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A NOTE ON RANDOM POLYNOMIALS

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1. Introduction. We consider the real polynomial

$$P_0(z) = z^n - a_{0,n-1} z^{n-1} + \cdots + (-1)^n a_{0,0}, \tag{1}$$

$$a_{0,n-k} = \sum_{i_1 < \cdots i_k} z_{i_1} z_{i_2} \cdots z_{i_k}, \qquad 1 \leqq k \leqq n, \tag{2}$$

the z_i being the zeros of P_0. Subsequently we will have coefficients $a_{i,j}$ and we will write

$$A_i = (a_{i,0}, a_{i,1}, \cdots, a_{i,r}). \tag{3}$$

The dimension $r+1$ of the vector A_i will be known in each case from the context.

The case where A_0 is a real random vector is of some interest in statistical communication theory. (1) is called minimum phase if all its zeros are in $|z| < 1$. One would like to know the probability that (1) is minimum phase. We will indicate a constructive way of writing down this probability when the joint density $f(A_0)$ of the coefficients is known. It will be obvious that we are not restricted to the unit disc but could in the same manner compute the probability that all the zeros lie in any given set of reasonable configuration. It will be equally clear that the complications severely limit practical applications.

2. Probability of minimum phase. We proceed now with the computation of the probability that (1) is minimum phase. We consider only the case $n = 2m+1$, the other case being similar. First, we prove a lemma of some independent interest. Throughout the lemma and its proof only we use the notation

$$P_r(z) = z^r - a_{r,r-1} z^{r-1} + \cdots + (-1)^r a_{r,0}.$$

LEMMA. *Suppose*

$$P_{m+n}(z) = P_m(z) P_n(z). \tag{i}$$

Equating coefficients of like powers of z in (i) *we may regard the equations expressing the* $a_{m+n,j}$'s *in terms of the* $a_{m,j}$'s *and the* $a_{n,j}$'s *as a transformation of variables from the former to the latter. The Jacobian of this transformation is*

$$\begin{aligned} J = J(A_m, A_n) &= |P_n(z_1) P_n(z_2) \cdots P_n(z_m)| \\ &= |P_m(z_1{}') P_m(z_2{}') \cdots P_m(z_n{}')|, \end{aligned} \tag{ii}$$

where $z_1, \cdots, z_m$ *are the zeros of* P_m *and* $z_1{}', \cdots, z_n{}'$ *are the zeros of* P_n.

PROOF. The last equality in (ii) is trivial.

The expansion of the determinant of the Jacobian is too hard. We make a

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sequence of transformations equal to the desired transformation and multiply their Jacobians together. Define polynomials by the formulae

(L.1) $$P_{m+n}(z) = (z - z_m)P_{m+n-1}(z),$$

(L.2) $$P_{m+n-1}(z) = (z - z_{m-1})P_{m+n-2}(z),$$

$\vdots$

(L.m) $$P_{n+1}(z) = (z - z_1)P_n(z).$$

In (L.1) we equate like powers of z and get a transformation from A_{m+n} to z_m and A_{m+n-1}. The Jacobian of this transformation is easily computed directly and is $|P_{m+n-1}(z_m)|$. In (L.2) in a similar way we get a transformation from A_{m+n-1}, z_m to A_{m+n-2}, z_{m-1}, z_m with z_m going into z_m. The Jacobian is $|P_{m+n-2}(z_{m-1})|$. In (L.3) A_{m+n-2}, z_{m-1}, z_m go into A_{m+n-3}, z_{m-2}, z_{m-1}, z_m, with z_{m-1} going into z_{m-1} and z_m going into z_m. The Jacobian is $|P_{m+n-3}(z_{m-2})|$. Finally, in (L.m) A_{n+1}, $z_m, \cdots, z_2$ go into A_n, $z_m, \cdots, z_1$ with z_j going into z_j, $2 \leqq j \leqq m$. The Jacobian is $|P_n(z_1)|$.

Finally, we must transform from A_n, $z_m, \cdots, z_1$ to A_n, A_m with A_n going under the identity transformation and A_m going under the elementary symmetric functions in the z_i. This Jacobian is well known and is

$$\left(\prod_{i<j} |z_i - z_j|\right)^{-1} = F.$$

So the Jacobian of the transformation of the lemma is

$$F \prod_{j=1}^{m} |P_{m+n-j}(z_{m+1-j})| = J.$$

But by definition, using (L.1) through (L.m),

$$P_{m+n-j}(z_{m+1-j}) = P_n(z_{m+1-j}) \prod_{r=j}^{m-1} (z_{m+1-j} - z_{m-r}).$$

Putting this back into J and canceling whatever we can proves the lemma.

In (1) $n = 2m+1$ so we can write

(4) $$P_0(z) = z^{2m+1} - a_{0,2m} z^{2m} + \cdots - a_{0,0} = (z-x)P_1(z),$$
$$P_1(z) = z^{2m} - a_{1,2m-1} z^{2m-1} + \cdots + a_{1,0},$$

x and the $a_{1,j}$'s being real. P_0 is minimum phase if and only if $|x| < 1$ and P_1 is minimum phase. From (4)

(5) $$a_{0,0} = xa_{1,0}, \qquad a_{0,j} = a_{1,j-1} + xa_{1,j}, \qquad 1 \leqq j \leqq 2m-1,$$
$$a_{0,2m} = a_{1,2m-1} + x.$$

According to the lemma the transformation (5) has the Jacobian

(6) $$J_1(A_1, x) = |x^{2m} - a_{1,2m-1} x^{2m-1} + \cdots + a_{1,0}|.$$

This being so, the probability we are calculating may be written

(7) $$\int_{K(0)} f(A_0)\, da_{0,0}\, da_{0,1} \cdots da_{0,n-1}$$
$$= \int_{-1}^{+1} dx \int_{K(1)} f(A_0) J_1(A_1, x)\, da_{1,0} \cdots da_{1,2m-1}.$$

A_0 in the right-hand side of (7) must be replaced by A_1 and x through (5). $K(0)$ is the set in A_0-space wherein P_0 is minimum phase and $K(1)$ is the set in A_1-space wherein P_1 is minimum phase.

P_1 has the real factorization

$$\begin{aligned} P_1(z) &= q_2(z)P_2(z), \qquad q_2(z) = z^2 - b_{2,1}z + b_{2,0}, \\ P_2(z) &= z^{2m-2} - a_{2,2m-3}\, z^{2m-3} + \cdots + a_{2,0}. \end{aligned} \tag{8}$$

From (8)

$$\begin{aligned} a_{1,0} &= b_{2,0}\, a_{2,0}, \qquad a_{1,1} = b_{2,0}\, a_{2,1} + b_{2,1}\, a_{2,0}, \\ a_{1,j} &= a_{2,j-2} + b_{2,1}\, a_{2,j-1} + b_{2,0}\, a_{2,j}, \qquad 2 \leqq j \leqq 2m-3 \\ a_{1,2m-2} &= b_{2,0} + b_{2,1}\, a_{2,2m-3} + a_{2,2m-4}, \\ a_{1,2m-1} &= b_{2,1} + a_{2,2m-3}. \end{aligned} \tag{9}$$

The transformation has the Jacobian (the lemma)

$$J_2(A_2, B_2) = |P_2(u)P_2(v)| \tag{10}$$

wherein $B_2 = (b_{2,0}, b_{2,1})$ and u and v are the zeros of q_2.

P_1 is minimum phase if and only if q_2 and P_2 are. It is easy to see that q_2 is minimum phase if and only if B_2 is in the triangle with vertices $B_2 = (-1, 0)$, $(1, 2)$, $(1, -2)$. Consequently, (7) may be written

$$\int_{-1}^{1} dx \int_{-1}^{1} db_{2,0} \int_{-b_{2,0}-1}^{b_{2,0}+1} db_{2,1} \int_{K(2)} f(A_0) J_1(A_1, x) J_2(A_2, B_2)\, da_{2,0} \cdots da_{2,2m-3}. \tag{11}$$

$K(2)$ is the set in A_2-space on which P_2 is minimum phase.

It is clear that we may deal with P_2 just as we dealt with P_1 in order to reduce the integral over $K(2)$ to an integral over B_3 and $K(3)$. Because P_1 is of even degree this process will soon terminate and we will be left with an iterated integral to be evaluated by conventional means.

We have not tried to evaluate this integral in any useful cases and do not know whether this procedure has any advantages over a sampling scheme for determining the desired probability.