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EFFICIENCY-ROBUST ESTIMATION OF LOCATION[1]

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1. Introduction. Let $U_1, \cdots, U_m$ and $V_1, \cdots, V_n$ be two independent samples from distributions with distribution functions F and G, where $F(x) = G(x-\Delta)$. The usual procedures for finding estimates of Δ which are "efficient" require knowledge of the function F. For many F, estimates are known whose asymptotic variance coincides with the Cramér-Rao lower bound; such estimates are sometimes called asymptotically efficient for F. It is usually the case that, if the populations from which the samples are taken are not in the translation family of F, the estimate computed on the assumption that they are, is not asymptotically efficient in the above sense.

Here, estimates are proposed which are asymptotically efficient, uniformly for all F in a large class; that is, each such estimate is a sequence of functions which can be constructed without knowledge of F (functions of the observations only) and whose asymptotic variance is the Cramér-Rao lower bound for F, no matter which F in a class $\mathscr{F}$ is the underlying population. That such estimates exist was indicated by Stein [6]. Bhattacharya [1] proposed estimates of Δ that are "universally almost efficient" for all F in a class $\mathscr{F}'$. By reducing one sample to a frequency distribution over a fixed set of intervals, he obtains a sequence of estimates (functions of the observations only) whose asymptotic variance equals the Cramér-Rao lower bound for the distribution of the grouped data, no matter which $F \in \mathscr{F}'$ is the underlying distribution. The classes $\mathscr{F}$ and $\mathscr{F}'$ are somewhat different. $\mathscr{F}'$ contains the Cauchy distribution, which is not contained in $\mathscr{F}$ and $\mathscr{F}$ contains the double exponential distribution which is not contained in $\mathscr{F}'$.

The estimates proposed here are based on Hájek's [2] uniformly asymptotically efficient test of the hypothesis $\Delta = 0$. A modification of this test, satisfying the conditions of Hodges and Lehmann [4], will be shown to lead to a uniformly asymptotically efficient estimate of Δ.

In Section 2 the definition of the estimates will be given, as well as their asymptotic distribution. The proofs of the results of Section 2 will be given in Section 3; Section 4 contains the analogous results for the one-sample problem.

2. The estimates and their asymptotic distribution. Let $\mathscr{F}$ be the set of all distribution functions F with the following properties

1. F has an absolutely continuous density f,

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(2.1) 2. $\int_0^1 \varphi^2(u,f)\,du < \infty$, where $\varphi(u,f) = -\dfrac{f'(F^{-1}(u))}{f(F^{-1}(u))}$

$(F^{-1}(u) = \inf\{x \mid F(x) \geqq u\})$ and f' is the derivative of f,

3. $\varphi(u,f)$ is nondecreasing in $u(0 < u < 1)$.

Let, for $\nu = 1, 2, \cdots$, $\{m_\nu, n_\nu\}$ be a sequence of pairs of integers with $\min(m_\nu, n_\nu) \to \infty$ and let $\{(U_\nu), (V_\nu)\} \equiv \{(U_{\nu,1}, \cdots, U_{\nu,m_\nu}), (V_{\nu,1}, \cdots, V_{\nu,n_\nu})\}$ be a sequence of pairs of independent samples from distributions with distribution functions F and G, where $F(x) = G(x-\Delta)$ and $F \in \mathscr{F}$. Let $\{K_\nu\}$ be a sequence of integers with

$$K_\nu \to \infty, \qquad K_\nu/\min(m_\nu, n_\nu) \to 0 \tag{2.2}$$

and let $(W_\nu) \equiv (U_{\nu,1}, \cdots, U_{\nu,K_\nu}), (Z_\nu) \equiv (V_{\nu,1}, \cdots, V_{\nu,K_\nu}), (X_\nu) \equiv (X_{\nu,1}, \cdots, X_{\nu,m_\nu - K_\nu}) \equiv (U_{\nu,K_\nu+1}, \cdots, U_{\nu,m_\nu})$ and $(Y_\nu) \equiv (Y_{\nu,1}, \cdots, Y_{\nu,n_\nu - K_\nu}) \equiv (V_{\nu,K_\nu+1}, \cdots, V_{\nu,n_\nu})$. Hájek [2] proposed an estimate of $\varphi(u,f)$ based on the observations (W_ν, Z_ν) and used this estimate and the observations (X_ν) and (Y_ν) to construct a sequence of tests of the hypothesis $\Delta = 0$ that is uniformly asymptotically efficient for a sequence of alternatives Δ_ν with $\Delta_\nu \to 0$. In order to obtain a sequence of estimates of Δ that is uniformly asymptotically efficient for each fixed Δ, an estimate of $\varphi(u,f)$ will be used that is an average of two estimates, one based on (W_ν) and one based on (Z_ν). The estimate of Hájek will be further modified by constructing it in such a way that it is nondecreasing in u for each (W_ν, Z_ν). This estimate of $\varphi(u,f)$ and the observations (X_ν) and (Y_ν) will be used to construct a sequence of estimates of Δ that is uniformly asymptotically efficient.

Let $\{p_\nu\}$ be a sequence of integers satisfying

$$K_\nu^{4/5} < p_\nu \leqq K_\nu^{4/5} + 1 \tag{2.3}$$

and let $\{0 = h_{\nu,0} < h_{\nu,1} < \cdots < h_{\nu,q_\nu-1} < h_{\nu,q_\nu} = K_\nu\}$ be a sequence of $(q_\nu+1)$-tuples of integers, satisfying

$$\begin{aligned} &\lim_{\nu\to\infty} \max_{0 \leqq j < q_\nu} |h_{\nu,j+1} - h_{\nu,j}|/K_\nu^{5/6} \\ &\qquad = \lim_{\nu\to\infty} \min_{0 \leqq j < q_\nu} |h_{\nu,j+1} - h_{\nu,j}|/K_\nu^{5/6} = 1. \end{aligned} \tag{2.4}$$

Let $U_\nu^{(1)} < \cdots < U_\nu^{(K_\nu)}$ be the order statistics of $U_{\nu,1}, \cdots, U_{\nu,K_\nu}$ and let $N_\nu = m_\nu + n_\nu$. Then Hájek's estimate $\tilde\varphi_\nu(u, W_\nu)$ of $\varphi(u,f)$ based on (W_ν), is given by

$$\begin{aligned} &\tilde\varphi_\nu\left(\frac{i}{N_\nu - 2K_\nu + 1}, W_\nu\right) \\ &= \tfrac{1}{2} K_\nu^{-1/30} \left\{ \frac{1}{U_\nu^{(h_{\nu,j}+p_\nu)} - U_\nu^{(h_{\nu,j}-p)_\nu}} - \frac{1}{U_\nu^{(h_{\nu,j+1}+p_\nu)} - U_\nu^{(h_{\nu,j+1}-p_\nu)}} \right\} \\ &\quad \text{for } \frac{h_{\nu,j}}{K_\nu} < \frac{i}{N_\nu - 2K_\nu + 1} \leqq \frac{h_{\nu,j+1}}{K_\nu}\ j = 1, \cdots, q_{\nu-1},\ i = 1, \cdots, N_\nu - 2K_\nu \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{2.5}$$

and the definition is completed by taking $\tilde{\varphi}_\nu(u, W_\nu)$ constant on the intervals $[(i-1)/(N_\nu-2K_\nu), i/(N_\nu-2K_\nu)], i = 1, \cdots, N_\nu-2K_\nu$.

For the same sequences $\{p_\nu\}$, $\{q_\nu\}$ and $\{h_{\nu,0}, \cdots, h_{\nu,q_\nu}\}$, let $\tilde{\varphi}_\nu(u, Z_\nu)$ be Hájek's estimate based on (Z_ν) and let

$$\tilde{\varphi}_\nu(u) = \tilde{\varphi}_\nu(u, W_\nu, Z_\nu) = \tfrac{1}{2}\{\tilde{\varphi}_\nu(u, W_\nu) + \tilde{\varphi}_\nu(u, Z_\nu)\} \qquad 0 \leqq u \leqq 1. \tag{2.6}$$

This estimate $\tilde{\varphi}_\nu(u)$ is a function that is, for $0 \leqq u \leqq 1$, constant on each of a finite number of intervals. Call these interval $I_{\nu,i}(i = 1, \cdots, Q_\nu)$, in such a way that, for each $i = 1, \cdots, Q_\nu - 1, u_i < u_{i+1}$ if $u_i \in I_{\nu,i}$ and $u_{i+1} \in I_{\nu,i+1}$. Let, for $i = 1, \cdots, Q_\nu, \tilde{\varphi}_{\nu,i}$ be the value of $\tilde{\varphi}_\nu(u)$ for $u \in I_{\nu,i}$ and let $l_{\nu,i}$ be the length of $I_{\nu,i}$. Then define

$$\begin{aligned} \hat{\varphi}_\nu(u) &= \hat{\varphi}_\nu(u, W_\nu, Z_\nu) \\ &= \max_{1 \leqq j \leqq i} \min_{i \leqq k \leqq Q_\nu} \frac{l_{\nu,j}\tilde{\varphi}_{\nu,j} + \cdots + l_{\nu,k}\tilde{\varphi}_{\nu,k}}{l_{\nu,j} + \cdots + l_{\nu,k}} \quad \text{for} \quad u \in I_{\nu,i}, \\ & \qquad i = 1, \cdots, Q_\nu. \end{aligned} \tag{2.7}$$

The estimate of $\varphi(u, f)$, used in the construction of the sequence of estimates of Δ, is then given by

$$\begin{aligned} \hat{\varphi}_\nu^*(u) = \hat{\varphi}_\nu^*(u, Z_\nu, W_\nu) &= \hat{\varphi}_\nu(u) - (N_\nu - 2K_\nu)^{-1} \\ &\quad \times \textstyle\sum_{i=1}^{N_\nu - 2K_\nu} \hat{\varphi}_\nu(i(N_\nu - 2K_\nu + 1)^{-1}) \\ &= \hat{\varphi}_\nu(u) - \int_0^1 \hat{\varphi}_\nu(u)\,du. \end{aligned} \tag{2.8}$$

Now define

$$\hat{h}_\nu^*(X_\nu, Y_\nu) = \textstyle\sum_{i=1}^{m_\nu - K_\nu} \hat{\varphi}_\nu^*(R_{\nu,i}(N_\nu - 2K_\nu + 1)^{-1}), \tag{2.9}$$

where, for $i = 1, \cdots, m_\nu - K_\nu$, $R_{\nu,i}$ is the rank of $X_{\nu,i}$ in $(X_{\nu,1}, \cdots, X_{\nu,m_\nu - K_\nu}, Y_{\nu,1}, \cdots, Y_{\nu,n_\nu - K_\nu})$. The statistic $\hat{h}_\nu^*(X_\nu, Y_\nu)$ satisfies the condition (see Section 3) that, for each (U_ν, V_ν), $\hat{h}^*(X_\nu - b, Y_\nu)$ is a nonincreasing function of b. Further (see Section 3) $\hat{h}^*(X_\nu - b, Y_\nu)$ satisfies, for each (U_ν, V_ν), one of the following two conditions.

$$\begin{aligned} &1.\ \hat{h}_\nu^*(X_\nu - b, Y_\nu) = 0 \quad \text{for all} \quad b \quad \text{or} \\ &2.\ \hat{h}_\nu^*(X_\nu - b, Y_\nu) > 0 \quad \text{for} \quad b < X_\nu^{(1)} - Y_\nu^{(n_\nu - K_\nu)} \\ &\qquad\qquad\qquad\qquad < 0 \quad \text{for} \quad b > X_\nu^{(m_\nu - K_\nu)} - Y_\nu^{(1)}. \end{aligned} \tag{2.10}$$

Let S_ν be the set of points (U_ν, V_ν) where (2.10.2) is satisfied, then (see Section 3), for every Δ and any $F \in \mathscr{F}$, $P_{\nu,\Delta}((U_\nu, V_\nu) \in S_\nu) \to 1$. For $(U_\nu, V_\nu) \in S_\nu$ the estimate $\hat{\Delta}_\nu(U_\nu, V_\nu)$ of Δ is defined as follows.
Let

$$\begin{aligned} \Delta_\nu^*(U_\nu, V_\nu) &= \sup\{b \mid \hat{h}_\nu^*(X_\nu - b, Y_\nu) > 0\} \\ \Delta_\nu^{**}(U_\nu, V_\nu) &= \inf\{b \mid \hat{h}_\nu^*(X_\nu - b, Y_\nu) < 0\} \end{aligned} \tag{2.11}$$

and let α be a fixed number with $0 \leqq \alpha \leqq 1$; then

$$\hat{\Delta}_\nu(U_\nu, V_\nu) = \alpha\Delta_\nu^*(U_\nu, V_\nu)+(1-\alpha)\Delta_\nu^{**}(U_\nu, V_\nu) \quad \text{for} \quad (U_\nu, V_\nu)\in S_\nu. \tag{2.12}$$

Because, for every Δ and any $F\in\mathscr{F}$, $P_{\nu,\Delta}((U_\nu, V_\nu)\notin S_\nu) \to 0$, the asymptotic distribution of the estimate $\hat{\Delta}_\nu(U_\nu, V_\nu)$ does not depend on the definition of $\hat{\Delta}_\nu(U_\nu, V_\nu)$ for $(U_\nu, V_\nu)\notin S_\nu$.

The following theorem will be proved in Section 3.

THEOREM 2.1. *For every fixed Δ and any $F\in\mathscr{F}$*

$$\begin{aligned}\lim_{\nu\to\infty} P_{\nu,\Delta}((m_\nu n_\nu)^{\frac{1}{2}}(N_\nu)^{-\frac{1}{2}}(\hat{\Delta}_\nu(U_\nu, V_\nu)-\Delta) \leqq u)\\ = \sigma^{-1}(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{u} \exp(-\tfrac{1}{2}\sigma^{-2}x^2)\,dx,\end{aligned} \tag{2.13}$$

where

$$\sigma^2 = [\int_0^1 \varphi^2(u,f)\,du]^{-1} = [\int_{-\infty}^{+\infty}(f'(x)/f(x))^2 f(x)\,dx]^{-1} \tag{2.14}$$

3. Proofs of the results of Section 2.

LEMMA 3.1. *For each (U_ν, V_ν), $\hat{h}_\nu^*(X_\nu-b, Y_\nu)$ is a nonincreasing function of b.*

PROOF. From the definition of $\hat{\varphi}_\nu^*(u)$ it follows that $\hat{\varphi}_\nu^*(u)$ is a nondecreasing function of u. Further $\hat{\varphi}_\nu^*(u)$ is independent of (X_ν) and (Y_ν) so that $\hat{h}_\nu^*(X_\nu-b, Y_\nu)$ depends on b only through the ranks $R_{\nu,i}(b)$ of $X_{\nu,i}-b$ among $(X_{\nu,1}-b, \cdots, X_{\nu,m_\nu-K_\nu}-b, Y_{\nu,1}, \cdots, Y_{\nu,n_\nu-K_\nu})$ and not through $\hat{\varphi}_\nu^*$ itself. It is easily seen that $R_{\nu,i}(b)$ is a nonincreasing function of b, so that $\hat{h}_\nu^*(X_\nu-b, Y_\nu)$ is a sum of $m_\nu-K_\nu$ nonincreasing functions, $\hat{\varphi}_\nu^*(R_{\nu,i}(b)(N_\nu-2K_\nu+1)^{-1})$, of b.

LEMMA 3.2. *For each (U_ν, V_ν), $\hat{h}_\nu^*(X_\nu-b, Y_\nu)$ satisfies one of the following two conditions*

$$\begin{aligned}&1.\ \hat{h}_\nu^*(X_\nu-b, Y_\nu) = 0 \quad \text{for all} \quad b \quad \text{or}\\ &2.\ \hat{h}_\nu^*(X_\nu-b, Y_\nu) > 0 \quad \text{for} \quad b < X_\nu^{(1)} - Y_\nu^{(n_\nu-K_\nu)}\\ &\qquad\qquad\qquad\qquad < 0 \quad \text{for} \quad b > X_\nu^{(m_\nu-K_\nu)} - Y_\nu^{(1)}.\end{aligned} \tag{3.1}$$

PROOF. $\hat{h}_\nu^*(X_\nu-b, Y_\nu) = 0$ for all b if $\hat{\varphi}_\nu^*(u) = 0$ for all u. If $\hat{h}_\nu^*(X_\nu-b, Y_\nu) \neq 0$ for some b, then $\hat{\varphi}_\nu^*(u) \neq 0$ for some u. Because $\hat{\varphi}_\nu^*(u)$ is nondecreasing and constant on each of the intervals $[(i-1)/(N_\nu-2K_\nu), i/(N_\nu-2K_\nu)]$ $(i = 1, \cdots, N_\nu-2K_\nu)$ and because $\sum_{i=1}^{N_\nu-2K_\nu} \hat{\varphi}_\nu^*(i(N_\nu-2K_\nu+1)^{-1}) = 0$, there exists then a value of i with $\hat{\varphi}_\nu^*(i(N_\nu-2K_\nu+1)^{-1}) < \hat{\varphi}_\nu^*((i+1)(N-2K_\nu+1)^{-1})$. Thus

$$\sum_{i=1}^{k} \hat{\varphi}^*(i(N_\nu-2K_\nu+1)^{-1}) < 0 < \sum_{i=k+1}^{N_\nu-2K_\nu} \hat{\varphi}^*(i(N_\nu-2K_\nu+1)^{-1}) \tag{3.2}$$
$$\text{for all} \quad k = 1, \cdots, N_\nu-2K_\nu-1$$

Further, if $b > X_\nu^{(m_\nu-K_\nu)} - Y_\nu^{(1)}$, $R_{\nu,i}(b) = i$ for $i = 1, \cdots, m_\nu-K_\nu$, thus (see (3.2))

$$\hat{h}_\nu^*(X_\nu-b, Y_\nu) = \sum_{i=1}^{m_\nu-K_\nu} \hat{\varphi}_\nu^*(i(N_\nu-2K_\nu+1)^{-1}) < 0; \tag{3.3}$$

if $b < X_\nu^{(1)} - Y_\nu^{(n_\nu - K_\nu)}$, $R_{\nu,i}(b) = n_\nu - K_\nu + i$ for $i = 1, \cdots, m_\nu - K_\nu$ and (see (3.2))

$$\text{(3.4)}\qquad \begin{aligned}\hat{h}_\nu^*(X_\nu - b, Y_\nu) &= \textstyle\sum_{i=1}^{m_\nu - K_\nu} \hat{\varphi}_\nu^*((n_\nu - K_\nu + i)(N_\nu - 2K_\nu + 1)^{-1}) \\ &= \textstyle\sum_{i=n_\nu - K_\nu + 1}^{N_\nu - 2K_\nu} \hat{\varphi}_\nu^*(i(N_\nu - 2K_\nu + 1)^{-1}) > 0.\end{aligned}$$

LEMMA 3.3. *For any $F \in \mathscr{F}$ and every fixed Δ, $\hat{\phi}_\nu^*(u)$ is a consistent estimate of $\phi(u, f)$ in the sense that*

$$\text{(3.5)}\qquad \lim_{\nu\to\infty} P_{\nu,\Delta}\{\textstyle\int_0^1 (\hat{\varphi}_\nu^*(u) - \varphi(u,f))^2\, du > \varepsilon\} = 0.$$

PROOF. Let Φ be the set of all functions $\varphi(u)$ ($0 \leqq u \leqq 1$) satisfying

(3.6) 1. $\int_0^1 \varphi^2(u)\, du < \infty$

2. $\varphi(u)$ is nondecreasing in u,

then it will first be shown that $\hat{\varphi}_\nu(u)$ is the function $\varphi(u)$ which minimizes $\int_0^1 (\varphi(u) - \tilde{\varphi}_\nu(u))^2\, du$ for $\varphi(u) \in \Phi$. This can be seen as follows.

$$\text{(3.7)}\qquad \textstyle\int_0^1 (\varphi(u) - \tilde{\varphi}_\nu(u))^2\, du = \sum_{i=1}^{Q_\nu} \int_{I_{\nu,i}} (\varphi(u) - \tilde{\varphi}_{\nu,i})^2\, du$$

Further, for each $i = 1, \cdots, Q_\nu$

$$\text{(3.8)}\qquad \textstyle\int_{I_{\nu,i}} (\varphi(u) - \tilde{\varphi}_{\nu,i})^2\, du \geqq \int_{I_{\nu,i}} (\bar{\varphi}_{\nu,i} - \tilde{\varphi}_{\nu,i})^2\, du, \qquad \text{where}$$

$\bar{\varphi}_{\nu,i} = l_{\nu,i}^{-1} \int_{I_{\nu,i}} \varphi(u)\, du$. Thus a function $\varphi(u)$ which minimizes $\int_0^1 (\varphi(u) - \tilde{\varphi}_\nu(u))^2\, du$ for $\varphi(u) \in \Phi$ is a function which is constant on each of the intervals $I_{\nu,i}$. Thus the problem of minimizing $\int_0^1 (\varphi(u) - \tilde{\varphi}_\nu(u))^2\, du$ for $\varphi(u) \in \Phi$ is reduced to the problem of finding Q_ν numbers $\hat{\varphi}_{\nu,1}, \cdots, \hat{\varphi}_{\nu,Q_\nu}$, satisfying $\hat{\varphi}_{\nu,1} \leqq \cdots \leqq \hat{\varphi}_{\nu,Q_\nu}$ such that $\sum_{i=1}^{Q_\nu} l_{\nu,i}(\hat{\varphi}_{\nu,i} - \tilde{\varphi}_{\nu,i})^2$ is a minimum. The problem of finding the $\hat{\phi}_{\nu,i}$ such that $\sum_{i=1}^{Q_\nu} l_{\nu,i}(\hat{\varphi}_{\nu,i} - \tilde{\varphi}_{\nu,i})^2$ is a minimum, subject to the conditions $\hat{\varphi}_{\nu,1} \leqq \cdots \leqq \hat{\varphi}_{\nu,Q_\nu}$, is a special case of a problem solved by van Eeden ([7] and [8]). It is proved there that this problem has a unique solution given by (2.7).

The consistency of $\hat{\varphi}_\nu(u)$ can now be proved as follows. Hájek [2] proved that, for any F satisfying (2.1.1) and (2.1.2), $\tilde{\varphi}_\nu(u, W_\nu)$ and $\tilde{\varphi}_\nu(u, Z_\nu)$ are consistent estimates of $\varphi(u,f)$. From the definition (2.6) it is then clear that for any such F and every fixed Δ, $\tilde{\varphi}_\nu(u)$ is a consistent estimate of $\varphi(u, f)$. Now suppose that $\int_0^1 (\varphi(u,f) - \tilde{\varphi}_\nu(u))^2\, du < \varepsilon$, then, because $\varphi(u,f) \in \Phi$ and because $\hat{\varphi}_\nu(u)$ minimizes $\int_0^1 (\varphi(u) - \tilde{\varphi}_\nu(u))^2\, du$ for $\varphi(u) \in \Phi$, we have

$$\text{(3.9)}\qquad \textstyle\int_0^1 (\varphi(u,f) - \tilde{\varphi}_\nu(u))^2\, du < \varepsilon \to \int_0^1 (\hat{\varphi}_\nu(u) - \tilde{\varphi}_\nu(u))^2\, du < \varepsilon$$

and thus

$$\text{(3.10)}\qquad \textstyle\int_0^1 (\varphi(u,f) - \tilde{\varphi}_\nu(u))^2\, du < \varepsilon \to \int_0^1 (\varphi(u,f) - \hat{\varphi}_\nu(u))^2\, du < 4\varepsilon$$

and the consistency of $\hat{\varphi}_\nu(u)$ then follows from the consistency of $\tilde{\varphi}_\nu(u)$. Now let $\bar{\varphi}_\nu = \int_0^1 \hat{\varphi}_\nu(u)\, du$, then

$$\text{(3.11)}\qquad \textstyle\int_0^1 (\varphi(u,f) - \hat{\varphi}_\nu^*(u))^2\, du = \int_0^1 (\varphi(u,f) - \hat{\varphi}_\nu(u) + \bar{\varphi}_\nu)^2, \qquad \text{where}$$

$$\text{(3.12)}\qquad \bar{\varphi}_\nu^2 = (\textstyle\int_0^1 \hat{\varphi}_\nu(u)\, du)^2 = (\int_0^1 (\hat{\varphi}_\nu(u) - \varphi(u,f))\, du)^2 \leqq \int_0^1 (\hat{\varphi}_\nu(u) - \varphi(u,f))^2\, du$$

so that

$$\int_0^1 (\varphi(u,f)-\hat{\varphi}_\nu(u))^2\,du < \varepsilon \rightarrow \int_0^1 (\varphi(u,f)-\hat{\varphi}_\nu{}^*(u))^2\,du < 4\varepsilon \tag{3.13}$$

and the consistency of $\hat{\varphi}_\nu{}^*(u)$ follows from that of $\hat{\varphi}_\nu(u)$.

LEMMA 3.4. *For every fixed* Δ *and any* $F \in \mathscr{F}$

$$\lim_{\nu\to\infty} P_{\nu,\Delta}(\hat{h}_\nu{}^*(X_\nu - b, Y_\nu) = 0 \quad for\ all \quad b) = 0. \tag{3.14}$$

PROOF. $\hat{h}_\nu{}^*(X_\nu - b, Y_\nu) = 0$ if and only if $\hat{\varphi}_\nu{}^*(u) = 0$ for all u. (See Lemma 3.2.) If $\hat{\varphi}_\nu{}^*(u)$ is identically 0 then

$$\int_0^1 (\hat{\varphi}_\nu{}^*(u) - \varphi(u,f))^2\,du = \int_0^1 \varphi^2(u,f)\,du > 0. \tag{3.15}$$

Thus

$$\begin{aligned} &P_{\nu,\Delta}\{\hat{h}^*(X_\nu - b, Y_\nu) = 0 \quad \text{for all} \quad b\} \\ &\qquad \leqq P_{\nu,\Delta}\{\textstyle\int_0^1 (\hat{\varphi}_\nu{}^*(u) - \varphi(u,f))^2\,du = \int_0^1 \varphi^2(u,f)\,du\} \end{aligned} \tag{3.16}$$

and this last probability tends to zero as $\nu \to \infty$, because of the consistency of $\hat{\varphi}_\nu{}^*(u)$.

LEMMA 3.5. *If, for* $\nu = 1, 2, \cdots, \Delta_\nu = b[N_\nu(m_\nu n_\nu)^{-1}]^{\frac{1}{2}}$ *is a sequence of values of* Δ, *then for any* $F \in \mathscr{F}$

$$\lim_{\nu\to\infty} P_{\nu,\Delta_\nu}(\sigma_\nu{}^{-1}(\hat{h}_\nu{}^*(X_\nu, Y_\nu) - \mu_\nu) \leqq u) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^u e^{-\frac{1}{2}x^2}\,dx, \tag{3.17}$$

$$\begin{aligned} \mu_\nu &= b[(m_\nu - K_\nu)(n_\nu - K_\nu)(N_\nu - 2K_\nu)^{-1}]^{\frac{1}{2}} \int_0^1 \varphi^2(u,f)\,du \\ \sigma_\nu{}^2 &= (m_\nu - K_\nu)(n_\nu - K_\nu)(N_\nu - 2K_\nu)^{-1} \int_0^1 \varphi^2(u,f)\,du \end{aligned} \tag{3.18}$$

PROOF. Let, for $i = 1, \cdots, N_\nu - 2K_\nu$

$$a_\nu(i,f) = E\varphi(T_\nu{}^{(i)}, f) \tag{3.19}$$

where $T_\nu{}^{(1)} < \cdots < T_\nu{}^{(N_\nu - 2K_\nu)}$ are the order statistics of a sample of size $N_\nu - 2K_\nu$ from a uniform distribution between 0 and 1. Further let

$$g_\nu(X_\nu, Y_\nu) = \sum_{i=1}^{m_\nu - K_\nu} a_\nu(R_{\nu,i}, f) \tag{3.20}$$

then it follows from Theorem VI.2.3 and Theorem V.1.4b of Hájek and Sidák [3] and the fact that $K_\nu/\min(m_\nu, n_\nu) \to 0$, that

$$\lim_{\nu\to\infty} P_{\nu,\Delta_\nu}(\sigma_\nu{}^{-1}(g_\nu(X_\nu, Y_\nu) - \mu_\nu) \leqq u) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^u e^{-\frac{1}{2}x^2}\,dx, \tag{3.21}$$

where μ_ν and σ_ν are given by (3.15). That

$$\sigma_\nu{}^{-1}(g_\nu(X_\nu, Y_\nu) - \mu_\nu) \quad \text{and} \quad \sigma_\nu{}^{-1}(\hat{h}_\nu{}^*(X_\nu, Y_\nu) - \mu_\nu)$$

have, for the sequence $\Delta_\nu = b[N_\nu(m_\nu n_\nu)^{-1}]^{\frac{1}{2}}$, the same asymptotic distribution follows from Hájek and Sidák ([3] pages 264–265). This proof applies in our case, since $(R_{\nu,1}, \cdots, R_{\nu,m_\nu - K_\nu})$ and (W_ν, Z_ν) are independent, $\hat{\varphi}_\nu{}^*(u)$ is a function of (W_ν, Z_ν) only and since $\hat{\varphi}_\nu{}^*(u)$ is a consistent estimate of $\varphi(u,f)$.

PROOF OF THEOREM 2.1. The proof is based on the results of Hodges and Lehmann [4]. They consider two sample statistics $h_\nu(X_\nu, Y_\nu)$ satisfying the conditions

A. for each (X_ν, Y_ν), $h_\nu(X_\nu - b, Y_\nu)$ is a nonincreasing function of b (and is not identically zero) and

B. if $\Delta = 0$, the distribution of $h_\nu(X_\nu, Y_\nu)$ is, for every continuous F, symmetric around 0.

Their estimates derived from these statistics are estimates of the form (2.12) with $\alpha = \frac{1}{2}$ and one of their results is the following inequality

$$P_{\nu,\Delta}(h_\nu(X_\nu - b, Y_\nu) < 0) \leqq P_{\nu,\Delta}(\hat{\Delta}_\nu(X_\nu, Y_\nu) \leqq b) \leqq P_{\nu,\Delta}(h_\nu(X_\nu - b, Y_\nu) \leqq 0) \tag{3.22}$$
for any continuous F.

From the proofs by Hodges and Lehmann it can be seen (see also Hoyland [5]) that condition B is not necessary for (3.22). Further it can easily be seen from their proofs that the estimates satisfy (3.22) for any $\alpha \in [0.1]$. From Lemma 3.1 and Lemma 3.2 it then follows that, for any $F \in \mathscr{F}$ and for every ν and every Δ,

$$\begin{aligned} &P_{\nu,\Delta}(\hat{h}_\nu^*(X_\nu - b, Y_\nu) < 0 \,|\, (U_\nu, V_\nu) \in S_\nu) \\ &\qquad \leqq P_{\nu,\Delta}(\hat{\Delta}_\nu(U_\nu, V_\nu) \leqq b \,|\, (U_\nu, V_\nu) \in S_\nu) \\ &\qquad \leqq P_{\nu,\Delta}(\hat{h}_\nu^*(X_\nu - b, Y_\nu) \leqq 0 \,|\, (U_\nu, V_\nu) \in S_\nu). \end{aligned} \tag{3.23}$$

Thus, independent of the definition of $\hat{\Delta}_\nu(U_\nu, V_\nu)$ for $(U_\nu, V_\nu) \notin S_\nu$,

$$\begin{aligned} &P_{\nu,\Delta}(\hat{h}_\nu^*(X_\nu - b, Y_\nu) < 0, (U_\nu, V_\nu) \in S_\nu) + P_{\nu,\Delta}(\hat{\Delta}_\nu(U_\nu, V_\nu) \leqq b, (U_\nu, V_\nu) \notin S_\nu) \\ &\qquad \leqq P_{\nu,\Delta}(\hat{\Delta}_\nu(U_\nu, V_\nu) \leqq b) \\ &\qquad \leqq P_{\nu,\Delta}(\hat{h}_\nu^*(X_\nu - b, Y_\nu) \leqq 0, (U_\nu, V_\nu) \in S_\nu) \\ &\qquad\qquad + P_{\nu,\Delta}(\hat{\Delta}_\nu(U_\nu, V_\nu) \leqq b, (U_\nu, V_\nu) \notin S_\nu). \end{aligned} \tag{3.24}$$

From Lemma 3.4 it follows that, for every $F \in \mathscr{F}$ and every fixed Δ,

$$\lim_{\nu \to \infty} P_{\nu,\Delta}(\hat{\Delta}_\nu(U_\nu, V_\nu) \leqq b, (U_\nu, V_\nu) \notin S_\nu) = 0 \tag{3.25}$$

and

$$\begin{aligned} &\lim_{\nu \to \infty} [P_{\nu,\Delta}(\hat{h}_\nu^*(X_\nu - b, Y_\nu) < 0, (U_\nu, V_\nu) \in S_\nu) \\ &\qquad - P_{\nu,\Delta}(\hat{h}_\nu^*(X_\nu - b, Y_\nu) < 0)] = 0 \\ &\lim_{\nu \to \infty} [P_{\nu,\Delta}(\hat{h}_\nu^*(X_\nu - b, Y_\nu) \leqq 0, (U_\nu, V_\nu) \in S_\nu) \\ &\qquad - P_{\nu,\Delta}(\hat{h}_\nu^*(X_\nu - b, Y_\nu) \leqq 0)] = 0. \end{aligned} \tag{3.26}$$

From Lemma 3.5 it then follows that, for any $F \in \mathscr{F}$ and every Δ, (see also Hodges and Lehmann [4])

$$
\begin{aligned}
&\lim_{\nu\to\infty} P_{\nu,\Delta}([N_\nu^{-1}(m_\nu n_\nu)]^{\frac{1}{2}}(\hat{\Delta}_\nu(U_\nu, V_\nu)-\Delta) \leqq b) \\
&= \lim_{\nu\to\infty} P_{\nu,0}([N_\nu^{-1}(m_\nu n_\nu)]^{\frac{1}{2}}\hat{\Delta}_\nu(U_\nu, V_\nu) \leqq b) \\
&= \lim_{\nu\to\infty} P_{\nu,0}(\hat{\Delta}_\nu(U_\nu, V_\nu) \leqq b[N_\nu(m_\nu n_\nu)^{-1}]^{\frac{1}{2}}) \\
&= \lim_{\nu\to\infty} P_{\nu,0}(\hat{h}_\nu^*(X_\nu - b[N_\nu(m_\nu n_\nu)^{-1}]^{\frac{1}{2}}, Y_\nu) \leqq 0) \qquad (3.27)\\
&= \lim_{\nu\to\infty} P_{\nu,-\Delta_\nu}(\hat{h}_\nu^*(X_\nu, Y_\nu) \leqq 0) \\
&= \lim_{\nu\to\infty} P_{\nu,-\Delta_\nu}(\sigma_\nu^{-1}(\hat{h}_\nu^*(X_\nu, Y_\nu)-\mu_\nu) \leqq -\sigma_\nu^{-1}\mu_\nu) \\
&= (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{b[\int_0^1 \varphi^2(u,f)\,du]^{1/2}} e^{-\frac{1}{2}x^2}\,dx = \sigma^{-1}(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{b} \exp(-\tfrac{1}{2}\sigma^{-2}x^2)\,dx,
\end{aligned}
$$

where $\sigma^2 = (\int_0^1 \phi^2(u, f)\,du)^{-1}$.

4. The one-sample location problem. In this section the results for the one-sample location problem will be given. The proofs are analogous to those for the two-sample case and will not be given.

Let $\mathscr{F}_1$ be defined by

$$\mathscr{F}_1 = \{F \in \mathscr{F} \mid F \quad \text{is symmetric around} \quad 0\}. \tag{4.1}$$

Let, for $\nu = 1, 2, \cdots$, $\{N_\nu\}$ be a sequence of integers with $N_\nu \to \infty$ and let $\{U_\nu\} = \{U_{\nu,1}, \cdots, U_{\nu,N_\nu}\}$ be a sequence of samples from a distribution with distribution function $F(x-\theta)$, where $F \in \mathscr{F}_1$. An estimate $\hat{\theta}_\nu(U_\nu)$ of θ that is uniformly asymptotically efficient for $F \in \mathscr{F}_1$ can be obtained as follows.

For every $F \in \mathscr{F}_1$, $\varphi(u, f)$ is nondecreasing in u and

$$\varphi(u,f) = -\varphi(1-u,f)\ 0 < u < 1. \tag{4.2}$$

An estimate $\hat{\varphi}_\nu^*(u)$ of $\varphi(u, f)$ satisfying these two conditions is obtained as follows. Let $\{K_\nu\}$ be a sequence of integers satisfying

$$K_\nu \to \infty, \qquad K_\nu/N_\nu \to 0, \tag{4.3}$$

and let $\{p_\nu\}$, $\{q_\nu\}$ and $\{h_{\nu,0}, \cdots, h_{\nu,q_\nu}\}$ be sequences satisfying (2.3) and (2.4). Further let $(W_\nu) \equiv (U_{\nu,1}, \cdots, U_{\nu,K_\nu})$, $(X_\nu) \equiv (X_{\nu,1}, \cdots, X_{\nu,N_\nu-K_\nu}) \equiv (U_{\nu,K_\nu+1}, \cdots, U_{\nu,N_\nu})$ and let $\tilde{\varphi}_\nu(u, W_\nu)$ be Hájek's estimate of $\varphi(u,f)$ based on (W_ν) (see (2.5)). This function $\tilde{\varphi}_\nu(u, W_\nu)$ is a function that is constant on each of a finite number of intervals $I_{\nu,1}, \cdots, I_{\nu,Q_\nu}$, where, for each $i = 1, \cdots, Q_\nu - 1$, $u_i < u_{i+1}$ if $u_i \in I_{\nu,i}$ and $u_{i+1} \in I_{\nu,i+1}$. Let, for $i = 1, \cdots, Q_\nu$, $\tilde{\varphi}_{\nu,i}$ be the value of $\tilde{\varphi}_\nu(u, W_\nu)$ for $u \in I_{\nu,i}$ and let $l_{\nu,i}$ be the length of $I_{\nu,i}$. Then define

$$
\begin{aligned}
\hat{\varphi}_\nu(u) &= \hat{\varphi}_\nu(u, W_\nu) \qquad (4.4)\\
&= \max_{1\leqq j\leqq i} \min_{i\leqq k\leqq Q_\nu} \frac{l_{\nu,j}\tilde{\varphi}_{\nu,j} + \cdots + l_{\nu,k}\tilde{\varphi}_{\nu,k}}{l_{\nu,j} + \cdots + l_{\nu,k}}, \quad u \in I_{\nu,i}, \\
&\qquad\qquad i = 1, \cdots, Q_\nu
\end{aligned}
$$

and

$$\hat{\varphi}_\nu^*(u) = \hat{\varphi}_\nu^*(u, W_\nu) = \tfrac{1}{2}(\hat{\varphi}_\nu(u) - \hat{\varphi}_\nu(1-u)) \qquad 0 \leqq u \leqq 1. \tag{4.5}$$

Then $\hat{\varphi}_\nu^*(u)$ is, for every W_ν, nondecreasing in u and

(4.6) $$\hat{\varphi}_\nu^*(u) = -\hat{\varphi}_\nu^*(1-u) \qquad 0 \leqq u \leqq 1.$$

Let, for $i = 1, \cdots, N_\nu$, $R_{\nu,i}^+$ be the rank of $|X_{\nu,i}|$ among $|X_{\nu,1}|, \cdots, |X_{\nu,N_\nu-K_\nu}|$ and let

(4.7) $$\hat{h}_\nu^*(X_\nu) = \sum_{i=1}^{N_\nu-K_\nu} \hat{\varphi}_\nu^*(\tfrac{1}{2}[R_{\nu,i}^+(N_\nu - K_\nu + 1)^{-1} + 1])X_{\nu,i}|X_{\nu,i}|^{-1},$$

then $\hat{h}_\nu^*(X_\nu - b)$ is, for every U_ν, nonincreasing in b and satisfies one of the following two conditions

(4.8) $$\begin{aligned} &1.\ \hat{h}_\nu^*(X_\nu - b) = 0 \quad \text{for all} \quad b \quad \text{or} \\ &2.\ \hat{h}_\nu^*(X_\nu - b) > 0 \quad \text{for} \quad b < X_\nu^{(1)} \\ &\qquad\qquad\qquad\quad < 0 \quad \text{for} \quad b > X_\nu^{(N_\nu - K_\nu)} \end{aligned}$$

with, for each fixed θ and any $F \in \mathscr{F}_1$,

(4.9) $$\lim_{\nu\to\infty} P_{\nu,\theta}(\hat{h}_\nu^*(X_\nu - b) = 0 \quad \text{for all} \quad b) = 0.$$

Let S_ν be the set of points (U_ν) such that (4.8.2) is satisfied, then the estimate $\hat{\theta}_\nu(U_\nu)$ for $(U_\nu) \in S_\nu$ is defined as follows. Let

(4.10) $$\begin{aligned} \theta_\nu^*(U_\nu) &= \sup\{b \mid \hat{h}_\nu^*(X_\nu - b) > 0\} \\ \theta_\nu^{**}(U_\nu) &= \inf\{b \mid \hat{h}_\nu^*(X_\nu - b) < 0\} \end{aligned}$$

and let α be a fixed number with $0 \leqq \alpha \leqq 1$. Then

(4.11) $$\hat{\theta}_\nu(U_\nu) = \alpha\theta_\nu^*(U_\nu) + (1-\alpha)\theta_\nu^{**}(U_\nu) \quad \text{for} \quad (U_\nu) \in S_\nu.$$

Further, because of (4.9), the asymptotic distribution of the estimate $\hat{\theta}_\nu(U_\nu)$ does not depend on the definition of the estimate for $(U_\nu) \notin S_\nu$. In the same way as for the two-sample problem, the following theorem can be proved

THEOREM 4.1. *For every fixed θ and any $F \in \mathscr{F}_1$*

(4.12) $$\lim_{\nu\to\infty} P_{\nu,\theta}(N_\nu^{\frac{1}{2}}(\hat{\theta}_\nu(U_\nu) - \theta) \leqq u) = \sigma^{-1}(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{u} \exp(-\tfrac{1}{2}\sigma^{-2}x^2)\,dx,$$

where

(4.13) $$\sigma^2 = [\int_0^1 \phi^2(\tfrac{1}{2}(u+1), f)\,du]^{-1} = [\int_{-\infty}^{+\infty} (f'(x)/f(x))^2 f(x)\,dx]^{-1}.$$

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