The Annals of Mathematical Statistics
1970, Vol. 41, No. 2, 706–709

CONVERGENCE PROPERTIES OF MARTINGALE TRANSFORMS

BY HARRO WALK

University of Stuttgart

1. Introduction. This note relates to papers of D. L. Burkholder [2], Y. S. Chow [4], B. Davis [5] and concerns the convergence behavior of martingales and, more generally, of martingale transforms. At first a generalization of Chow's convergence theorem ([4], Theorem 10) for sums of means of independent random variables is obtained by a modification of his proof. To three equivalent convergence assertions on martingale transforms (see Burkholder [2], Theorem 4; compare J. L. Doob [6] page 320, (iv)) this result yields two other equivalent assertions, one of them concerning the notion of essential convergence (compare [9] 5). This notion is also treated in the last theorem, the second part of which may be regarded as a statement on random power series (compare [1] (L. Arnold) page 229; [10] 4, 5; [9] 5; [11] 3).

Throughout the note we use the following definitions and notations in which all numbers and functions are real or complex. Let $\{\sum_{k=1}^{n} y_k, n \geqq 1\}$ be a martingale on a probability space $(\Omega, \mathscr{A}, P)$. According to Burkholder [2] we say that $\{\sum_{k=1}^{n} s_k\}$ is a *transform* of $\{\sum_{k=1}^{n} y_k\}$ if $s_m = v_m y_m$, where v_m is an $\mathscr{A}_{m-1}$-measurable function, $m \geqq 1$, and $\mathscr{A}_0 \subset \mathscr{A}_1 \subset \cdots \mathscr{A}$ are σ-fields such that $\{\sum_{k=1}^{n} y_k, \mathscr{A}_n, n \geqq 1\}$ is a martingale. Let $\{b_n\}$ be a sequence of numbers and $x_n = y_n + b_n$, $r_n = s_n + b_n = v_n y_n + b_n$; if $E|v_n y_n|$ is finite, then $Er_n = b_n$ $(n \geqq 2)$. A series $\sum g_n$ of $\mathscr{A}$-measurable functions is said to be essentially convergent (essentially divergent) on $\Omega' \in \mathscr{A}$, if there exists a number sequence $\{c_n\}$ with convergence of $\sum(g_n - c_n)$ a.e. on Ω' (if $\sum(g_n - c_n)$ is divergent a.e. on Ω' for every choice of the number sequence $\{c_n\}$) (compare [8] page 250). For any sequence $\{h_n\}$ of functions on Ω let h^* be defined by $h^*(\omega) = \sup_n |h_n(\omega)|$, $\omega \in \Omega$. The α-quantile $(0 < \alpha < 1)$ of an $\mathscr{A}$-measurable function h is denoted by $\mu_\alpha h$ $(= \mu_\alpha h' + i\mu_\alpha h''$ for a complex $h = h' + ih'')$.

2. Theorems. The following theorems obviously yield convergence assertions on martingales themselves ($v_n = 1$ for all n and thus $s_n = y_n$, $r_n = x_n$, $Er_n = Ex_n = b_n$).

THEOREM 1. *Suppose* $Ey^* < \infty$. $P[\sum r_n$ *convergent*; $v^* < \infty] > 0$ *implies convergence of* $\sum b_n$. $P[\sup_n |\sum_{k=1}^{n} r_k| < \infty;\ v^* < \infty] > 0$ *implies* $\sup_n |\sum_{k=1}^{n} b_k| < \infty$; *furthermore* $\sum s_n$ *converges* a.e. *on the set where* $\sup_n |\sum_{k=1}^{n} r_k| < \infty$, $v^* < \infty$.

THEOREM 2. *Let* $\Omega' \in \mathscr{A}$, α *a real number with* $0 < \alpha < P(\Omega')$, $v^* < \infty$ a.e. *on* Ω', *and* $Ey^* < \infty$. *With the essential convergence of* $\sum s_n$ (*or* $\sum r_n$) *on* Ω' *there are equivalent the convergence of* $\sum s_n$ a.e. *on* Ω'—*therefore* (*according to* [2]) *the validity of* $\sup_n |\sum_{k=1}^{n} s_k| < \infty$ a.e. *on* Ω' *and* $\sum |s_n|^2 < \infty$ a.e. *on* Ω', *too—and the convergence of* $\sum |s_n - \mu_\alpha s_n|^2$ (*or* $\sum |r_n - \mu_\alpha r_n|^2$) a.e. *on* Ω'.

Received April 16, 1969.

REMARK 1. (a) Into the list of equivalent assertions in Theorem 2 the "essential boundedness" of the partial sums of $\sum s_n$ in Ω', i.e. the existence of a number sequence $\{c_n\}$ with $\sup_n |\sum_{k=1}^n (s_k - c_k)| < \infty$ a.e. on Ω', can be taken up.

(b) Theorem 2 remains valid, if everywhere "convergence," "$<\infty$," and "essential convergence" are replaced by "(for every $\alpha \in (0, 1)$ existing) divergence" resp. "$=\infty$" resp. "essential divergence."

THEOREM 3. (a) *Let* $\Omega' \in \mathscr{A}$, $v^* < \infty$ a.e. *on* Ω', *and* $Ey^* < \infty$. *If* $\sum r_n$ *is essentially convergent on* Ω', *then also for every bounded number sequence* $\{d_n\}$ *the series* $\sum d_n r_n$ *is essentially convergent on* Ω'.

(b) *Let* $v^* < \infty$ a.e., $Ey^* < \infty$. *There exists a set* $\Omega' \in \mathscr{A}$, *determined up to a set of P-measure zero, such that* $\sum r_n e^{in\varphi}$ *is essentially convergent on* Ω' *for all* $\varphi \in [0, 2\pi)$ *and essentially divergent on* $\Omega - \Omega'$ *for all* $\varphi \in [0, 2\pi)$. *At this* $\sum s_n(\omega) e^{in\varphi}$ *and* $\sum (r_n(\omega) - \mu_\alpha r_n) e^{in\varphi}$ *converge for P-almost all* $\omega \in \Omega'$, *L-almost all* $\varphi \in [0, 2\pi)$ (*if* $0 < \alpha < P(\Omega')$) *and diverge for P-almost all* $\omega \in \Omega - \Omega'$, *L-almost all* $\varphi \in [0, 2\pi)$ (*if* $0 < \alpha < P(\Omega - \Omega')$).

REMARK 2. In Theorem 1—Theorem 3 and Remark 1 the supposition $Ey^* < \infty$ may be replaced by the weaker supposition that there exists an L^1 bounded martingale $\{f_n, \mathscr{A}_n, n \geqq 1\}$ on $(\Omega, \mathscr{A}, P)$ with $f^* \geqq y^*$ (compare Davis [5], Theorem 2).

3. Proofs. For the proof of Theorem 1 the following lemmas will be needed. By an indirect proof—like that of [9], Theorem 3b with footnote—we obtain

LEMMA 1. *Let* $t_1, t_2, \cdots$ *be real nonnegative random variables on* $(\Omega, \mathscr{A}, P)$ *and* $0 < \alpha < q \leqq 1$. *Then* $P[\sup_n t_n < \infty] = q$ *implies* $\sup_n \mu_\alpha t_n < \infty$.

LEMMA 2. *Let* $t_1, t_2, \cdots$ *be random variables* $\in L^1(\Omega, \mathscr{A}, P)$ *and* $0 < \alpha < 1$. *Then the two assertions* $E \sup_n |t_n - Et_n| < \infty$ *and* $E \sup_n |t_n - \mu_\alpha t_n| < \infty$ *are equivalent.*

PROOF OF LEMMA 2. From $E \sup_n |t_n - Et_n| < \infty$ the relation $P[\sup_n |t_n - Et_n| < \infty] = 1$ follows, from this by Lemma 1 the relation $\sup_n \mu_\alpha |t_n - Et_n| < \infty$ and thus $\sup_n |\mu_\alpha t_n - Et_n| < \infty$; we get

$$E \sup_n |t_n - \mu_\alpha t_n| \leqq E \sup_n |t_n - Et_n| + \sup_n |\mu_\alpha t_n - Et_n| < \infty.$$

As to the converse, from $E \sup_n |t_n - \mu_\alpha t_n| < \infty$ the relation $\sup_n E|t_n - \mu_\alpha t_n| < \infty$ follows and thus $\sup_n |Et_n - \mu_\alpha t_n| < \infty$; we get

$$E \sup_n |t_n - Et_n| \leqq E \sup_n |t_n - \mu_\alpha t_n| + \sup_n |Et_n - \mu_\alpha t_n| < \infty.$$

PROOF OF THEOREM 1. The proof can be reduced—compare Burkholder [2] page 1498–1499—to the case $v_n = 1$, $r_n = x_n$, $Ey^* < \infty$ with $b_n = Ex_n$ $(n \geqq 1)$. For $P[\sum (v_n y_n + b_n)$ convergent; $v^* < \infty] > 0$ implies the existence of a $c > 0$ with $P[\sum (\hat{v}_n y_n + b_n)$ convergent$] > 0$ where $\hat{v}_n(\omega) = v_n(\omega)$ if $|v_n(\omega)| < c$, $= 0$ otherwise $(\omega \in \Omega)$ and thus $\{\sum_{k=1}^n \hat{v}_k y_k\}$ is a martingale with $E \sup_n |\hat{v}_n y_n| < \infty$ because of

$Ey^* < \infty$; in a similar way we can treat the second part and—by letting $c \to \infty$—the third part of the theorem.

Now we turn to the case $v_n = 1$, $r_n = x_n$ $(n \geqq 1)$, $Ey^* < \infty$. Under the stronger assumption $Ey^{*2} < \infty$ the assertions could be proved by symmetrizing $\{x_n\}$ and using [6] page 320, (v). In the following we modify a proof which has been given by Y. S. Chow [4] page 1492, as a conclusion of the convergence of $\sum Ex_n$ from $P[\sum x_n \text{ convergent}] = 1$ (!) for a stochastically independent sequence $\{x_n\}$ with $Ex^* < \infty$ and which also holds without altering for the case that $\{\sum_{k=1}^n (x_k - Ex_k)\}$ is a martingale.

Suppose $P[\sup_n |\sum_{k=1}^n x_k| < \infty] > 0$. With $0 < \alpha < P[\sup_n |x_n| < \infty]$ and $Ey^* < \infty$ we obtain

$$Ex^* \leqq E \sup_n |x_n - \mu_\alpha x_n| + \sup_n |\mu_\alpha x_n| < \infty$$

using Lemma 1 and Lemma 2. There exists a $d > 0$ with $P[\sup_n |\sum_{k=1}^n x_k| < d] > 0$. If we put $x_n' = g_n x_n$ with $g_1(\omega) = 1$ $(\omega \in \Omega)$ and $g_n(\omega) = 1$ if $\sup_{m \leqq n-1} |\sum_{k=1}^m x_k(\omega)| < d$, $= 0$ otherwise $(n \geqq 2;\ \omega \in \Omega)$, then, because $\{\sum_{k=1}^n (x_k - Ex_k)\}$ is a martingale and $P[|\sum_{k=1}^n x_k'| < d + x^*] = 1$, we get

$$\left|\sum_{k=1}^n Ex_k\, Eg_k\right| = \left|E \sum_{k=1}^n x_k'\right| < d + Ex^* \qquad (n \geqq 1).$$

Thus the sequence $\{\sum_{k=1}^n Ex_k\, Eg_k\}$ is bounded. From this and the monotoneity and boundedness of the sequence $\{1/Eg_n\}$ we obtain the boundedness of $\{\sum_{k=1}^n Ex_k\}$ by using a variant of Abel's criterion which can be proved analogically to [7] 184, °1. Furthermore a.e. on the set where $\sup_n |\sum_{k=1}^n x_k| < \infty$ we thus have $\sup_n |\sum_{k=1}^n (x_k - Ex_k)| < \infty$ and therefore by [6] page 320, (iv), convergence of the series $\sum (x_n - Ex_n)$.

By this result the supposition $P[\sum x_n \text{ convergent}] > 0$ implies $P[\sum x_n \text{ convergent}, \sum (x_n - Ex_n) \text{ convergent}] > 0$ and therefore the convergence of $\sum Ex_n$. □

Proof of Theorem 2. The last part of Theorem 1 immediately yields that $\sum s_n$ converges a.e. on Ω' if $\sum s_n$ is essentially convergent on Ω'. The converse is trivial. Now it will be shown that convergence of $\sum |s_n - \mu_\alpha s_n|^2$ a.e. on Ω' is a further equivalent assertion. We start with $\sum |s_n|^2 < \infty$ a.e. on Ω', obtain by [9], Theorem 3(a) with footnote, $\sum \mu_\alpha |s_n|^2 < \infty$ and thus $\sum |\mu_\alpha s_n|^2 < \infty$ and conclude $\sum |s_n - \mu_\alpha s_n|^2 < \infty$ a.e. on Ω'. In order to prove the converse we use the Rademacher functions ϕ_n on $[0, 1)$ and successively obtain convergence of $\sum \phi_n(t)(s_n - \mu_\alpha s_n)$ a.e. on Ω' and—by Theorem 1—convergence of $\sum \phi_n(t) \mu_\alpha s_n$ for L-almost all $t \in [0, 1)$, then $\sum |\mu_\alpha s_n|^2 < \infty$, and finally $\sum |s_n|^2 < \infty$ a.e. on Ω'. □

Remark 1 (a) is proved analogously to the first part of the proof of Theorem 2. Remark 1 (b) is proved by using Theorem 2 (and—as to the $\alpha \in (0, 1)$—its proof).

Proof of Theorem 3. We only remark that—according to a theorem of Carleson [3]—for an $\omega \in \Omega$ from $\sum |r_n(\omega) - \mu_\alpha r_n|^2 < \infty$ the convergence of $\sum (r_n(\omega) - \mu_\alpha r_n) e^{in\varphi}$ for L-almost all $\varphi \in [0, 2\pi)$ follows, and that all the other conclusions can be made by Theorem 2, Remark 1 (b), and the Fubini theorem. □

As to Remark 2 it suffices to prove the statements which relate to Theorem 1 and the parenthesis of Theorem 2 because the other proofs do not alter. As in the proof of Theorem 1 resp. as in [2] page 1498–1499, we make a reduction to the case $v_n = 1$, $r_n = x_n$ $(n \geqq 1)$, $f^* \geqq y^*$. Without loss of generality we may assume $\{f_n, \mathscr{A}_n\}$ as a nonnegative martingale (compare [5] page 2143) and all the present numbers and functions as real ones. Thus we have come to an assertion reducible to Theorem 1 (with $v_n = 1$, $r_n = x_n$ $(n \geqq 1)$, $Ey^* < \infty$) by a stopping time argument like that in Davis' [5] proof of his Theorem 2 resp. we have come to this theorem itself.

REFERENCES

[1] ARNOLD, LUDWIG (1967). Zur Konvergenz und Nichtfortsetzbarkeit zufälliger Potenzreihen. *Trans. Fourth Prague Conf. Information Theory, Statist. Decision Functions, Random Processes* 1965. Czechoslovak Academy, Prague. 223–234.

[2] BURKHOLDER, D. L. (1966). Martingale transforms. *Ann. Math. Statist.* **37** 1494-1504.

[3] CARLESON, LENNART (1966). On convergence and growth of partial sums of Fourier series. *Acta Math.* **116** 135–157.

[4] CHOW, Y. S. (1966). Some convergence theorems for independent random variables. *Ann. Math. Statist.* **37** 1482–1493.

[5] DAVIS, BURGESS (1968). Comparison tests for the convergence of martingales. *Ann. Math. Statist.* **39** 2141–2144.

[6] DOOB, J. L. (1953). *Stochastic Processes.* Wiley, New York.

[7] KNOPP, KONRAD (1947). *Theorie und Anwendung der unendlichen Reihen* (4.Aufl.). Springer, Berlin and Heidelberg.

[8] LOÈVE, MICHEL (1963). *Probability Theory* (3rd ed.). Van Nostrand, Princeton.

[9] WALK, HARRO (1967). Symmetrisierte und zentrierte Folgen von Zufallsvariablen. *Math. Z.* **102** 44–45.

[10] WALK, HARRO (1968). Über das Randverhalten zufälliger Potenzreihen. *J. Reine Angew. Math.* **230** 66–103.

[11] WALK, HARRO (1969). Wachtumsverhalten zufälliger Potenzreihen. *Z. Wahrscheinlichkeitstheorie verw. Geb.* **12** 293–306.