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SOME STRUCTURE THEOREMS FOR THE SYMMETRIC STABLE LAWS

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1. Introduction. The Spectral Representation Theorem for stationary Gaussian processes

$$x_l = \int_{-\frac{1}{2}}^{\frac{1}{2}} \exp[2\pi i l\lambda]\, dF(\lambda) \tag{1.1}$$

where $F(\lambda)$ is an independent increments Gaussian process has proved exceedingly useful in the statistical and probabilistic analysis of these processes (see Doob [1] chapter X or Karhunen [4], for example). In this paper it will be shown that a similar representation can be given if x_l, $l = 1, 2, \cdots, n$ is a finite set of random variables with a stable distribution of type α. It will be shown that there is an independent increments process, $F(\lambda)$, of type α, and a set of functions, $f_l(\lambda)$, such that

$$x_l = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_l(\lambda)\, dF(\lambda) \tag{1.2}$$

where the stochastic integral of (1.2) will be defined. Some elementary properties of this representation will also be derived. An interesting by-product of the theory presented here is that there is an isometric isomorphism between the sets of symmetric stable variables of type α with a natural norm and the usual L^p spaces on the interval $[-\frac{1}{2}, \frac{1}{2}]$ where $p = \alpha$.

2. The stable laws. In this paper a random variable x will be said to have a symmetric stable distribution of type α and scale factor $|b|$ if the characteristic function of x, ch.f.$(x; u)$, is of the form

$$\text{ch.f.}(x, u) = \exp[-|b|\,|u|^\alpha], \qquad b \text{ real}, \quad 0 < \alpha \leqq 2. \tag{2.1}$$

The stable variables considered in this paper therefore have a symmetric distribution around the origin, with median zero and a scale or variance parameter $|b|$.

It can be shown (see Loève [6] for example) that $E\{x^2\} = \infty$ if $0 < \alpha < 2$, and that $E\{|x|\} = \infty$ if $0 < \alpha \leqq 1$ where $E\{x\}$ denotes the expected value of x. Therefore the only stable variable with finite mean and variance is the normal distribution—the stable distribution with $\alpha = 2$. Since (1.1) is derived with the implicit assumption of finite variance for the $\{x_l\}$ process, it follows that different methods must be used to derive (1.2).

If x_1 and x_2 are stable variables of type α, then it can be shown that $x_3 = c_1 x_1 + c_2 x_2$ is also a stable variable of type α (hence the name stable) if c_1 and c_2 are constants. If $x_1, \cdots, x_n$ are random variables, then $x_1, \cdots, x_n$ are said to have a multidimensional stable distribution of type α if every linear combination of $x_1, \cdots, x_n$ is stable of type α.

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Therefore all linear combinations of $x_1, \cdots, x_n$ generate a linear space, if $x_1, \cdots, x_n$ have a multidimensional stable distribution.

A metric is now defined on this space.

DEFINITION 2.1. If x has a symmetric stable distribution of type α, $1 \leqq \alpha \leqq 2$, then the length of x, $||x||$, is defined as $|b|^{1/\alpha}$. If $0 < \alpha < 1$, then the length of x is defined by $|b|$, where b is the scale factor defined previously.

THEOREM 2.1. *The metric defined by Definition* (2.1) *is a true metric.*

PROOF. P. Lévy [5] (see also Rvaceva [7]) has shown that if $x_1, x_2, \cdots, x_n$ are stable variables of type α that the joint characteristic function of $x_1, x_2, \cdots, x_n$ can be written in the form

$$\text{(2.2)} \quad \text{ch.f.}(x_1, \cdots, x_n; \mu_1, \cdots, \mu_n) = \exp\left[-\int |\mu_1 y_1 + \cdots + \mu_n y_n|^\alpha \, dG(y_1, \cdots, y_n)\right]$$

where $dG(y_1, \cdots, y_n)$ is a measure with all its mass on the surface of the n dimensional unit sphere.

From (2.2) it follows that, for x_1 and x_2 stable type α and $1 \leqq \alpha \leqq 2$

$$\text{(2.3)} \quad ||x_1 + x_2|| = [-\log[\text{ch.f.}(x_1, x_2; 1, 1)]]^{1/\alpha} = [\int |y_1 + y_2|^\alpha \, dG(y_1, y_2)]^{1/\alpha}.$$

From Loève [6], page 161, it follows that

$$[\int |y_1 + y_2|^\alpha dG(y_1, y_2)]^{1/\alpha} \leqq [\int |y_1|^\alpha dG(y_1, y_2)]^{1/\alpha} + [\int |y_2|^\alpha dG(y_1, y_2)]^{1/\alpha}$$
$$= ||x_1|| + ||x_2||.$$

For $0 < \alpha < 1$, (2.3) becomes $||x_1 + x_2|| = \int |y_1 + y_2|^\alpha \, dG(y_1, y_2)$.

From Loève [6] page 161, it again follows that

$$[\int |y_1 + y_2|^\alpha dG(y_1, y_2)] \leqq \int |y_1|^\alpha G(y_1, y_2) + \int |y_2|^\alpha dG(y_1, y_2) = ||x_1|| + ||x_2||$$

which shows the triangle inequality. If $||x|| = 0$, then the characteristic function of x is 1 and x therefore has the same characteristic function as the zero variable. The only stable variable that has the same characteristic function as the zero variable is the zero variable itself. Therefore if $||x|| = 0$, $x = 0$. Since the other axioms of a metric space are easily verified, the theorem is proved.

COROLLARY 2.1. *If* $\alpha \geqq 1$, *then the linear space defined above with the metric defined above is a linear normed space.*

PROOF. It is sufficient to show that for $\alpha \geqq 1$ $||cx|| = |c| \, ||x||$. But

$$||cx|| = [-\log[\text{ch.f.}(cx; 1)]]^{1/\alpha}$$
$$= [-\log(\textstyle\int_{-\infty}^{\infty} \exp[i\theta] \, dP_{cx}(\theta))]^{1/\alpha}$$
$$= [-\log(\textstyle\int_{-\infty}^{\infty} \exp[ic\theta] \, dP_x(\theta))]^{1/\alpha}$$
$$= [-\log(\exp[-|c|^\alpha ||x||^\alpha]]^{1/\alpha} = |c| \, ||x||,$$

where $dP_{cx}(\theta)$ and $P_x(\theta)$ are the distribution functions of cx and x respectively.

3. Stable integrals. In this section a stochastic integral for stable processes will be defined which is similar to the stochastic integral defined by Wiener [9] for the Brownian motion process. See also Doob [1] or Ito [3].

DEFINITION 3.1. $F(\lambda)$ is an independent increments process of type α on $[-\frac{1}{2}, \frac{1}{2}]$ if $F(\lambda)$ has a symmetric stable distribution of type α for every λ between $-\frac{1}{2}$ and $\frac{1}{2}$; if $F(\lambda_2)-F(\lambda_1)$ is independent of $F(\lambda_1)$, whenever $-\frac{1}{2} \leqq \lambda_1 < \lambda_2 \leqq \frac{1}{2}$, if $F(-\frac{1}{2}) = 0$, and if $||F(\frac{1}{2})|| < \infty$.

Doob [1] page 422, shows that except for a set of measure zero the paths of stable independent increments processes are continuous except for jump discontinuities. The integral (1.2) $\int_{-\frac{1}{2}}^{\frac{1}{2}} f(\lambda)\,dF(\lambda)$ is defined in the classical Stieljes sense only if one or the other of $f(\lambda)$ or $F(\lambda)$ is continuous and the other is of bounded variation. We will need (see Section 4) to define this integral when $f(\lambda)$ is not continuous (or of bounded variation). Since $F(\lambda)$ is in general not continuous, it follows that the classical approach cannot be used to define (1.2). However, Wiener and later Ito (see above) showed that integrals of the type (1.2) could be defined as a random variable for $F(\lambda)$ a finite variance process. We now extend Ito's method to symmetric stable processes.

LEMMA 3.1. *If x_1 and x_2 are symmetric stable variables of type α, and if x_1 is independent of x_2, then $||x_1+x_2|| = ||x_1||+||x_2||$ if $0 < \alpha < 1$, and $||x_1+x_2||^\alpha = ||x_1||^\alpha+||x_2||^\alpha$ if $1 < \alpha \leqq 2$.*

PROOF. Since x_1 and x_2 are independent, the joint characteristic function of x_1 and x_2 is the product of the characteristic functions x_1 and x_2. Therefore if $0 < \alpha < 1$

$$\begin{aligned} ||x_1+x_2|| &= -\log[\text{ch.f.}(x_1+x_2; 1)] \\ &= \log[\text{ch.f.}(x_1, x_2; 1, 1)] \\ &= -\log[\text{ch.f.}(x_1; 1)\,\text{ch.f.}(x_2; 1)] \\ &= -\log[\text{ch.f.}(x_1; -1)]-\log[\text{ch.f.}(x_2; 1)] \\ &= ||x_1||+||x_2||. \end{aligned}$$

A similar proof holds in case $1 \leqq \alpha \leqq 2$.

LEMMA 3.2. *If $0 < \alpha < 1$, then $||F(\lambda)||$ is a bounded monotonically increasing function. If $1 \leqq \alpha \leqq 2$, then $||F(\lambda)||^\alpha$ is a bounded monotonically increasing function.*

PROOF. This lemma follows directly from the definitions and Lemma 3.1 since for $-\frac{1}{2} \leqq \lambda_1 < \lambda_2 \leqq \frac{1}{2}$ and $0 < \alpha < 1$

$$||F(\lambda_2)|| = ||F(\lambda_2)-F(\lambda_1)+F(\lambda_1)|| = ||F(\lambda_1)||+||F(\lambda_2)-F(\lambda_1)||.$$

Since $||F(\lambda_2)-F(\lambda_1)|| \geqq 0$, it follows that $||F(\lambda_2)|| \geqq ||F(\lambda_1)||$, and the lemma follows. A similar proof applies if $1 \leqq \alpha \leqq 2$.

Since $||F(\lambda)||^\alpha$ is a monotonically increasing function for $1 \leqq \alpha \leqq 2$ and

$-\frac{1}{2} \leqq \lambda \leqq \frac{1}{2}$, it follows that it can be used as a measure on $[-\frac{1}{2}, \frac{1}{2}]$ in the usual Lebesgue–Stieltjes sense. We now define (as usual) L^α to be the set of functions $f(\lambda)$, which are measurable with respect to the measure $d||F(\lambda)||^\alpha$ and for which the integral $\int_{-\frac{1}{2}}^{\frac{1}{2}} |f(\lambda)|^\alpha d\,||F(\lambda)||^\alpha$ is finite if $1 \leqq \alpha \leqq 2$. If $0 < \alpha < 1$, then L^α is the set of functions which are measurable with respect to $d||F(\lambda)||$ and for which the integral $\int_{-\frac{1}{2}}^{\frac{1}{2}} |f(\lambda)|^\alpha d\,||F(\lambda)||$ is finite. See Loève [6] for an exposition of this kind of space.

Let now $g(\lambda)$ be a step function on $[-\frac{1}{2}, \frac{1}{2}]$, i.e. suppose $g(\lambda) = g_j$ if $z_{j-1} < \lambda \leqq z_j$ $j = 1, 2, \cdots, k$. For $0 < \alpha \leqq 2$ we now define the integral $\int_{-\frac{1}{2}}^{\frac{1}{2}} g(\lambda)\, dF(\lambda)$ as

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} g(\lambda)\, dF(\lambda) = \sum_{j=1}^{k} g_j[F(z_j - 0) - F(z_{j-1} + 0)]$$

(keeping in mind that $F(\alpha)$ is left continuous). It follows easily from the definition that $\int_{-\frac{1}{2}}^{\frac{1}{2}} g(\lambda)\, dF(\lambda)$ is a stable variable of type α. From Lemma 3.1 for $1 \leqq \alpha \leqq 2$ it follows that

$$\begin{aligned}(||\textstyle\int_{-\frac{1}{2}}^{\frac{1}{2}} g(\lambda)\, dF(\lambda)||)^\alpha &= ||\textstyle\sum_{j=1}^{k} g_j(F(z_j) - F(z_{j-1}))||^\alpha \\ &= \textstyle\sum_{j=1}^{k} |g_j|^\alpha [||F(z_j) - F(z_{j-1})||]^\alpha = \int_{-\frac{1}{2}}^{\frac{1}{2}} |g(\lambda)|^\alpha d\,||F(\lambda)||^\alpha.\end{aligned}$$

It therefore follows for step functions anyway if $1 \leqq \alpha \leqq 2$ that

$$||\int_{-\frac{1}{2}}^{\frac{1}{2}} g(\lambda)\, dF(\lambda)|| = (\int_{-\frac{1}{2}}^{\frac{1}{2}} |g(\lambda)|^\alpha d\,||F(\lambda)||^\alpha)^{1/\alpha}.$$

In a similar manner it can be shown that if $0 < \alpha < 1$

$$||\int_{-\frac{1}{2}}^{\frac{1}{2}} g(\lambda)\, dF(\lambda)|| = \int_{-\frac{1}{2}}^{\frac{1}{2}} |g(\lambda)|^\alpha d\,||F(\lambda)||.$$

We therefore have the result

LEMMA 3.3. *The norm of the stable variable* $\int_{-\frac{1}{2}}^{\frac{1}{2}} g(\lambda)\, dF(\lambda)$ *is the same as the* L^α *norm of* g *with respect to the measure* $d||F(\lambda)||^\alpha$ *if* $1 \leqq \alpha \leqq 2$ *or with respect to the measure* $d||F(\lambda)||$ *if* $0 < \alpha < 1$ *if* $g(\lambda)$ *is a step function.*

Let now $f(\lambda)$ be an arbitrary function in L^α and $1 \leqq \alpha \leqq 2$. It follows either by or from the definition of L^α that the step functions are dense in L^α, and therefore there is a sequence of step functions $g_m(\lambda)$ such that

$$\lim_{m\to\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} |g_m(\lambda) - f(\lambda)|^\alpha d\,||F(\lambda)||^\alpha = 0.$$

It follows that the sequence of stable variables, $\int_{-\frac{1}{2}}^{\frac{1}{2}} g_m(\lambda)\, dF(\lambda)$, is a Cauchy sequence, since the $\{g_m(\lambda)\}$ is in L^α and the norms are the same. We now define $\int_{-\frac{1}{2}}^{\frac{1}{2}} f(\lambda)\, dF(\lambda)$ to be

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f(\lambda)\, dF(\lambda) = \lim_{m\to\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} g_m(\lambda)\, dF(\lambda).$$

Since the characteristic functions of $\int_{-\frac{1}{2}}^{\frac{1}{2}} g_m(\lambda)\, dF(\lambda)$ are

$$\exp[-\int_{-\frac{1}{2}}^{\frac{1}{2}} |g_m(\lambda)|^\alpha d\,||F(\lambda)||^\alpha |\mu|^\alpha]$$

[Lemma 3.3], it follows that the characteristic functions of $\int_{-\frac{1}{2}}^{\frac{1}{2}} g_m(\lambda)\, dF(\lambda)$ are converging to the characteristic function

$$\exp[-\int_{-\frac{1}{2}}^{\frac{1}{2}} |f(\lambda)|^\alpha dF\,||(\lambda)||^\alpha |M|^\alpha].$$

Since this characteristic function is continuous at zero, it follows from P. Lévy's continuity theorem, Loève [6], page 191, that $\int_{-\frac{1}{2}}^{\frac{1}{2}} f(\lambda)\, dF(\lambda)$ has characteristic function

$$\exp\left[-\int_{-\frac{1}{2}}^{\frac{1}{2}} |f(\lambda)|^{\alpha} d\,||F(\lambda)||^{\alpha} |\mu^{\alpha}|\right].$$

It has therefore been shown that $\int_{-\frac{1}{2}}^{\frac{1}{2}} f(\lambda)\, dF(\lambda)$ is a stable variable of type α and norm $(\int_{-\frac{1}{2}}^{\frac{1}{2}} |f(\lambda)|^{\alpha} d\,||F(\lambda)||^{\alpha})^{1/\alpha}$. A similar argument shows for $0 < \alpha < 1$ that $\int_{-\frac{1}{2}}^{\frac{1}{2}} f(\lambda)\, dF(\lambda)$ can be defined if $f(\lambda)$ is in L^{α} and that it is of type α and norm $\int_{-\frac{1}{2}}^{\frac{1}{2}} |f(\lambda)|^{\alpha} d\,||F(\lambda)||$. The proceeding results will be stated as a theorem.

THEOREM 3.1. *If* $1 \leqq \alpha \leqq 2$, *and* $f(\lambda)$ *is in* L^{α} *of* $d||F(\lambda)||^{\alpha}$, *then the stochastic integral* $\int_{-\frac{1}{2}}^{\frac{1}{2}} f(\lambda)\, dF(\lambda)$ *can be defined. It is a symmetric stable variable of type* α *and norm*

$$\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} |f(\lambda)|^{\alpha} d\,||F(\lambda)||^{\alpha}\right]^{1/\alpha}.$$

If $0 < \alpha < 1$ *then* $\int_{-\frac{1}{2}}^{\frac{1}{2}} f(\lambda)\, dF(\lambda)$ *is also a symmetric stable variable of type* α *and norm*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |f(\lambda)|^{\alpha} d\,||F(\lambda)||.$$

COROLLARY 3.1. *The space* L^{α} *of* $d||F(\lambda)||^{\alpha}$ *and the space of stable variables of the form* $\int_{-\frac{1}{2}}^{\frac{1}{2}} f(\lambda)\, dF(\lambda)$ *where* $f(\lambda) \in L^{\alpha}$ *are isometrically isomorphic if* $0 < \alpha < 1$. *If* $1 \leqq \alpha \leqq 2$ *then the space* L^{α} *of* $d||F(\lambda)||$ *and the space of stable variables of the form* $\int_{-\frac{1}{2}}^{\frac{1}{2}} f(\lambda)\, dF(\lambda)$ *where* $f(\lambda) \in L^{\alpha}$ *are isometrically isomorphic.*

PROOF. The correspondence for $f(\lambda) \in L^{\alpha}$ with $\int_{-\frac{1}{2}}^{\frac{1}{2}} f(\lambda)\, dF(\lambda)$ is easily seen to be one to one and linear. Theorem 3.1 shows that it is norm preserving.

4. A representation theorem. In this section a representation theorem will be given for a finite number of stable variables of type α. First, however, a lemma is needed.

LEMMA 4.1. *If* x_i, $i = 1, \cdots, n$ *are stable variables type* α, *then for* $1 \leqq \alpha \leqq 2$

$$\sum_{l}^{n} ||x_l||^{\alpha} \geqq \int dG(y_1, \cdots, y_n),$$

and for $0 < \alpha < 1$

$$\sum_{l}^{n} ||x_l|| \geqq \int dG(y_1, \cdots, y_n),$$

where $dG(y_1, \cdots, y_n)$ *is the measure defined by* (2.2).

PROOF. (2.2) states that the joint characteristic function of $x_1, \cdots, x_n$ is

$$\exp\left[-\int |\mu_1 y_1 + \cdots + \mu_n y_n|^{\alpha} dG(y_1, \cdots, y_n)\right).$$

It follows that the characteristic function of x_l can be written as

$$\text{ch.f.}(x_l, \mu_l) = \exp\left[-\int |\mu_l y_l|^{\alpha} dG(y_1, \cdots, y_n)\right).$$

By definition for $1 \leqq \alpha \leqq 2$ ch.f. $(x_l; \mu_l) = \exp[-||x_l||^{\alpha} |\mu_l|^{\alpha}]$. It follows that

$$\sum_{l}^{n} ||x_l||^{\alpha} = \sum_{l}^{n} \int |y_l|^{\alpha} dG(y_1, \cdots, y_n).$$

Since all the mass of $dG(y_1, \cdots, y_n)$ is on the n dimensional unit sphere, it can be assumed $|y_l| \leqq 1$. Since $\alpha \leqq 2$, it therefore follows that $|y_l|^\alpha \geqq |y_l|^2$. Therefore

$$\begin{aligned} \textstyle\sum_l^n | y_l|^\alpha dG(y_1, \cdots, y_n) &\geqq \textstyle\int \sum_l^n |y_l|^2 dG(y_1, \cdots, y_n) \\ &= \textstyle\int dG(y_1, \cdots, y_n) \end{aligned}$$

since all the mass of $dG(y_1, \cdots, y_n)$ is on $\sum_l^n |y_l|^2 = 1$. Since a similar argument holds if $0 < \alpha < 1$ the lemma is proved.

We now prove the main theorem of this section.

THEOREM 4.1.[1] *If $x_1, x_2, \cdots, x_n$ is a set of stable variables of type α, then there is an independent increments process of type α, $F(\lambda)$, and a set of functions $f_l(\lambda)$ in L^α such that*

$$W_l = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_l(\lambda)\, dF(\lambda) \qquad l = 1, 2, \cdots, n$$

have the same joint distributions as the x_l. Therefore the W_l process is probabilistically equivalent to the x_l process.

PROOF. From (2.2) it follows the characteristic function of $x_1, \cdots, x_n$ may be written in the form

$$\text{ch.f.}\,(x_1, \cdots, x_n; \mu_1, \cdots, \mu_n) = \exp\left[-\textstyle\int |\mu_1 y_1 + \cdots + \mu_n y_n|^\alpha dG(y_1, \cdots, y_n)\right].$$

Let $T(\lambda) = f_1(\lambda), \cdots, f_n(\lambda)$ be a 1–1 measurable and measure preserving mapping from $[-\frac{1}{2}, \frac{1}{2}]$ to the unit sphere in n dimensions; for existence of this kind of mapping see for example Halmos [2] page 153.

We can now write the characteristic function of $x_1, \cdots, x_n$ as (see Halmos [2] page 163)

$$\exp\left[-\textstyle\int_{-\frac{1}{2}}^{\frac{1}{2}} |\mu_1 f_1(\lambda) + \cdots + \mu_n f_n(\lambda)|^\alpha dG(f_1(\lambda), \cdots, f_n(\lambda))\right).$$

If we let for $1 \leqq \alpha \leqq 2$ (Halmos [2] page 179 and Lemma (4.1)] $dG(f_1(\lambda), \cdots, f_n(\lambda)) = dF^*(\lambda)$ then we may write

$$\text{(4.1)} \quad \text{ch.f.}\,(x_1, \cdots, x_n; \mu_1, \cdots, \mu_n) = \exp\left[-\textstyle\int_{-\frac{1}{2}}^{\frac{1}{2}} |\mu_1 f_1(\lambda) + \cdots + \mu_n f(\lambda)|^\alpha dF^*(\lambda)\right].$$

Define now an independent increments process $F(\lambda)$ of type α by $F(-\frac{1}{2}) = 0$ and

$$\|F(\lambda_2) + F(\lambda_1)\|^\alpha = F^*(\lambda_2) - F^*(\lambda_1) \qquad \text{for} \quad -\tfrac{1}{2} \leqq \lambda_1 < \lambda_2 \leqq \tfrac{1}{2}.$$

This gives the transition probabilities for the Markov process $F(\lambda)$ and therefore defines it as a process.

Since by construction the $f_l(\lambda)$ are measurable and integrable with respect to $dF^*(\lambda)$, it follows by Theorem 3.1 that we may define $W_l = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_l(\lambda)\, dF(\lambda)$.

[1] Proof of this theorem for even countably many x's has eluded the most vigorous efforts of the author.

The characteristic function of $W_l, \cdots, W_n$ can be written as

$$\begin{aligned} &\text{ch.f.}(W_1, \cdots, W_n; \mu_1, \cdots, \mu_n) \\ &\quad = \text{ch.f.}(\mu_1 W_1 + \cdots + \mu_n W_n; 1) \\ &\quad = \exp[-\|\mu_1 W_1 + \cdots + \mu_n W_n\|^\alpha] \\ &\quad = \exp[-\|\mu_1 \textstyle\int_{-\frac{1}{2}}^{\frac{1}{2}} f_1(\lambda)\, dF(\lambda) + \cdots + \mu_n \int_{-\frac{1}{2}}^{\frac{1}{2}} f_n(\lambda)\, dF(\lambda)\|^\alpha]. \end{aligned}$$

By an easy exercise in integration theory, it can be shown that the finite summation and the integral signs may be interchanged and therefore the joint characteristic function of $W_1, \cdots, W_n$ may be written as

$$\exp[-\|\textstyle\int_{-\frac{1}{2}}^{\frac{1}{2}} (\mu_1 f_1(\lambda) + \cdots + \mu_n f_n(\lambda))\, dF(\lambda)\|^\alpha].$$

By Theorem 3.1 this is

$$\exp[-\textstyle\int_{-\frac{1}{2}}^{\frac{1}{2}} |\mu_1 f_1(\lambda) + \mu_2 f_2(\lambda) + \cdots + \mu_n f_n(\lambda)|^\alpha d\|F(\lambda)\|^\alpha].$$

Since by construction $d\|F(\lambda)\|^\alpha = dF^*(\lambda)$ the last expression is

$$\exp[-\textstyle\int_{-\frac{1}{2}}^{\frac{1}{2}} |\mu_1 f_1(\lambda) + \cdots + \mu_n f_n(\lambda)|^\alpha dF^*(\lambda)].$$

But this expression is the same as (4.1), the joint characteristic function of $x_1, \cdots, x_n$, and therefore the W and x variables have the same joint characteristic function which shows they have the same probability structure. Since a similar argument may be used if $0 < \alpha < 1$, the theorem is proved.

5. Some properties of symmetric stable variables. In this section some elementary consequences of the definitions and theorems of this paper will be given.

THEOREM 5.1. *If $0 < \alpha < 2$ and if*

$$x_1 = \textstyle\int_{-\frac{1}{2}}^{\frac{1}{2}} f_1(\lambda)\, dF(\lambda) \quad \textit{and} \quad x_2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_2(\lambda)\, dF(\lambda),$$

where $F(\lambda)$ is an independent increments process of type α, then x_1 is independent of x_2 if and only if $f_1(\lambda)f_2(\lambda) = 0$ except (possibly) on a set of $d\|F(\lambda)\|^\alpha$ measure zero ($d\|F(\lambda)\|$ measure if $0 < \alpha < 1$).

PROOF. x_1 and x_2 are independent if and only if their joint characteristic function factors. Therefore x_1 is independent of x_2 if and only if

$$\begin{aligned} &\textstyle\int_{-\frac{1}{2}}^{\frac{1}{2}} |\mu_1 f_1(\lambda) + \mu_2 f_2(\lambda)|^\alpha d\|F(\lambda)\|^\alpha \\ &\quad = \|\mu_1 \textstyle\int_{-\frac{1}{2}}^{\frac{1}{2}} f_1(\lambda)\, dF(\lambda) + \mu_2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f_2(\lambda)\, dF(\lambda)\|^\alpha \\ (5.1) &\quad = -\log[\text{ch.f.}(\mu_1 x_1 + \mu_2 x_2; 1)] = -\log[\text{ch.f.}(x_1, x_2; \mu_1, \mu_2)] \\ &\quad = -\log[\text{ch.f.}(x_1; \mu_1)\, \text{ch.f.}(x_2; \mu_2)] \\ &\quad = -\log[\text{ch.f.}(x_1; \mu_1)] - \log[\text{ch.f.}(x_2; \mu_2)] \\ &\quad = |\mu_1|^\alpha \textstyle\int_{-\frac{1}{2}}^{\frac{1}{2}} |f_1(\lambda)|^\alpha d\|F(\alpha)\|^\alpha + |\mu_2|^\alpha \int_{-\frac{1}{2}}^{\frac{1}{2}} |f_2(\lambda)|^\alpha d\|F(\lambda)\|^\alpha \end{aligned}$$

for $1 \leqq \alpha < 2$ and all real μ_1 and μ_2.

If $f_1(\lambda)f_2(\lambda) = 0$ except on a set of measure zero, then clearly (5.1) holds and the if part of the theorem follows. A similar proof holds if $0 < \alpha \leqq 1$. For the only if part, we note for $1 \leqq \alpha \leqq 2$

$$|\mu_1 f_1(\lambda)+\mu_2 f_2(\lambda)|^\alpha \leqq |\mu_1|^\alpha |f_1(\lambda)|^\alpha + |\mu_2|^\alpha |f_2(\lambda)|^\alpha.$$

Suppose $f_1(\lambda)f_2(\lambda) \neq 0$ on the set of positive measure $d||F(\lambda)||^\alpha$. If $f_1(\lambda) > 0, f_2(\lambda) < 0$ then choose $\mu_1 = \mu_2 = 1$, then the right-hand side is $|f_1(\lambda)+f_2(\lambda)|^\alpha$ which is strictly less than $|f_1(\lambda)|^\alpha + |f_2(\lambda)|^\alpha$ and hence

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |\mu_1 f_1(\lambda)+\mu_2 f_2(\lambda)|^\alpha d||F(\lambda)||^\alpha < |\mu_1|^\alpha \int_{-\frac{1}{2}}^{\frac{1}{2}} |f_1(\lambda)|^\alpha d||F(\lambda)||^\alpha + |\mu_2|^\alpha \int_{-\frac{1}{2}}^{\frac{1}{2}} |f_2(\lambda)|^\alpha d||F(\lambda)||^\alpha$$

for such μ_1, μ_2. If $f_1(\lambda) > 0$, $f_2(\lambda) > 0$ or $f_1(\lambda) < 0$, $f_2(\lambda) < 0$; choosing $\mu_1 = -\mu_2 = 1$ we have the same inequality. This contradicts (5.1). A similar proof holds if $0 < \alpha < 1$.

If $\alpha = 2$, then the stable variables become Gaussian (normal) and, as is easily seen, the condition for the independence of $x_1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_1(\lambda)\, dF(\lambda)$ and $x_2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_2(\lambda)\, dF(\lambda)$ is that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f_1(\lambda) f_2(\lambda)\, d||F(\lambda)||^2 = 0,$$

(their correlation coefficient is zero). This is an example of the fact that analysis of stable variables for $\alpha < 2$ is considerably different than for Gaussian variables. It is clearly much harder for stable variables to be independent. (Independence is no longer a unitary invariant.)

It is shown in most elementary books on statistics that if x_1 and x_2 are Gaussian variables then x_1 and x_2 may be written as

$$x_l = \sum_{j=1}^{2} a_{lj}\, y_j \qquad l = 1,2, \tag{5.2}$$

where y_1 and y_2 are independent. It might be conjectured that (5.2) holds for stable variables as well and therefore that the complicated representation Theorem 4.1 is unnecessary, for at least a finite number of x_l's anyway. Let, however, $1 \leqq \alpha < 2$ and let $F(\lambda)$ be an independent increments process of type α such that $||F(\lambda_2) - F(\lambda_1)||^\alpha = \lambda_2 - \lambda_1$ for $-\frac{1}{2} \leqq \lambda_1 \leqq \lambda_2 \leqq \frac{1}{2}$. Let $x_1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} dF(\lambda)$ and $x_2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \lambda\, dF(\lambda)$. Then reasoning as before it follows that

$$\begin{aligned} \text{ch.f.}(x_1, x_2; \mu_1, \mu_2) &= \exp[-||x_1 \mu_1 + x_2 x_2||^\alpha] \\ &= \exp[-||\textstyle\int_{-\frac{1}{2}}^{\frac{1}{2}} (\mu_1 + \lambda\mu_2)\, dF(\lambda)||^\alpha \\ &= \exp[-\textstyle\int_{-\frac{1}{2}}^{\frac{1}{2}} |\mu_1 + \lambda\mu_2|^\alpha d||F(\lambda)||^\alpha] \\ &= \exp[-\textstyle\int_{-\frac{1}{2}}^{\frac{1}{2}} |\mu_1 + \lambda\mu_2|^\alpha d\lambda]. \end{aligned} \tag{5.3}$$

Suppose now that x_1 and x_2 have a representation of the type

$$x_l = \sum_{j=1}^{n} a_{lj}\, y_j \qquad l = 1,2$$

and where n is some finite number. Then from Lemma 3.1 it follows that

$$\begin{aligned} \text{ch.f.}(x_1, x_2; \mu_1, \mu_2) &= \exp[-||x_1 \mu_1 + x_2 \mu_2||^\alpha] \\ &= \exp[-||\textstyle\sum_j^n (a_{1j}\mu_1 + a_{2j}\mu_2) y_j||^\alpha] \qquad (5.4) \\ &= \exp[-\textstyle\sum_j^n |a_{1j}\mu_1 + a_{2j}\mu_2|^\alpha ||y_j||^\alpha]. \end{aligned}$$

If x_1 and x_2 had the two representations, it would follow that the derivatives of the right-hand sides of (5.3) and (5.4) would be the same but it can be seen by explicit evaluation of the integral in the exponent of the right-hand side of (5.3) that it has two continuous derivatives with respect to μ_2 if $\mu_1 \neq 0$. On the other hand the second derivative of the term in the exponent of (5.4) with respect to μ_2 for $a_{2j} \neq 0$ has discontinuities at $\mu_2 = -a_{1j}\mu_1/a_{2j}$.

Since at least one a_{2j} is not zero and since the same type of argument holds if $0 < \alpha < 1$, it follows that *in general representation of even two stable variables of type* α, $0 < \alpha < 2$ *as the linear combination of a finite number of independent variables of the same type is impossible.*

If $x_1, \cdots, x_n, y$ are Gaussian variables, then it can be shown (in many ways) that there are numbers $a_1, \cdots, a_n$ which minimize $||y - \sum_l^n x_l a_l||$ and that $\omega = y - \sum_l^n x_l a_l$ is independent of all the x_l's. It is an easy corollary of Theorem 5.1 that in general $a_1, \cdots, a_n$ cannot be found so that ω is independent of $x_1, x_2, \cdots, x_n$ if $0 < \alpha < 2$. However, they can be found so that $||y - \sum x_l a_l||$ is minimized.

THEOREM 5.2. *Suppose*

$$x_l = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_l(\lambda)\, dF(\lambda) \qquad l = 1, \cdots, n+1$$

where $F(\lambda)$ *is an independent increments process of type* α, $1 \leqq \alpha \leqq 2$ *and suppose the* $f_l(\lambda)$ *are linearly independent with respect to the measure* $d||F(\lambda)||^\alpha$ *then in order that the random variable* $P_n = \sum x_l a_l$ *should deviate least in norm from* x_{n+1} *it is sufficient and (for* $\alpha = 1$ *in the case where the difference* $f_{n+1}(\lambda) - \sum x_l f_l(\lambda)$ *is different from zero almost everywhere* $d||F(\lambda)||^\alpha$ *measure) it is also necessary that for any* $f_l(\lambda)$ $l = 1, \cdots, n$ *the equality*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} (f_l(\lambda)\, |f_{n+1}(\lambda) - \textstyle\sum a_l f_l(\lambda)|^{\alpha-1} \operatorname{sgn}(f_{n+1}(\lambda) - \sum a_l f_l(\lambda))\, d\, ||F(\lambda)||^\alpha = 0$$

should hold.

PROOF. The proof follows immediately from a theorem proved in Timan [7] page 64, where a similar theorem for L^α spaces is proved, since there is an isometric isomorphism between L^α and the linear space generated by the x_l's.

A number of the other results from the second order theory have analogies for the symmetric stable process.

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