The Annals of Mathematical Statistics
1970, Vol. 41, No. 1, 44–58

SPECTRAL ESTIMATION WITH RANDOM TRUNCATION

BY JAMES PICKANDS III[1]

Virginia Polytechnic Institute

0. Summary. Let $f(\omega)$, $-\pi \leqq \omega \leqq \pi$ be the spectral density function of a discrete coordinate real-valued time series, stationary to order four. Assume that the covariance function $r(k)$ is such that $-\log r^2(k) \sim Ck^\gamma$, and $-\log(r^2(k+1)/r^2(k)) \sim C\gamma k^{\gamma-1}$, as $k \to \infty$, for some C, γ, $0 < C, \gamma < \infty$. Then there exists a non-random sequence $t(n)$ which is such that the estimator $f^*(\omega) \equiv (2\pi)^{-1} \sum_{k=-t(n)}^{t(n)} (1-n^{-1}|k|)\, \hat{r}(k)\, e^{ik\omega}$ is efficient where $\hat{r}(k) \equiv (n-|k|)^{-1} \sum_{j=1}^{n-|k|}(X(j)-\bar{X})(X(j+|k|)-\bar{X})$, $\bar{X} = n^{-1}\sum_{j=1}^{n} X(j)$, and an estimator $\hat{f}(\omega)$ is said to be efficient if

$$\lim_{n\to\infty} 2\pi E \int_{-\pi}^{\pi} (\hat{f}(\omega) - f(\omega))^2\, d\omega / I_{\min}^2(n) = 1,$$

where $I_{\min}^2(n)$ is the smallest integrated mean squared error which can be achieved using an estimator of the form $\tilde{f}(\omega) = (2\pi)^{-1} \sum_{k=-(n-1)}^{n-1} a(k,n)\hat{r}(k)\, e^{ik\omega}$, where $a(k,n)$ is nonrandom. In general a sequence $t(n)$ which is efficient for one covariance function is inefficient for another. A class of estimators $\hat{f}(\omega)$ is presented which are of the form $\hat{f}(\omega) = (2\pi)^{-1}\sum_{k=-\hat{t}(n)}^{\hat{t}(n)}(1-n^{-1}|k|)\hat{r}(k)\, e^{ik\omega}$, where $\hat{t}(n)$ is a function of the observations. In an appropriate sense $\hat{t}(n)$ "estimates" $t(n)$. For any covariance function satisfying the above conditions $\sup_{-\pi\leqq\omega\leqq\pi}|\hat{f}(\omega) - \hat{f}^*(\omega)|/I_{\min}(n) \to 0$, in probability, where $f^*(\omega)$ is the unattainable efficient truncation estimator.

1. Introduction. The problem of estimating the spectral density function of a stationary time series is of considerable practical importance. Let $\{X(n), 1 \leqq n < \infty\}$ be a sequence of random variables, and assume that all moments of order four are finite. Let

$$\mu \equiv EX(n), \tag{1.1}$$

$$r(k) \equiv E(X(n)-\mu)(X(n+k)-\mu), \qquad -\infty < k < \infty, \qquad \text{and} \tag{1.2}$$

$$q(k,l,m) \equiv E(X(n)-\mu)(X(n+k)-\mu)(X(n+l)-\mu)(X(n+m)-\mu), \qquad -\infty < k,l,m < \infty. \tag{1.3}$$

These are, respectively, the mean, the covariance function, and the fourth moment function. It is assumed that they do not depend upon n. In other words, the sequence $X(n)$ is stationary to the fourth order. It is also assumed that the terms of the sequence are real valued. We can conclude from this, that

$$r(-k) = r(k), \qquad -\infty < k < \infty. \tag{1.4}$$

The function

$$f(\omega) = (2\pi)^{-1} \sum_{k=-\infty}^{\infty} r(k)\, e^{ik\omega}, \qquad -\pi \leqq \omega \leqq \pi,$$

Received April 3, 1969.

[1] Now at the University of Pennsylvania.

if it exists, is called the spectral density function. It follows from (1.4) that $f(\omega)$ is real valued. By Bochner's Theorem, in order that a function $r(k)$ be a covariance function, it is necessary and sufficient that it be positive definite. But this is true if and only if $f(\omega) \geqq 0$ for all ω, $-\pi \leqq \omega \leqq \pi$. So, in order that a function $f(\omega)$ be a spectral density function, it is necessary and sufficient that $f(\omega) \geqq 0$, and that $f(\omega)$ is integrable.

All of the spectral density functions considered in this work will be square integrable. That is $\int_{-\pi}^{\pi} f^2(\omega)\,d\omega = (2\pi)^{-1}\sum_{k=-\infty}^{\infty} r^2(k) < \infty$. The same is true of the estimators $\hat{f}(\omega)$. The most common criterion for error or "figure of merit" for an estimator is

$$I^2(n) \equiv 2\pi E \int_{-\pi}^{\pi} (\hat{f}(\omega) - f(\omega))^2\,d\omega. \tag{1.5}$$

An estimator is said to be consistent (in integrated mean square) if $I^2(n) \to 0$, as $n \to \infty$. Two obvious estimators for $f(\omega)$ are $\hat{f}(\omega) = (2\pi)^{-1}\sum_{k=-(n-1)}^{n-1} \hat{r}(k)\,e^{ik\omega}$, and $\hat{f}(\omega) = (2\pi)^{-1}\sum_{k=-(n-1)}^{n-1}(1-n^{-1}|k|)\hat{r}(k)\,e^{ik\omega}$,

where

$$\hat{r}(k) \equiv (n-|k|)^{-1}\sum_{j=1}^{n-|k|}(X(j)-\bar{X})(X(j+|k|)-\bar{X}), \quad \text{and} \tag{1.6}$$

$$\bar{X} = n^{-1}\sum_{j=1}^{n} X(j). \tag{1.7}$$

The second of these is the well known periodogram estimator. Unlike most of the estimators in the literature and unlike the first above, the second guarantees that $\hat{f}(\omega) \geqq 0$. Strangely, as natural as these estimators are, neither of them is consistent.

The class of estimators most widely employed consists of those which can be written

$$\hat{f}(\omega) = (2\pi)^{-1}\sum_{k=-(n-1)}^{n-1} a(k,n)\hat{r}(k)\,e^{ik\omega}, \tag{1.8}$$

where $a(k,n)$ does not depend upon the observations and $a(k,n) = a(-k,n)$. The conditions which are necessary and sufficient in order to insure consistency are well known. See, for example, Parzen [4] and [5], and Lomnicki and Zaremba [3].

In this paper, we are not concerned with consistency but rather with efficiency. The function $a(k,n)$ can be chosen so as to minimize $I^2(n)$. The corresponding value of $I^2(n)$ is called $I^2_{\min}(n)$. The problem is that in order to minimize $I^2(n)$ it is necessary to know the covariances $r(k)$. But if we knew this, we would not need an estimator. The asymptotic efficiency of an estimator is

$$\text{eff} \equiv \lim_{n\to\infty} I^2_{\min}(n)/I^2(n),$$

if it exists.

When estimating a vector-valued parameter, it is usually possible to find an estimator which has efficiency 1, for every value of the parameter except possibly for those in some set having Lebesgue measure zero. See Bahadur [1]. The same is not true for an estimator of the spectral density function. An estimator which has efficiency 1 for one process may well have efficiency 0 for another. The same problem arises in estimating a probability density function. See, for example, Watson and Leadbetter [7] and the author [6].

In Section 2, it is shown that under certain conditions, there exists a "truncation estimator" which is efficient, that is which has efficiency 1. A "truncation estimator" is one which is of the form

$$\hat{f}^*(\omega) = (2\pi)^{-1} \sum_{k=-t(n)}^{t(n)} (1-n^{-1}|k|)\hat{r}(k)\, e^{ik\omega}, \tag{1.9}$$

for some integer-valued function $t(n)$. Clearly this is a special case of an estimator of the form (1.8). Again, of course, the problem arises. In order to know what sequence $t(n)$ will yield an efficient estimator, it is necessary to know a great deal about the covariance function. So, it is not generally possible to obtain the estimator $\hat{f}^*(\omega)$.

In Section 3, a class of estimators is introduced, wherein the truncation point is allowed to depend on the observations. A fairly general class of covariance functions is considered. For any such covariance function there exists a truncation function $t(n)$ and an estimator $\hat{f}^*(\omega)$ which is efficient. Furthermore, if the estimator $\hat{f}(\omega)$ is any estimator in our class of estimators, and $r(k)$ is any covariance function in our class of covariance functions, then

$$\sup_{-\pi \leqq \omega \leqq \pi} |\hat{f}(\omega) - \hat{f}^*(\omega)| / I_{\min}(n) \to 0, \tag{1.10}$$

in probability as $n \to \infty$, where $\hat{f}^*(\omega)$ is the efficient truncation estimator.

A method of estimating a spectral density function, using a random truncation point has been given by Leppink [2]. His method of determining the truncation point is entirely different from the present one, and it does not appear to enjoy the property (1.10).

As is true of most of the estimators considered in the statistical literature, those considered here do not guarantee that $\hat{f}(\omega) \geqq 0$. But if a negative estimate occurs for some ω, the result can be replaced by 0. Since we know that $f(\omega) \geqq 0$, this can only reduce $I^2(n)$.

In Section 4, it is indicated that the results hold for a process in continuous time.

The approach taken in this work is similar to that taken by the author with respect to probability density functions (see [6]).

For a discussion of the problem of truncation, see Zaremba [8].

2. Minimum mean squared error and truncation estimation. In this section, conditions are given, under which there exists a truncation estimator which is efficient. The truncation function $t(n)$ is given explicitly in terms of the covariance function $r(k)$. First let us collect some facts and give some definitions concerning the fourth moment function $q(k, l, m)$. These are given in Parzen [3]. Let

$$q(k,l,m) = q_G(k,l,m) + q_{NG}(k,l,m), \tag{2.1}$$

where

$$q_G(k,l,m) \equiv r(k)r(m-l) + r(l)r(k-m) + r(m)r(l-k). \tag{2.2}$$

We call $q_G(k,l,m)$ the Gaussian part, since it is the fourth moment function for a stationary Gaussian sequence. Not surprisingly $q_{NG}(k,l,m)$ is called the non-

Gaussian part since for a stationary Gaussian process, it is identically zero. The absolute summability of the non-Gaussian part will be a condition assumed in what follows.

THEOREM 2.1. *Let* $\{X(n), 1 \leqq n < \infty\}$ *be a stationary time series with finite moments to the fourth order. Assume that it is stationary to the fourth order with a covariance function* $r(k)$ *which is such that*

$$R^l < \infty, \qquad l = 1, 2, \tag{2.3}$$

where

$$R^l = \sum_{k=-\infty}^{\infty} |r(k)|^l, \tag{2.4}$$

and that the non-Gaussian part of the fourth moment function is absolutely summable, that is that $\sum_{k,l,m=-\infty}^{\infty} |q_{NG}(k,l,m)| < \infty$. *Then* $E(\hat{r}(k) - r(k))^2$ *satisfies the inequality*

$$A(m) - B(k) \leqq mE(\hat{r}(k) - r(k))^2 \leqq A(m) + B(k), \tag{2.5}$$

where $m = n - |k|$, *and* $A(m)$ *and* $B(k)$ *are functions, which are such that*

$$\lim_{m\to\infty} A(m) = R^2 = (2\pi)\int_{-\pi}^{\pi} f^2(\omega)\, d\omega, \tag{2.6}$$

$$A(m) \leqq R^2, \tag{2.7}$$

for all m, and

$$\lim_{k\to\infty} B(k) = 0. \tag{2.8}$$

The equality on the right side of (2.6) *holds by the Parseval identity.*

PROOF. By definition (1.6),

$$\begin{aligned} \hat{r}(k) &= (n-|k|)^{-1}\sum_{j=1}^{n-|k|}((X(j)-\bar{X}')+(\bar{X}'-\bar{X}))((X(j+|k|)-\bar{X}'')+(\bar{X}''-\bar{X})) \\ &= (n-|k|)^{-1}\sum_{j=1}^{n-|k|}(X(j)-\bar{X}')(X(j+|k|)-\bar{X}'')+(\bar{X}'-\bar{X})(\bar{X}''-\bar{X}), \end{aligned} \tag{2.9}$$

where

$$\bar{X}' \equiv (n-|k|)^{-1}\sum_{j=1}^{n-|k|} X(j), \tag{2.10}$$

and

$$\bar{X}'' \equiv (n-|k|)^{-1}\sum_{j=|k|+1}^{n} X(j), \tag{2.11}$$

and, of course, $\bar{X}$ is given by (1.7). But

$$\begin{aligned} &(n-|k|)^{-1}\sum_{j=1}^{n-|k|}(X(j)-\bar{X}')(X(j+|k|)-\bar{X}'') \\ &\qquad = (n-|k|)^{-1}\sum_{j=1}^{n-|k|} X(j)X(j+|k|)-\bar{X}'\bar{X}''. \end{aligned} \tag{2.12}$$

Combining (2.9) and (2.12), it follows that

$$\hat{r}(k) = \hat{r}'(k) - \bar{X}(\bar{X}'+\bar{X}'')+\bar{X}^2, \tag{2.13}$$

where

$$\hat{r}'(k) \equiv (n-|k|)^{-1}\sum_{j=1}^{n-|k|} X(j)X(j+|k|), \tag{2.14}$$

and it is assumed, without loss of generality that $\mu \equiv 0$. First, let us consider $E(\hat{r}'(k)-r(k))^2$. Clearly, $E(\hat{r}'(k))^2 = m^{-2}\sum_{i,j=1}^{m} EX(i)X(i+|k|)X(j)X(j+|k|)$

where $m = n-|k|$. But, by definition (1.3), $EX(i)X(i+|k|)X(j)X(j+|k|) = q(|k|,j-i,j+|k|-i)$. So, by (2.1).

$$\begin{aligned} E(\hat{r}'(k))^2 &= m^{-1}\textstyle\sum_{l=-m}^{m}(1-m^{-1}|l|)q(|k|,l,|k|+l) \\ &= m^{-1}\textstyle\sum_{l=-m}^{m}(1-m^{-1}|l|)q_G(|k|,l,|k|+l) \\ &\quad + m^{-1}\textstyle\sum_{l=-m}^{m}(1-m^{-1}|l|)q_{NG}(|k|,l,|k|+l). \end{aligned} \tag{2.15}$$

But

$$\begin{aligned} &m^{-1}\textstyle\sum_{l=-m}^{m}(1-m^{-1}|l|)q_G(|k|,l,|k|+l) \\ &= m^{-1}\textstyle\sum_{l=-m}^{m}(1-m^{-1}|l|)r^2(k)+m^{-1}\sum_{l=-m}^{m}(1-m^{-1}|l|)r^2(l) \\ &\quad + m^{-1}\textstyle\sum_{l=-m}^{m}(1-m^{-1}|l|)r(|k|-l)r(|k|+l). \end{aligned} \tag{2.16}$$

Consider the first term on the right side of (2.16). Clearly $m^{-1}\sum_{l=-m}^{m}(1-m^{-1}|l|)r^2(k) = r^2(k)$. The second term is defined to be $A(m)/m$ and clearly $\lim_{m\to\infty} A(m) = R^2$, where R^2 is given by (2.4). Furthermore, clearly $A(m) \leqq R^2$. The absolute value of the third term is dominated by $C(k)/m$, where $C(k) \equiv \sum_{l=-\infty}^{\infty} |r(k-l)r(k+l)|$, and clearly $\lim_{k\to\infty} C(k) = 0$. Recalling (2.1), consider the non-Gaussian part. Clearly

$$\left|m^{-1}\textstyle\sum_{l=-m}^{m}(1-m^{-1}|l|)q_{NG}(|k|,l,|k|+l)\right| \leqq m^{-1}\sum_{l=-\infty}^{\infty}|q_{NG}(|k|,l,|k|+l)|,$$

which is finite, by assumption. Furthermore, it follows by the same assumption, that $\lim_{k\to\infty}\sum_{l=-\infty}^{\infty}|q_{NG}(|k|,l,|k|+l)| = 0$. So, the theorem would hold provided that $\hat{r}(k)$ were replaced by $\hat{r}'(k)$.

Now, we evaluate the term $E(\hat{r}(k)-\hat{r}'(k))^2$. By definition, recalling (2.13),

$$\hat{r}(k)-\hat{r}'(k) = \bar{X}^2 - \bar{X}(\bar{X}'+\bar{X}''). \tag{2.17}$$

Consider the second term on the right side of (2.17). Clearly

$$\begin{aligned} E(\bar{X}\bar{X}')^2 &= n^{-2}m^{-2}\textstyle\sum_{i,j=1}^{n}\sum_{k,l=1}^{m} EX(i)X(j)X(k)X(l) \\ &= n^{-2}m^{-2}\textstyle\sum_{i,j=1}^{n}\sum_{k,l=1}^{m}(q_G(j-i,k-i,l-i)+q_{NG}(j-i,k-i,l-i)). \end{aligned}$$

But, recalling (2.2), it follows that

$$n^{-2}m^{-2}\textstyle\sum_{i,j=1}^{n}\sum_{k,l=1}^{m} q_G(j-i,k-i,l-i) \leqq 3(m^{-1}\sum_{l=-m}^{m}|r(l)|)^2.$$

So, $E(\bar{X}\bar{X}')^2 \leqq 3(m^{-1}\sum_{k=-m}^{m}|r(k)|)^2 + m^{-3}\sum_{k,l,m=-\infty}^{\infty}|q_{NG}(k,l,m)|$. By identical reasoning it follows that the same inequality holds for $E(\bar{X}\bar{X}'')^2$ and $E\bar{X}^2$. By the Schwarz inequality, the cross-terms are similarly bounded and $E(\hat{r}(k)-\hat{r}'(k))^2 \leqq 27(m^{-1}\sum_{k=-m}^{m}|r(k)|)^2 + 9m^{-3}\sum_{k,l,m=-\infty}^{\infty}|q_{NG}(k,l,m)|$, which depends only on m. By the condition (2.3), with $l = 1$, it follows that $\lim_{m\to\infty} mE(\hat{r}(k)-\hat{r}'(k))^2 = 0$, uniformly for all k. Recalling the Schwarz inequality and the fact that the theorem has already been proved for $\hat{r}'(k)$, the result (2.5) follows, and the theorem is proved.

THEOREM 2.2. *Under the conditions of Theorem* 2.1,

$$0 \leqq |E\hat{r}(k)-r(k)| \leqq 3m^{-1}R^1,$$

where $m = n-|k|$, *and* R^1 *is given by* (2.4).

PROOF. Recalling (2.13) and (2.14), $E\hat{r}(k)-r(k) = E(\bar{X}^2-\bar{X}(\bar{X}'+\bar{X}''))$, where $\bar{X}, \bar{X}'$, and $\bar{X}''$ are given respectively by (1.7), (2.10) and (2.11), since $E\hat{r}(k)-r(k) = 0$. First, let us consider the term $E\bar{X}\bar{X}^1$. Clearly $E\bar{X}\bar{X} \leqq n^{-1}m^{-1}\sum_{i=1,j=1}^{n}{}^{m} EX(i)X(j)$. So $|E\bar{X}\bar{X}'| \leqq m^{-1}R^1$. By the same reasoning, this inequality is valid for the other two terms as well. The theorem is proved.

THEOREM 2.3. *Among all estimators of the form* (1.8), *the one which minimizes the mean square error* $I^2(n)$ *is the one for which*

$$a(k,n) = r(k)E\hat{r}(k)/E\hat{r}^2(k). \tag{2.18}$$

In this case

$$I^2(n) = I^2_{\min}(n) \equiv \sum_{k=-(n-1)}^{i-1} r^2(k)\operatorname{Var}\hat{r}(k)/E\hat{r}^2(k),$$

where, of course, $I^2_{\min}(n)$ *is the smallest possible value of* $I^2(n)$.

The proof is direct. The terms $E(a(k,n)\hat{r}(k)-r(k))^2$ are individually minimized by differentiation and equating to zero. When the mean is known, assuming without loss of generality that it is zero, $\hat{r}(k)$ is replaced by $\hat{r}'(k)$, as defined in (2.14). Then $E\hat{r}(k) = r(k)$ and the result of Theorem 2.3 would correspond to the result given by Lomnicki and Zaremba [3].

THEOREM 2.4. *Let* $r(k)$ *be a covariance sequence, which is non-zero for an infinite number of integers* k. *Assume that*

$$\lim_{k\to\infty} k\log(r^2(k+1)/r^2(k)) = -\infty. \tag{2.19}$$

Then there exists an integer k^* *which is such that* $r^2(k)$ *is nonincreasing for all* $k \geqq k^*$. *Let* $t(n)$ *be the smallest integer greater than or equal to* k^*+1, *which is such that* $r^2(k) \leqq n^{-1}$. *Then* $I^2_{\min}(n) \sim 2n^{-1}t(n)R^2$, *as* $n\to\infty$, *where* R^2, *given by* (2.4), *is finite.*

Before proving the theorem, two lemmas are presented.

LEMMA 2.1. *Under the condition* (2.19) *of Theorem* 2.4, *for any* $\varepsilon > 0$ *there exists a* k_0 *which is such that if* $k_0 \leqq k_1 < k_2$, $r^2(k_2)/r^2(k_1) \leqq (k_2/k_1)^{-1/\varepsilon}$. *It follows immediately from this, that* $r^2(k)$ *is nonincreasing for all sufficiently large* k.

PROOF. By assumption (2.19), for any $\varepsilon > 0$ there exists a k_0 which is such that for any $k \geqq k_0$, $k\log(r^2(k+1)/r^2(k)) < -1/\varepsilon$. Equivalently, $(r^2(k+1)/r^2(k)) \leqq \exp(-\varepsilon^{-1}k^{-1})$. So for any k_1, k_2 such that $k_0 \leqq k_1 < k_2 < \infty$,

$$\begin{aligned}(r^2(k_2)/r^2(k_1)) &\leqq \exp\{-\varepsilon^{-1}(k_1{}^{-1}+(k_1+1)^{-1}+\cdots+(k_2-1)^{-1})\}\\ &\leqq \exp(-\varepsilon^{-1}\textstyle\int_{k_1}^{k_2} dt/t) = (k_2/k_1)^{-1/\varepsilon}.\end{aligned}$$

The lemma is proved.

LEMMA 2.2. *Under the conditions of Theorem* 2.4,

$$\lim_{n\to\infty} t(n) = \infty, \tag{2.20}$$

and for any $\varepsilon > 0$,

$$\lim_{n\to\infty} n^{-\varepsilon}t(n) = 0, \tag{2.21}$$

where $t(n)$ *is as defined in the statement of Theorem* 2.4.

PROOF. Recall that by Lemma 2.2, $r^2(k)$ is nonincreasing for all k greater than or equal to some k^*. Recall, also, that it is assumed that infinitely many of the covariances are non-zero. It follows from this that for any positive integer $l > k^*$, $\min_{k^* \leqq |k| \leqq l} r^2(k) > 0$. For any $n > 1/\min_{k^* \leqq |k| \leqq l} r^2(k)$, clearly, $t(n) \geqq l$. Thus (2.20) follows.

Now, we prove (2.21). For any $\varepsilon > 0$, by Lemma 2.1 there exists a finite positive constant C_1, such that $r^2(k) \leqq C_1 k^{-1/\varepsilon}$, for all sufficiently large k. By definition $n^{-1} \leqq r^2(t(n)-1) \leqq C_1(t(n)-1)^{-1/\varepsilon}$. Equivalently, $t(n) \leqq 1+(C_1 n)^\varepsilon$. The result (2.21) is proven, since ε was arbitrary.

PROOF OF THEOREM 2.4. The first part of the theorem follows from Lemma 2.1. We proceed to prove the second part. Let $\varepsilon > 0$ be arbitrarily chosen. By Theorem 2.3,

$$I^2_{\min}(n) = \sum_{l=1}^4 A(l,n), \tag{2.22}$$

where

$$A(1,n) \equiv \sum_{k=-[t(n)\varepsilon]}^{[t(n)\varepsilon]} r^2(k) \operatorname{Var} \hat{r}(k)/E\hat{r}^2(k),$$

$$A(2,n) \equiv 2\sum_{k=[t(n)\varepsilon]+1}^{[t(n)(1-\varepsilon)]} r^2(k) \operatorname{Var} \hat{r}(k)/E\hat{r}^2(k),$$

$$A(3,n) \equiv 2\sum_{k=[t(n)(1-\varepsilon)]+1}^{[t(n)(1+\varepsilon)]} r^2(k) \operatorname{Var} \hat{r}(k)/E\hat{r}^2(k),$$

and

$$A(4,n) \equiv 2\sum_{k=[t(n)(1+\varepsilon)]+1}^{n-1} r^2(k) \operatorname{Var} \hat{r}(k)/E\hat{r}^2(k).$$

Let us consider the individual summands. Clearly

$$\begin{aligned} \operatorname{Var} \hat{r}(k) &= E(\hat{r}(k)-r(k))^2-(E\hat{r}(k)-r(k))^2 \\ &\leqq m^{-1}(A(m)+B(k))+9m^{-2}(R^1)^2, \end{aligned} \tag{2.23}$$

where $m = n-|k|$, by Theorems 2.1 and 2.2, where $A(m)$ and $B(k)$ satisfy (2.6), (2.7), and (2.8). Similarly

$$\operatorname{Var} \hat{r}(k) \geqq m^{-1}(A(m)+B(k))-9m^{-2}(R^1)^2. \tag{2.24}$$

Also,

$$\begin{aligned} E\hat{r}^2(k) &= E(r(k)+(\hat{r}(k)-r(k)))^2 \\ &= r^2(k)+E(\hat{r}(k)-r(k))^2+2r(k)E(\hat{r}(k)-r(k)) \\ &\leqq r^2(k)+m^{-1}(A(m)+B(k))+6m^{-1}|r(k)|R^1. \end{aligned} \tag{2.25}$$

Similarly,

$$E\hat{r}^2(k) \geqq r^2(k)+m^{-1}(A(m)+B(k))-6m^{-1}|r(k)|R^1. \tag{2.26}$$

Let us consider the term $A(2,n)$. By Theorem 2.3 and Lemma 2.2, for sufficiently large n, $[t(n)\varepsilon] > k^*$, where k^* is defined in the statement of Theorem 2.3. So uniformly for all k, $[t(n)\varepsilon] < k \leqq [t(n)(1-\varepsilon)]$, by the condition (2.19) and Lemma 2.1, $r^2(k)$ becomes large in comparison to n^{-1}, equivalently m^{-1}. Furthermore $\operatorname{Var} \hat{r}(k) \sim m^{-1}R^2 \sim n^{-1}R^2$, as $n \to \infty$, uniformly for all such k. Therefore,

$$A(2,n) \sim n^{-1}t(n)(1-2\varepsilon)R^2, \qquad \text{as} \quad n \to \infty. \tag{2.27}$$

Now consider the term $A(4,n)$. By the condition (2.19) and Lemma 2.2, $r^2(k)$ becomes small with respect to $E\hat{r}^2(k)$ as $n \to \infty$, uniformly for all $k \geqq [t(n)(1+\varepsilon)]$. Therefore $A(4,n) \sim \sum_{k=[t(n)(1+\varepsilon)]+1}^{n-1} r^2(k)$, as $n \to \infty$. But by definition, for sufficiently large n,

$$\begin{aligned}\sum_{k=[t(n)(1+\varepsilon)]}^{n-1} r^2(k) &\leqq \sum_{k=t(n)+1}^{\infty} r^2(k) \leqq r^2(t(n)) \sum_{k=t(n)+1}^{\infty} (r^2(k)/r^2(t(n))) \\ &\leqq n^{-1} \sum_{k=t(n)+1}^{\infty} (r^2(k)/r^2(t(n))) \leqq n^{-1} \sum_{k=t(n)+1}^{\infty} (k/t(n))^{-1/\varepsilon} \\ &\leqq n^{-1} \int_{t(n)}^{\infty} (s/t(n))^{-1/\varepsilon}\, ds = n^{-1}t(n) \int_1^{\infty} u^{-1/\varepsilon}\, du \\ &= n^{-1}t(n)\varepsilon(1-\varepsilon)^{-1}.\end{aligned}$$

That is

$$A(4,n) \leqq n^{-1}t(n)\varepsilon(1-\varepsilon)^{-1}(1+o(1)), \qquad \text{as} \quad n \to \infty. \tag{2.28}$$

Each term $r^2(k)\operatorname{Var}\hat{r}(k)/E\hat{r}^2(k)$, is of the form $E(A\hat{r}(k)-r(k))^2$ where A is chosen so as to minimize it. Clearly, then, it is at least as small as it would be for $A = 1$. So $r^2(k)\operatorname{Var}\hat{r}(k)/E\hat{r}^2(k) \leqq E(\hat{r}(k)-r(k))^2 \leqq m^{-1}(R^2+C_1)$, where $C_1 = \sup_{k\geqq 1} B(k)$. Thus $A(1,n)+A(3,n) \leqq 4n^{-1}t(n)(R^2+C_1)\varepsilon(1+o(1))$, as $n \to \infty$. Combining this with (2.22), (2.27), and (2.28), and recalling the fact that ε was arbitrarily chosen, the theorem is proved.

THEOREM 2.5. *Assume that the conditions of Theorem* 2.4 *hold. Let* $\hat{f}^*(\omega)$ *be given by* (1.9), *where* $t(n)$ *is as defined in the statement of Theorem* 2.4. *Then* $\hat{f}^*(\omega)$ *has efficiency* 1, *in the sense that*

$$I^2(n) \sim 2n^{-1}t(n)R^2,$$

as $n \to \infty$, *where* $I^2(n)$ *is given by* (1.5), *for* $\hat{f}^*(\omega)$, *and* R^2 *if given by* (2.4).

PROOF. By definition,

$$\begin{aligned} I^2(n) &\equiv (2\pi)E\int_{-\pi}^{\pi} (\hat{f}^*(\omega)-f(\omega))^2\, d\omega \\ &= \sum_{k=-t(n)}^{t(n)} E((1-n^{-1}|k|)\hat{r}(k)-r(k))^2 + 2\sum_{k=t(n)+1}^{n-1} r^2(k).\end{aligned}$$

But

$$\begin{aligned} &\sum_{k=-t(n)}^{t(n)} E((1-n^{-1}|k|)\hat{r}(k)-r(k))^2 \\ &= \sum_{k=-t(n)}^{t(n)} E((1-n^{-1}|k|)(\hat{r}(k)-r(k))+r(k)((1-n^{-1}|k|)-1))^2 \\ &\doteq \sum_{k=-t(n)}^{t(n)} (1-n^{-1}|k|)^2 E(\hat{r}(k)-r(k))^2 + \sum_{k=-t(n)}^{t(n)} n^{-2}|k|^2 r^2(k) \\ &\quad -2\sum_{k=-t(n)}^{t(n)} n^{-1}|k|(1-n^{-1}|k|)r(k)E(\hat{r}(k)-r(k)).\end{aligned} \tag{2.29}$$

Observe that by the condition (2.19) and Lemma 2.1, it follows that $R^1 < \infty$, where R^1 is given by (2.4) with $l = 1$. So, we can apply Theorem 2.1. Consider the first term on the right side of (2.29). Clearly $\sum_{k=-t(n)}^{t(n)} (1-n^{-1}|k|)^2 E(\hat{r}(k)-r(k))^2 \leqq \sum_{k=-t(n)}^{t(n)} A(n-|k|)/(n-|k|) + \sum_{k=-t(n)}^{t(n)} B(k)/(n-|k|) \sim n^{-1}(2t(n)+1)R^2 + o(n^{-1}t(n))$ as $n \to \infty$. Furthermore, recalling Lemma 2.2, $\sum_{k=-t(n)}^{t(n)} (1-n^{-1}|k|)^2 E(\hat{r}(k)-r(k))^2 \geqq (1-n^{-1}t(n))^2 \sum_{k=-t(n)}^{t(n)} A(n-|k|)/(n-|k|) + \sum_{k=-t(n)}^{t(n)} B(k)/(n-|k|) \sim n^{-1}(2t(n)+1)R^2$

$+o(n^{-1}t(n))$, as $n\to\infty$. So $\sum_{k=-t(n)}^{t(n)}(1-n^{-1}|k|)^2E(\hat r(k)-r(k))^2\sim 2n^{-1}t(n)R^2$, as $n\to\infty$. Let us now consider the second term on the right side of (2.29). Clearly $\sum_{k=-t(n)}^{t(n)}n^{-2}|k|^2\,r^2(k)\leqq n^{-2}t^2(n)R^2=o(n^{-1}t(n))$, as $n\to\infty$. Now, consider the third term. Before doing so, however, we should examine the term $E(\hat r(k)-r(k))$. But by Theorem 2.2, $|E(\hat r(k)-r(k))|\leqq 3(n-|k|)^{-1}R^1$. Returning to the third term on the rightmost side of (2.29), $|\sum_{k=-t(n)}^{t(n)}n^{-1}|k|(1-n^{-1}|k|)r(k)E(\hat r(k)-r(k))|\leqq 2n^{-2}t(n)(R^1)^2(1-n^{-1}t(n))^{-1}=o(n^{-1}t(n))$, as $n\to\infty$. But, for any $\varepsilon>0$, $\sum_{k=t(n)+1}^{n-1}r^2(k)\leqq r^2(t(n))\sum_{k=t(n)+1}^{\infty}(r^2(k)/r^2(t(n)))\leqq n^{-1}\sum_{k=t(n)+1}^{\infty}(k/t(n))^{-1/\varepsilon}\leqq n^{-1}\int_{t(n)}^{\infty}(s/t(n))^{-1/\varepsilon}ds=n^{-1}t(n)\int_1^\infty u^{-1/\varepsilon}du=n^{-1}t(n)\varepsilon(1-\varepsilon)^{-1}$, as $n\to\infty$. But ε was arbitrarily chosen. The theorem is proven.

THEOREM 2.6. *Let $\hat f_i(\omega)$, $i=1,2$, be two estimators which are efficient in the sense that*

$$2\pi E\int_{-\pi}^{\pi}(\hat f_i(\omega)-f(\omega))^2\,d\omega/I_{\min}^2(n)\to 1,$$

as $n\to\infty$, $i=1,2$. Then

$$2\pi E\int_{-\pi}^{\pi}(\hat f_1(\omega)-f(\omega))(\hat f_2(\omega)-f(\omega))\,d\omega/I_{\min}^2(n)\to 1, \tag{2.30}$$

as $n\to\infty$, and

$$2\pi E\int_{-\pi}^{\pi}(\hat f_1(\omega)-\hat f_2(\omega))^2\,d\omega/I_{\min}^2(n)\to 0,\qquad \textit{as } n\to\infty. \tag{2.31}$$

This theorem obviously applies when $\hat f_1(\omega)$ is given by (1.8), where $a(k,n)$ is given by (2.18), and $\hat f_2(\omega)=\hat f^*(\omega)$, given by (1.9). The reasoning in the proof is very similar to that used in proving the analogous result relating two efficient estimators of a vector-valued parameter.

PROOF. Let us define the estimator $\hat f_3(\omega)\equiv\frac12(\hat f_1(\omega)+\hat f_2(\omega))$. Then $E\int_{-\pi}^{\pi}(\hat f_3(\omega)-f(\omega))^2\,d\omega=\frac14E\int_{-\pi}^{\pi}(\hat f_1(\omega)-f(\omega))^2\,d\omega+\frac14E\int_{-\pi}^{\pi}(\hat f_2(\omega)-f(\omega))^2d\omega+\frac12E\int_{-\pi}^{\pi}(\hat f_1(\omega)-f(\omega))(\hat f_2(\omega)-f(\omega))\,d\omega$. Multiplying throughout by $2\pi/I_{\min}^2(n)$, it follows that $\liminf_{n\to\infty}2\pi E\int_{-\pi}^{\pi}(\hat f_1(\omega)-f(\omega))(\hat f_2(\omega)-f(\omega))\,d\omega/I_{\min}^2(n)\geqq 1$. That the lim sup is $\leqq 1$ follows by the Schwarz inequality and the definition of $I_{\min}^2(n)$. So (2.30) holds. Now $E\int_{-\pi}^{\pi}(\hat f_1(\omega)-\hat f_2(\omega))^2\,d\omega=E\int_{-\pi}^{\pi}(\hat f_1(\omega)-f(\omega))^2\,d\omega+E\int_{-\pi}^{\pi}(\hat f_2(\omega)-f(\omega))^2\,d\omega-2E\int_{-\pi}^{\pi}(\hat f_1(\omega)-f(\omega))(\hat f_2(\omega)-f(\omega))\,d\omega$. Multiplying by $2\pi/I_{\min}^2(n)$, (2.31) follows. The theorem is proved.

3. Estimators with random truncation. We begin by proving the following theorem.

THEOREM 3.1. *Let $\{X(n),1\leqq n<\infty\}$ be a time series with finite moments of order four, stationary to order four with covariance function $r(k)$ satisfying the conditions of Theorem 2.4. Let $\hat t(n)$ be a function of the observations, which is such that*

$$\lim_{n\to\infty}P\{\hat t(n)\leqq t(n)\}=0,\qquad \textit{and} \tag{3.1}$$

$$\lim_{n\to\infty}P\{\hat t(n)\geqq t(n)(1+\varepsilon)\}=0, \tag{3.2}$$

for any $\varepsilon > 0$, *where* $t(n)$ *is as defined in the statement of Theorem* 2.4. *Let* $\hat{f}(\omega) \equiv (2\pi)^{-1} \sum_{k=-\hat{t}(n)}^{\hat{t}(n)} (1-n^{-1}|k|)\hat{r}(k)\, e^{ik\omega}$. *Then*

$$\sup_{-\pi \leqq \omega \leqq \pi} |\hat{f}(\omega)-\hat{f}^*(\omega)|/I_{\min}(n) \to 0, \tag{3.3}$$

in probability, as $n \to \infty$, *where* $\hat{f}^*(\omega)$ *is given by* (1.9) *and* $I_{\min}^2(n)$ *is the minimum mean squared error, the minimum taken among all estimators of the form* (1.8).

PROOF. Let

$$A(n) \equiv (2\pi)^2 \sup_{-\pi \leqq \omega \leqq \pi} (\hat{f}(\omega)-\hat{f}^*(\omega))^2, \tag{3.4}$$

and for every $\varepsilon > 0$, let $A(\varepsilon, n) \equiv A(n)$, if $t(n) < \hat{t}(n) \leqq t(n)(1+\varepsilon)$, 0 otherwise. By definition (1.9), $A(\varepsilon, n) \leqq \sum_{k=t(n)+1}^{[t(n)(1+\varepsilon)]} (1-n^{-1}|k|)^2 \hat{r}^2(k) \leqq \sum_{k=t(n)+1}^{[t(n)(1+\varepsilon)]} \hat{r}^2(k)$, where $[x]$ is the greatest integer less than or equal to x. By the Schwarz inequality,

$$\begin{aligned} EA(\varepsilon, n) \leqq \sum_{k=t(n)+1}^{[t(n)(1+\varepsilon)]} r^2(k) &+ 2\sum_{k=t(n)+1}^{[t(n)(1+\varepsilon)]} r(k)(E(\hat{r}(k)-r(k))^2)^{\frac{1}{2}} \\ &+ \sum_{k=t(n)+1}^{[t(n)(1+\varepsilon)]} E(\hat{r}(k)-r(k))^2. \end{aligned} \tag{3.5}$$

Let us consider the first term on the right side of (3.5). Clearly $\sum_{k=t(n)+1}^{[t(n)(1+\varepsilon)]} r^2(k) \leqq ([t(n)\varepsilon]+1)r^2(t(n)+1) \leqq n^{-1}([t(n)\varepsilon]+1) \sim n^{-1}t(n)\varepsilon$, as $n \to \infty$, by the definition of $t(n)$. Now, let us consider the third term on the right side of (3.5). By Theorem 2.1, $\sum_{k=t(n)+1}^{[t(n)(1+\varepsilon)]} E(\hat{r}(k)-\hat{r}(k))^2 \leqq \sum_{k=t(n)+1}^{[t(n)(1+\varepsilon)]} A(n-|k|)/(n-|k|)+\sum_{k=t(n)+1}^{[t(n)(1+\varepsilon)]} B^*(k)/(n-|k|)$,

where

$$B^*(k) \equiv \sup_{l \geqq k} B(l), \tag{3.6}$$

and $B(k)$ satisfies (2.8). From this, it follows that $\lim_{k\to\infty} B^*(k) = 0$. So $\sum_{k=t(n)+1}^{[t(n)(1+\varepsilon)]} B(k)/(n-|k|) = o(n^{-1}t(n))$ as $n \to \infty$. Furthermore by (2.7) of Theorem 2.1,

$$\begin{aligned} &\sum_{k=t(n)+1}^{[t(n)(1+\varepsilon)]} A(n-|k|)/(n-|k|) \\ &\qquad \leqq n^{-1}([t(n)\varepsilon]+1)(1-n^{-1}[t(n)(1+\varepsilon)])R^2 \sim n^{-1}t(n)\varepsilon R^2, \end{aligned} \tag{3.7}$$

as $n \to \infty$, by Lemma 2.2. So, it follows that

$$\sum_{k=t(n)+1}^{[t(n)(1+\varepsilon)]} E(\hat{r}(k)-r(k))^2 \leqq n^{-1}t(n)\varepsilon R^2(1+o(1)), \tag{3.8}$$

as $n \to \infty$. Now consider the middle term on the right side of (3.5). By the Schwarz inequality, the term is dominated by the square root of the product of the other two terms. That is

$$\sum_{k=t(n)+1}^{[t(n)(1+\varepsilon)]} r(k)(E(\hat{r}(k)-r(k))^2)^{\frac{1}{2}} \leqq n^{-1}t(n)(R^2)^{\frac{1}{2}}\varepsilon(1+o(1)), \tag{3.9}$$

as $n \to \infty$. Recalling (3.5), (3.7), (3.8), and (3.9), it follows that $EA(\varepsilon, n) \leqq n^{-1}t(n)\varepsilon(1+(R^2)^{\frac{1}{2}})^2(1+o(1))$, as $n \to \infty$. By Theorem 2.3, then $EA(\varepsilon, n)/I_{\min}^2(n) \leqq \varepsilon(1+(R^2)^{\frac{1}{2}})^2(1+o(1))$, as $n \to \infty$. So by the Markov inequality, $P\{A(\varepsilon, n)/I_{\min}^2(n) > x\} \leqq (\varepsilon(1+(R^2)^{\frac{1}{2}})^2/x)(1+o(1))$ as $n \to \infty$, for any $x, 0 < x < \infty$. But by the assumptions (3.1) and (3.2) it follows that for any $\varepsilon > 0$, $\lim_{n\to\infty} P\{A(n) \neq A(\varepsilon, n)\} = 0$. Since ε was arbitrarily chosen, the result (3.3) follows.

COROLLARY 3.1. *Theorem* 3.1 *would still be valid if $\hat{f}^*(\omega)$ were replaced by $\hat{f}(\omega)$, given by* (1.8), *where $a(k, n)$ is given by* (2.18).

The proof follows by combining the results of Theorems 2.4, 2.5, 2.6, and 3.1.

COROLLARY 3.2. *Under the conditions of Theorem* 3.1, *for any real c, $0 < c < \infty$,*

$$\limsup_{n\to\infty} E\{\min c, 2\pi \textstyle\int_{-\pi}^{\pi} (\hat{f}(\omega) - f(\omega))^2 \, d\omega / I_{\min}^2(n)\} \leqq 1, \tag{3.10}$$

where $I_{\min}^2(n)$ is as defined in Section 1, *and in the statement of Theorem* 2.3.

PROOF: Clearly

$$\begin{aligned} E\textstyle\int_{-\pi}^{\pi} (\hat{f}(\omega) - f(\omega))^2 \, d\omega &= E\textstyle\int_{-\pi}^{\pi} (\hat{f}^*(\omega) - f(\omega))^2 \, d\omega \\ &\quad + E\textstyle\int_{-\pi}^{\pi} (\hat{f}(\omega) - \hat{f}^*(\omega))^2 \, d\omega + 2E\textstyle\int_{-\pi}^{\pi} (\hat{f}^*(\omega) - f(\omega))(\hat{f}(\omega) - \hat{f}^*(\omega)) \, d\omega, \end{aligned} \tag{3.11}$$

where $\hat{f}^*(\omega)$ is given by (1.9). By Theorem 2.5,

$$2\pi E\textstyle\int_{-\pi}^{\pi} (\hat{f}^*(\omega) - f(\omega))^2 \, d\omega / I_{\min}^2(n) \to 1, \tag{3.12}$$

as $n \to \infty$. Let c, $0 < c < \infty$, be arbitrarily chosen. Then clearly, by Theorem 3.1,

$$\limsup_{n\to\infty} E\{\min (c, 2\pi \textstyle\int_{-\pi}^{\pi} (\hat{f}(\omega) - \hat{f}^*(\omega))^2 \, d\omega / I_{\min}^2(n)\} = 0, \tag{3.13}$$

since for a sequence of uniformly bounded positive random variables convergence in probability implies convergence in expectation. Now consider the third term on the right side of (3.11). By the Schwarz inequality,

$$\begin{aligned} &|2\pi \textstyle\int_{-\pi}^{\pi} (\hat{f}^*(\omega) - f(\omega))(\hat{f}(\omega) - \hat{f}^*(\omega)) \, d\omega / I_{\min}^2(n)| \\ &\qquad \leqq (2\pi \textstyle\int_{-\pi}^{\pi} (\hat{f}^*(\omega) - f(\omega))^2 \, d\omega / I_{\min}^2(n))^{\frac{1}{2}} \\ &\qquad \cdot (2\pi \textstyle\int_{-\pi}^{\pi} (\hat{f}(\omega) - \hat{f}^*(\omega))^2 \, d\omega / I_{\min}^2(n))^{\frac{1}{2}}. \end{aligned} \tag{3.14}$$

Let $\varepsilon_1, \varepsilon_2 > 0$ be arbitrarily chosen. But $P\{2\pi \int_{-\pi}^{\pi} (\hat{f}^*(\omega) - f(\omega))^2 \, d\omega / I_{\min}^2(n) > 2/\varepsilon_1\} \leqq (\varepsilon_1/2) + o(1)$, as $n \to \infty$, by the Markov inequality. Equivalently

$$P\{(2\pi \textstyle\int_{-\pi}^{\pi} (\hat{f}^*(\omega) - f(\omega))^2 \, d\omega)^{\frac{1}{2}} / I_{\min}^2(n) > (2/\varepsilon_1)^{\frac{1}{2}}\} \leqq (\tfrac{1}{2}\varepsilon_1) + o(1), \qquad \text{as } n \to \infty.$$

For all sufficiently large n, by Theorem 2.5, $P\{(2\pi \int_{-\pi}^{\pi} (\hat{f}(\omega) - \hat{f}^*(\omega))^2 \, d\omega)^{\frac{1}{2}} / I_{\min}(n) > \varepsilon_2(\frac{1}{2}\varepsilon_1)^{\frac{1}{2}}\} \leqq \frac{1}{2}\varepsilon_1$. In order that the term on the right side of (3.14) be greater than ε_2, it is necessary that either $(2\pi \int_{-\pi}^{\pi} (\hat{f}^*(\omega) - f(\omega))^2 \, d\omega)^{\frac{1}{2}} / I_{\min}(n) > (2/\varepsilon_1)^{\frac{1}{2}}$ or $(2\pi \int_{-\pi}^{\pi} (\hat{f}(\omega) - \hat{f}^*(\omega))^2 \, d\omega)^{\frac{1}{2}} / I_{\min}(n) > \varepsilon_2(\frac{1}{2}\varepsilon_1)^{\frac{1}{2}}$. So

$$\begin{aligned} &P\{2\pi \textstyle\int_{-\pi}^{\pi} (\hat{f}^*(\omega) - f(\omega))(\hat{f}(\omega) - \hat{f}^*(\omega)) \, d\omega / I_{\min}^2(n) > \varepsilon_2\} \\ &\qquad \leqq P\{(2\pi \textstyle\int_{-\pi}^{\pi} (\hat{f}^*(\omega) - f(\omega))^2 \, d\omega)^{\frac{1}{2}} / I_{\min}(n) > (2/\varepsilon_1)^{\frac{1}{2}}\} \\ &\qquad\quad + P\{(2\pi \textstyle\int_{-\pi}^{\pi} (\hat{f}(\omega) - \hat{f}^*(\omega))^2 \, d\omega)^{\frac{1}{2}} / I_{\min}(n) > \varepsilon_2(\tfrac{1}{2}\varepsilon_1)^{\frac{1}{2}}\} \\ &\qquad = \tfrac{1}{2}\varepsilon_1 + \tfrac{1}{2}\varepsilon_1 + o(1) = \varepsilon_1 + o(1). \end{aligned}$$

Since ε_1 and ε_2 were arbitrarily chosen, it follows that the term on the left side of (3.14) converges to zero in probability. Let us again recall the fact that for a uniformly bounded sequence of random variables convergence in probability implies convergence in expectation. Thus

(3.15) $\limsup_{n\to\infty} E\{\min c, 2\pi \int_{-\pi}^{\pi} (\hat{f}^*(\omega)-f(\omega))(\hat{f}(\omega)-\hat{f}^*(\omega))\, d\omega / I_{\min}^2(n)\} = 0.$

Recalling (3.11) through (3.15) then the result (3.10) follows. The corollary is proved.

THEOREM 3.2. *Assume that the covariance function* $r(k)$ *is such that for some* $C, \gamma, 0 < C, \gamma < \infty$,

$$-\log r^2(k) \sim Ck^\gamma, \qquad \text{and} \tag{3.16}$$

$$-\log(r^2(k+1)/r^2(k)) \sim C\gamma k^{\gamma-1}, \tag{3.17}$$

as $k \to \infty$. *Then* $\hat{t}(n)$ *satisfies the conditions of Theorem* 3.1, *provided that it is defined as follows. We let*

$$\hat{t}(n) \equiv [(\hat{s}(n)+2)(1+d(n))]+1,$$

where $[x]$ *is defined to be the greatest integer, less than or equal to* x, $\hat{s}(n)$ *is the number of integers from* 0 *to* $a(n)$ *which are smaller than* k^*, *or are such that* $\hat{r}^2(k) \geqq a(n)/n$, *the integer* k^*, *as defined in the statement of Theorem* 2.4, *is the smallest integer which is such that* $r^2(k)$ *is nonincreasing for all* $k \geqq k^*$ *and* $a(n)$ *and* $d(n)$ *are functions which satisfy the following conditions. The function* $a(n)$ *is integer valued and is such that*

$$\lim_{n\to\infty} \log a(n)/\log n = 0, \tag{3.18}$$

and for every $\alpha, 0 < \alpha < \infty$,

$$\lim_{n\to\infty} a(n)/(\log n)^\alpha = \infty. \tag{3.19}$$

The function $d(n)$ *is such that for some* $\beta > 1$,

$$\lim_{n\to\infty} d(n)\log n/(\log\log n)^\beta = \infty, \tag{3.20}$$

$$\lim_{n\to\infty} d(n)\log n/\log a(n) = \infty, \qquad \text{and} \tag{3.21}$$

$$\lim_{n\to\infty} d(n) = 0. \tag{3.22}$$

The conditions of Theorem 3.2 are satisfied if, for example

$$a(n) \equiv [\exp(\log n)^{\theta_1}]+1, \quad \text{and} \quad d(n) \equiv (\log n)^{\theta_2-1},$$

where $0 < \theta_1 < \theta_2 < 1$.

Clearly, the condition (2.19) is satisfied if (3.16) and (3.17) hold, for some $\gamma, 0 < \gamma < \infty$.

Before proving the theorem, several lemmas are given.

LEMMA 3.1. *Under the condition* (3.16),

$$t(n) \sim C^{-1/\gamma}(\log n)^{1/\gamma}, \qquad \text{as } n \to \infty. \tag{3.23}$$

PROOF. Let $\varepsilon > 0$ be arbitrarily chosen. By the condition (3.16), it follows that for all sufficiently large n, $-\log r^2(t(n)) \leqq (C+\varepsilon)(t(n))^\gamma$. By the definition of $t(n)$ as given in the statement of Theorem 2.4, it follows that $-\log r^2(t(n)) \geqq \log n$. Combining the two inequalities, $t(n) \geqq (C+\varepsilon)^{-1/\gamma}(\log n)^{1/\gamma}$ for all sufficiently large n. By similar reasoning $-\log r^2(t(n)-1) \geqq (C-\varepsilon)(t(n)-1)^\gamma$, and $-\log r^2(t(n)-1) \leqq \log n$. So $t(n) \leqq (C-\varepsilon)^{-1/\gamma}(\log n)^{1/\gamma}+1$, for all sufficiently large n. Since ε was arbitrarily chosen, the result (3.23) follows. The lemma is proved.

LEMMA 3.2. *Let $a(n)$ be a sequence which is such that* (3.18) *and* (3.19) *hold. Let k^* be defined as in Theorem* 2.4. *Let $s(n)$ be the smallest integer k, which is such that $k \geqq k^*$ and $r^2(k) \leqq n^{-1}a(n)$. Then, if the conditions of Theorem* 3.2 *are satisfied,*

$$s(n) \sim t(n), \tag{3.24}$$

as $n \to \infty$. Furthermore,

$$(t(n)-s(n)-1)/(t(n)-1) \leqq (1+o(1))\log a(n)/\gamma \log n, \qquad \text{as } n \to \infty. \tag{3.25}$$

PROOF. By exactly the same reasoning as that used in Lemma 3.1, $s(n) \sim C^{-1/\gamma}$ $(\log (n/a(n)))^{1/\gamma} = C^{-1/\gamma}(\log n - \log a(n))^{1/\gamma} = C^{1/\gamma}(\log n)^{1/\gamma}(1-\log a(n)/\log n)^{1/\gamma} \sim$ $t(n)$, as $n \to \infty$, by Lemma 3.1 and the condition (3.18). That is (3.24) holds.

Now, let us establish that (3.25) holds. By definition of $t(n)$, $r^2(t(n)-1) \geqq n^{-1}$, and so $-\log r^2(t(n)-1) \leqq \log n$. By definition of $s(n)$, $r^2(s(n)) \leqq n^{-1}a(n)$, and so $-\log r^2(s(n)) \geqq \log n - \log a(n)$. Thus $-\log r^2(t(n)-1)-(-\log r^2(s(n))) \leqq \log a(n)$. But $-\log r^2(t(n)-1)-(-\log r^2(s(n))) \geqq (t(n)-s(n)-1)C\gamma \min_{s(n) \leqq k \leqq t(n)-1} k^{\gamma-1}$. Therefore, $(t(n)-s(n)-1) \leqq \log a(n)/C\gamma \min_{s(n)\leqq k \leqq t(n)-1} k^{\gamma-1}$. Clearly $C\min_{s(n)\leqq k\leqq t(n)-1} k^{\gamma-1} \geqq C(s(n))^\gamma/(t(n)-1)$. So $(t(n)-s(n)-1)/(t(n)-1) \leqq \log a(n)/C\gamma(s(n))^\gamma \sim \log a(n)/\gamma \log n$, as $n \to \infty$. The Lemma is proved.

LEMMA 3.3. *If the covariance sequence satisfies the conditions* (3.16) *and* (3.17) *and is such that $(s(n)-1)e(n)$ is an integer for every n, and*

$$\lim_{n\to\infty} e(n) = 0, \tag{3.26}$$

then

$$-\log \left|r^2(s(n)-1)/r^2((s(n)-1)(1-e(n)))\right| \sim \gamma e(n)\log n, \qquad \textit{and} \tag{3.27}$$

$$-\log \left|r^2((s(n)-1)(1+e(n)))/r^2(s(n)-1)\right| \sim \gamma e(n)\log n, \qquad \text{as } n \to \infty. \tag{3.28}$$

PROOF. By (3.17), $(s(n)-1)e(n)C\gamma \min_{(s(n)-1)(1-e(n))\leqq k \leqq s(n)-1} k^{\gamma-1} \leqq |-\log r^2(s(n)-1)-(-\log r^2((s(n)-1)(1-e(n))))| \leqq (s(n)-1)e(n)C\gamma \max_{(s(n)-1)(1-e(n))\leqq k\leqq s(n)-1} k^{\gamma-1}$. By (3.26) $|-\log(r^2(s(n)-1)/r^2((s(n)-1)(1-e(n))))| \sim (s(n)-1)e(n)C\gamma(s(n)-1)^{\gamma-1} = e(n)C\gamma(s(n)-1)^\gamma \sim e(n)C\gamma(s(n))^\gamma \sim \gamma e(n)\log(n/a(n)) \sim \gamma e(n)\log n$, as $n \to \infty$. Thus (3.27) is proven. The result (3.28) follows by the same reasoning with appropriate but obvious modifications. The lemma is proved.

LEMMA 3.4. *Under the conditions of Theorem* 3.2 *if $e(n)$ is a sequence, which is such that for every integer n, $(s(n)-1)e(n)$ is an ever integer, and*

(3.29) $$\lim_{n\to\infty} e(n) = 0, \qquad \textit{and}$$

(3.30) $$\liminf_{n\to\infty} (\gamma e(n)\log n + \log e(n)) = \infty, \qquad \textit{then}$$

(3.31) $$\lim_{n\to\infty} P\{\hat{s}(n) \leqq (s(n)-1)(1-e(n))\} = 0.$$

PROOF. Observe that by (3.19) and Lemmas 3.1 and 3.2, for all sufficiently large n,

(3.32) $$a(n) \geqq s(n).$$

From here on it will be assumed that n is sufficiently large so that (3.32) holds. If $\hat{s}(n) \leqq (s(n)-1)(1-e(n))$, then, for all sufficiently large n, $\sum_{k=-a(n)}^{a(n)} (\hat{r}(k)-r(k))^2 \geqq 2\sum_{k=(s(n)-1)(1-e(n))}^{s(n)-1} (r(k)-r(s(n)-1))^2 \geqq 2\sum_{k=(s(n)-1)(1-e(n))}^{(s(n)-1)(1-\frac{1}{2}e(n))} (r(k)-r(s(n)-1))^2 \geqq (s(n)-1)e(n)r^2(s(n)-1)f(n)$, where $f(n) \equiv ((r((s(n)-1)(1-\frac{1}{2}e(n))/r(s(n)-1))-1)^2$. But, by definition, $r^2(s(n)-1) \geqq n^{-1}a(n)$. So $\sum_{k=-a(n)}^{a(n)} (\hat{r}(k)-r(k))^2 \geqq (s(n)-1) e(n)n^{-1}a(n)f(n)$. By the Markov inequality, $P\{\hat{s}(n) \leqq (s(n)-1)(1-e(n))\} \leqq E\sum_{k=-a(n)}^{a(n)} (\hat{r}(k)-r(k))^2/(s(n)-1)e(n)n^{-1}a(n)f(n)$. But, by Theorem 2.1, $E(\sum_{k=-a(n)}^{a(n)} (\hat{r}(k)-r(k))^2) \leqq n^{-1}(2a(n)+1)(R^2+B^*(n-a(n))) \sim n^{-1}(2a(n)+1)R^2$, where R^2 is given by (2.4) and $B^*(k)$ by (3.6). Then $E(\sum_{k=-a(n)}^{a(n)} (\hat{r}(k)-r(k))^2) \leqq n^{-1}(2a(n)+1)R^2(1+o(1))$, as $n\to\infty$, and $P\{s(n) \leqq (s(n)-1)(1-e(n))\} \leqq C_1 n^{-1}(2a(n)+1)R^2/s(n)e(n)n^{-1}a(n)f(n) \to 0$, as $n\to\infty$, provided

(3.33) $$\lim_{n\to\infty} e(n)f(n) = \infty,$$

since, by definition, $\lim_{n\to\infty} s(n) = \infty$. By Lemma 3.3, (3.33) holds provided that (3.30) does. Thus, the result (3.31) is established. The lemma is proved.

LEMMA 3.5. *Under the conditions of Theorem* 3.2, *for any* $\varepsilon > 0$,

(3.34) $$\lim_{n\to\infty} P\{\hat{s}(n) \geqq [(s(n)-1)(1+\varepsilon)] = 0.$$

PROOF. The proof is similar to that for Lemma 2.4. Let $\varepsilon > 0$ be arbitrarily chosen. If $\hat{s}(n) \geqq [(s(n)-1)(1+\varepsilon)]$, then, for all sufficiently large n, $\sum_{k=-a(n)}^{a(n)} (\hat{r}(k)-r(k))^2 \geqq 2\sum_{k=s(n)}^{[(s(n)-1)(1+\varepsilon)]} (\hat{r}(k)-r(k))^2 \geqq 2\sum_{k=[(s(n)-1)(1+\varepsilon/2)]}^{[(s(n)-1)(1+\varepsilon)]} (\hat{r}(k)-r(k))^2 \geqq ([s(n)\varepsilon]+1)n^{-1}a(n)(1+o(1))$, as $n\to\infty$, since by Lemma 2.1, for any $\theta > 1$, $\lim_{k\to\infty} (r^2([k\theta]+1)/r^2(k)) = 0$. By the Markov inequality, then, $P\{\hat{s}(n) \geqq [(s(n)-1)(1+\varepsilon)]\} \leqq 2n^{-1}a(n)R^2(1+o(1))/s(n)\varepsilon n^{-1}a(n) = 2R^2(1+o(1))/s(n)\varepsilon \to 0$, as $n\to\infty$. The lemma is proved.

PROOF OF THEOREM 3.2. Let $e(n)$ be a sequence satisfying the conditions of Lemma 3.4 for all γ, $0 < \gamma < \infty$, then $\hat{s}(n)+2 \geqq (s(n)+1)(1-e(n))$, with probability approaching 1 as $n\to\infty$. Then $\lim_{n\to\infty} P\{\hat{t}(n) \geqq (s(n)+1)(1-e(n))(1+d(n))\} = 1$. But by Lemma 3.2, for any $\varepsilon > 0$ and for all sufficiently large n, $s(n) \geqq (t(n)-1)(1-(1+\varepsilon)\log a(n)/\log n)$. But, by the condition (3.18) it follows that for sufficiently large n, $s(n)+1 \geqq t(n)(1-(1+\varepsilon)\log a(n)/\log n)$. Thus $\hat{t}(n) \geqq t(n)(1-e(n))(1+d(n))(1-(1+\varepsilon)\log a(n)/\log n)$, with probability approaching 1, as $n\to\infty$. So, in order that the condition (3.1) be satisfied, it is sufficient that, for any γ, $0 < \gamma < \infty$,

(3.35) $$\lim_{n\to\infty} d(n)/(e(n)+(1+\varepsilon)\log a(n)/\gamma\log n) = \infty,$$

where $e(n)$ is some sequence satisfying (3.30). Let $e(n) \equiv (\log\log n)^{\frac{1}{2}(1+\beta)}/\log n$, for some $\beta > 1$. But, by the conditions (3.20) and (3.21), (3.35) holds for any γ, $0 < \gamma < \infty$. So, the condition (3.1) of Theorem 3.1 is satisfied.

By Lemma 3.5, in order that the condition (3.2) hold, (3.22) must hold. The theorem is proved.

4. Continuous coordinate time series. Let $\{X(\tau),\ 0 < \tau < \infty\}$ be a continuous coordinate time series with finite moments to order four, and assume that the time series is stationary to order four. The results of Sections 2 and 3 are still valid, provided appropriate but obvious modifications are made. In general, the following exceptions are involved. Sums are replaced by integrals, and integrals involving the spectral density function, and its estimators are taken from $-\infty$, to ∞, rather than from $-\pi$ to π. The inequalities involved in the definition of $s(n)$ and $t(n)$ become equalities. In defining the class of covariance functions, the condition (2.19) is replaced by the condition that

$$\lim_{\tau\to\infty} \tau\partial \log r^2(\tau)/\partial\tau = -\infty,$$

and (3.16) and (3.17) are replaced by the conditions that $-\log r^2(\tau) \sim C\tau^\gamma$, and $-\partial \log r^2(\tau)/\partial\tau \sim C\gamma\tau^{\gamma-1}$, as $\tau \to \infty$. The term $\hat{s}(n)$ is replaced by $\hat{s}(T)$ which is defined using Lebesgue measure instead of summation. The result (1.10) is replaced by the conclusion that $\sup_{-\infty \leqq \omega \leqq \infty} |\hat{f}(\omega) - \hat{f}^*(\omega)|/I_{\min}(T) \to 0$, in probability as $T \to \infty$, where, of course, observation is made over $[0, T]$.

REFERENCES

[1] BAHADUR, R. R. (1964). On Fisher's bound for asymptotic variances. *Ann. Math. Statist.* **35** 1545–1552.

[2] LEPPINK, G. J. (1961). On the estimation of the spectral density function by the periodogram truncated at an estimated point. Ph.D. thesis, Rijksuniversiteit te Utrecht.

[3] LOMNICKI, Z. A. and ZAREMBA, S. K. (1957). On estimating the spectral density function of a stochastic process. *J. Roy. Statist. Soc. Ser. B* **19** 13–17.

[4] PARZEN, E. (1957a). On consistent estimates of the spectrum of a stationary time series. *Ann. Math. Statist.* **28** 329–348.

[5] PARZEN, E. (1957b). On choosing an estimate of the spectral density function of a stationary time series. *Ann. Math. Statist.* **28** 921–932.

[6] PICKANDS, J. (1969). Efficient estimation of a probability density function. *Ann. Math. Statist.* **40** 854–864.

[7] WATSON, G. S. and LEADBETTER, M. R. (1963). On the estimation of a probability density I. *Ann. Math. Statist.* **43** 480–491.

[8] ZAREMBA, S. K. (1966). Quartic statistics in spectral analysis. *Advanced Seminar on Spectral Analysis of Time Series.* Wiley, New York, 47–79.