \bar{\mathscr{B}}(Y_{n-1}, I(Z \geqq Y_{n-1})) \quad (\text{cf., } (8)) \\ &= \bar{\mathscr{B}}(Z_{n-1}, I(Z \geqq Z_{n-1})) \quad (\text{cf., } (1), (2), (3), (4)) \\ &= \bar{\mathscr{F}}_t.\end{aligned}$$

(7) follows from

$$I(Z \geqq Y_j)(Y_k - Y_j) = E^{\mathscr{G}_k} I(Z \geqq Y_j)(Z - Y_j) \geqq 0 \quad \text{a.s.} \tag{9}$$

and

$$I(Z < Y_j)(Y_k - Y_j) = E^{\mathscr{G}_k} I(Z < Y_j)(Z - Y_j) < 0 \quad \text{a.s. on} \quad [Z < Y_j]. \tag{10}$$

(8) is true for $j = 0$ and if true for $j = 0, \cdots, \alpha - 1 < k - 1$, then

$$\bar{\mathscr{G}}_\alpha = \bar{\mathscr{B}}(Y_i, I(Y_k \geqq Y_i); i = 0, \cdots, \alpha - 1) \subset \bar{\mathscr{B}}(Y_k)$$

and, hence, (8) is true for $j = \alpha$.

(ii) is immediate from (1) and (2). Preliminary to showing (iii), we observe that for $0 \leqq j < k$,

$$\begin{aligned}E|Z - Y_j| &= EE^{\mathscr{G}_k}(I(Z \geqq Y_j) - I(Z < Y_j))(Z - Y_j) \\ &= E(I(Z \geqq Y_j) - I(Z < Y_j))(Y_k - Y_j) \\ &= E(I(Y_k \geqq Y_j) - I(Y_k < Y_j))(Y_k - Y_j) = E|Y_k - Y_j|.\end{aligned} \tag{11}$$

It is easily seen that $\sup E|Y_n| \leqq E|Z| < \infty$ and that the Y_n are uniformly integrable. Hence, by the martingale convergence theorem, there is a random variable Y_∞ with $Y_n \to Y_\infty$ a.s. and $E|Y_\infty - Y_n| \to 0$ as $n \to \infty$. In view of (6) and (11), (iii) clearly follows.

REMARKS.

(A) It is easy to see that if one of the Z_k is decomposed as we have Z into a sequence Z_{kn} $(n = 0, 1, \cdots)$, say, then $Z_{kn} = Z_{k \wedge n}$ a.s. where $k \wedge n$ is the smaller of k and n.

(B) The search for this theorem was primarily motivated by the work of Lester Dubins [2]. In particular, the decomposition leads to an obvious procedure for embedding Z into Brownian motion when $EZ = 0$ and, more generally, for embedding zero mean martingales. The idea is the following: If Z is not almost surely equal to zero (a trivial case), then Z_1 assumes one of two possible values a or b, say, with $a < 0 < b$. The law of Z_1 is identical to the law of $W(t)$ where $(W(s), s \geqq 0)$ is Brownian motion and t is the first time $s > 0$ such that $W(s) = a$ or b (cf., Skorokhod [3] page 163). Having embedded Z_1, one embeds $Z_2, Z_3, \cdots$ successively (cf., Strassen [4] page 318). The embedding of Z is accomplished by proceeding to limits. This procedure, which does not require external randomization, is identical to the one suggested by Dubins [2].

(C) I am indebted to Professor Dubins for bringing to the author's attention the recent work of Paul Meyer appearing in the form of University of Strasbourg seminar notes. These include a section on "Un Théorème de Dubins" in which he clarifies a difficult point in Dubins' paper. There is some overlap between Meyer's work and the author's.

REFERENCES

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[4] Strassen, Volker (1967). Almost sure behavior of sums of independent random variables and martingales. *Proc. Fifth Berkeley Symp. Math. Statist.* **2** 315–343.