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ON THE MATRIX RENEWAL FUNCTION FOR MARKOV RENEWAL PROCESSES[1]

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1. Introduction. A Markov renewal process [7, 8], $[N_1(t), N_2(t), \cdots, N_R(t)]$ is an extension of an ordinary renewal process having considerable practical importance. Whereas the ordinary renewal process describes the number of recurrences (renewals) in the interval $(0, t]$ for a single recurrent class of epochs $\mathcal{E}_1$ of interest, the Markov renewal process describes the recurrence statistics for intermingling classes of epochs $\{\mathcal{E}_j ; j = 1, 2, \cdots R\}$ of an underlying semi-Markov process. The process is characterized by a stochastic transition matrix p_{ij} for the Markov chain governing the sequence of successive epochs, and a matrix of probability distributions $F_{ij}(x)$ for the time elapsing between epochs of class $\mathcal{E}_i$ and epochs of class $\mathcal{E}_j$, whenever an $\mathcal{E}_j$ epoch follows an $\mathcal{E}_i$ epoch. We adopt the convention that an epoch of a given class may be succeeded by another epoch of that class.

Let $N_{ij}(t)$ be the number of epochs of class $\mathcal{E}_j$ appearing in the interval $(0, t]$, when it is known that at $t = 0$ there was an epoch of class $\mathcal{E}_i$. Let $H_{ij}(t) = E[N_{ij}(t)]$ and let $\mathbf{H}(t)$ be the $R \times R$ matrix with elements $H_{ij}(t)$. We will call this matrix the matrix renewal function.

When $R = 1$, $N_{11}(t)$ is the ordinary renewal process, and $H_{11}(t) = H(t)$ is the renewal function. It is well known [6] that if an interval distribution $F(x)$ has finite first and second moments μ_1 and μ_2, and does not have arithmetic support, then the associated renewal function $H(t)$ has the behavior

$$(1.1) \qquad H(t) = \mu_1^{-1}t + \tfrac{1}{2}\mu_1^{-2}(\mu_2 - 2\mu_1^2) + \epsilon(t)$$

where $\epsilon(t)$ is bounded and goes to zero as $t \to \infty$.

The literature on semi-Markov processes has dealt largely with the theoretical structure of such processes, and pathology distinguishing such processes from Markov chains, in continuous time. The statistics for such processes, e.g. the mean and variance of the renewal time between epochs of the same class, and the mean passage time from an epoch of class $\mathcal{E}_i$ to an epoch of class $\mathcal{E}_j$, have received less attention.[2] For finite semi-Markov processes ($R < \infty$), much of this statistical information is available from the following direct analogue of equation (1.1).

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[2] Other procedures and results may be found in W. S. Jewell, Markov–renewal programming I, II, *Operations Res.*, **11** (1963) 938–971; Kshirsagar and Gupta, Asymptotic values of the two moments in Markov renewal processes, *Biometrika*, **54** (1967); and P. J. Schweitzer, Perturbation theory and Markovian decision processes, MIT ScD Thesis, Department of Physics, 1965. I am indebted to D. R. Cox and P. J. Schweitzer for these references.

THEOREM 1. *Let $N(t)$ be a finite semi-Markov process. If* (a) *the transition matrix* $\mathbf{P} = \{p_{ij}\}$ *for the governing chain is irreducible*; (b) $\int x^2\, dF_{ij}(x) < \infty$ *for all* $i, j \leqq R$; (c) *the distributions* $F_{ij}(x)$ *are not all lattice distributions with a common span; then the matrix renewal function* $\mathbf{H}(t)$ *for the associated Markov renewal process has the form*

$$\mathbf{H}(t) = \mathbf{a}_2 t + \mathbf{a}_1 + \boldsymbol{\varepsilon}(t) \tag{1.2}$$

where

$$\mathbf{a}_2 = m^{-1}\mathbf{J}_0, \tag{1.3}$$

$$\mathbf{a}_1 = m^{-1}\mathbf{J}_0\{-\mathbf{B}_1 + \tfrac{1}{2}m^{-1}\mathbf{B}_2\mathbf{J}_0\} + \{\mathbf{Z} - m^{-1}\mathbf{J}_0\mathbf{B}_1\mathbf{Z}\}\{\mathbf{B}_0 - m^{-1}\mathbf{B}_1\mathbf{J}_0\}, \tag{1.4}$$

and $\boldsymbol{\varepsilon}(t) \to \mathbf{0}$ *as* $t \to \infty$. In the above $(\mathbf{B}(x))_{ij} = p_{ij}F_{ij}(x)$; $\mathbf{B}_k = \int x^k\, d\mathbf{B}(x)$; $\mathbf{e}$ is the left real positive eigenvector of $\mathbf{B}_0$ with $\sum e_j = 1$; $(\mathbf{J}_0)_{ij} = e_j$, $m = \sum_{ij} e_i B_{1ij}$, and $\mathbf{Z} = [\mathbf{I} - \mathbf{B}_0 + \mathbf{J}_0]^{-1}$ is the fundamental matrix for the governing chain [4].

From (1.1) and Theorem 1 we will exhibit the mean and variance of the renewal times for the epochs of any class, the parameters needed for the Central Limit Theorem for the corresponding renewal process. The mean passage time $E(\tau_{ij})$ from an epoch of class $\mathcal{E}_i$ to the first subsequent epoch of class $\mathcal{E}_j$ will also be exhibited.

The matrices $\mathbf{a}_1$ and $\mathbf{a}_2$ appearing in (1.2) are obtained from a study of the spectral decomposition of the matrix $[I - \boldsymbol{\beta}(s)]^{-1}$ where $\boldsymbol{\beta}(s)$ is the Laplace-Stieltjes transform of $\mathbf{B}(x)$. This part of the analysis is an extension of results obtained earlier with D. M. G. Wishart [2, 3].

2. The asymptotic behavior of the matrix renewal function. We first wish to demonstrate the validity of the form of the matrix renewal function $\mathbf{H}(t)$ exhibited in (1.2) under the conditions of Theorem 1.

LEMMA 2.1. *If the transition matrix* $\mathbf{B}_0 = \mathbf{p}$ *governing the finite imbedded Markov chain is irreducible, and if* $\mathbf{B}_2$ *is finite, then the random interval* τ_{ij} *between an epoch of class* $\mathcal{E}_i$ *and the first subsequent epoch of class* $\mathcal{E}_j$ *has a finite second moment.*

PROOF. Let K_{ij} be the random number of epochs of all classes from an epoch of the class $\mathcal{E}_i$ to the first subsequent epoch of class $\mathcal{E}_j$. The irreducibility of $\mathbf{B}_0$ implies that $E[K_{ij}^2]$ and $E[K_{ij}]$ are finite. The finiteness of $\mathbf{B}_2$ (and hence of $\mathbf{B}_1$) and the finiteness of R permits one to infer that $\mu_{1ij} = \int x\, dF_{ij}(x) < A$, all i, j, and that $\mu_{2ij} = \int x^2\, dF_{ij}(x) < B$, all i, j. Any sample path in continuous time from an epoch of $\mathcal{E}_i$ to the first subsequent epoch in $\mathcal{E}_j$ will be associated with the corresponding path for the imbedded chain $i = j_0 \to j_1 \to j_2 \cdots \to j_M = j$, and the duration of the sample path in continuous time will be

$$\tau_{ij} = T_1 + T_2 \cdots + T_{K_{ij}}$$

where T_k is the duration of the kth step in the path. Hence,[3] $E[\tau_{ij}] = E\left[\sum_1^{K_{ij}} T_k\right]$

[3] It is known that $E[K_{ij}]$ and $E[K_{ij}^2]$ are finite for finite, irreducible chains. See Kemeny and Snell [4].

$\leqq AE[K_{ij}] < \infty$. Similarly, since $E[T_jT_k] < B$, from the Schwarz Inequality, $E[\tau_{ij}^2] \leqq BE[K_{ij}^2] < \infty$. ▯

LEMMA 2.2. *Under the conditions of Theorem 1, $H_{ij}(t)$ has the form for $i = j$,*

$$H_{jj}(t) = t/E[\tau_{jj}] + (E[\tau_{jj}^2] - 2E^2[\tau_{jj}])/(2E^2[\tau_{jj}]) + \epsilon_{jj}(t) \tag{2.1}$$

and for $i \neq j$

$$H_{ij}(t) = t/E[\tau_{jj}] + \{1 + (E[\tau_{jj}^2] - 2E^2[\tau_{jj}])/(2E^2[\tau_{jj}]) - E[\tau_{ij}]/E[\tau_{jj}]\} + \epsilon_{ij}(t) \tag{2.2}$$

where $\epsilon_{ij}(t)$ and $\epsilon_{jj}(t) \to 0$ as $t \to \infty$, and are bounded for all t.

PROOF. Equation (2.1) follows from Lemma (2.1) and the result (1.1) of ordinary renewal theory when increments have finite second moments and do not have arithmetic support [6]. The non-arithmetic support of τ_{jj} follows from the irreducibility of $\mathbf{B}_0$ and the assumption that not all the distributions $F_{ij}(x)$ are lattice distributions with a common span.

The proof of (2.2) proceeds from (2.1) and the observation that

$$\begin{aligned} H_{ij}(t) &= \int_0^t H_{jj}(t - t')\, dS_{ij}(t') + S_{ij}(t) \\ &= H_{jj}(t) * S_{ij}(t) + S_{ij}(t) \end{aligned} \tag{2.3}$$

where $S_{ij}(t)$ is the pdf for τ_{ij}, i.e. $S_{ij}(t) = P\{\tau_{ij} \leqq t\}$. To obtain (2.2) one only requires the Dominated Convergence Theorem. Thus let

$$F(t) = \{at + b + \epsilon(t)\} * S(t),$$

i.e.,

$$F(t) = b\int_0^t dS(t') + at\int_0^t dS(t') - a\int_0^t t'\, dS(t') + \int_0^t \epsilon(t - t')\, dS(t').$$

If we write $\int_0^t dS(t')$ as $1 - \int_t^\infty dS(t')$, and note that $t\int_t^\infty dS(t') \leqq \int_t^\infty t'\, dS(t') \to 0$ as $t \to \infty$, we need only consider the fourth term. We see from (2.1) that $\epsilon(t)$ is bounded, i.e. $|\epsilon(t)| < M < \infty$ for all t and $\epsilon(t) \to 0$ as $t \to \infty$. If we write our integral as $\int_0^\infty \{\epsilon(t - t')\, U(t - t')\}\, dS(t')$ the term in curly brackets is dominated by M, and the Dominated Convergence Theorem permits us to infer that $\epsilon(t) * S(t) \to 0$ as $t \to \infty$. Hence $F(t) = at + b - a\int_0^\infty t'\, dS(t') + o(t)$. Since $S_{ij}(t) \to 1$, (2.2) follows ▯.

Equations (2.1) and (2.2) may be written in the combined form

$$\mathbf{H}(t) = \mathbf{a}_2 t + \mathbf{a}_1 + \boldsymbol{\varepsilon}(t). \tag{2.4}$$

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[4] For let path α be a chain path $(j, i_{\alpha 1}, \cdots i_{\alpha N-1}, j)$ starting at j and returning to j after N steps. These paths are enumerable and we may write $R_{jj}(x) = P\{\tau_{jj} \leqq x\} = \sum_\alpha P\{\text{path } \alpha\}\, F_{j i_{\alpha 1}}(x) * \cdots * F_{i_{\alpha N-1}\, j}(x)$. Because of the irreducibility any transition permitted by the chain must appear in at least one path contributing to the summation. It then follows from a simple characteristic function argument that if τ_{jj} has span h, every $F_{kl}(x)$ must have the same span. Note that if ξ_{ij} is the rv for $F_{ij}(x)$, and $\xi_{ij} = K_i - K_j + N_{kj}h$, then τ_{jj}/h is an integer lattice random variable when K_i and K_j are constants, and N_{ij} is an integer random variable. This possibility preventing $\boldsymbol{\varepsilon}(t)$ from going to zero was suggested to the author by J. Th. Runnenburg.

The constant matrices $\mathbf{a}_2$ and $\mathbf{a}_1$ are known from (2.1) and (2.2) in terms of $E[\tau_{ij}]$ and $E[\tau_{ij}^2]$. An expression is needed for $\mathbf{a}_2$ and $\mathbf{a}_1$ in terms of the elements of $\mathbf{B}(x)$. This expression may be obtained by a comparison of (2.4) with the alternate form of $\mathbf{H}(t)$ given by

$$\mathbf{H}(t) = \textstyle\sum_{k=1}^{\infty} \mathbf{B}^{(k)}(t) \tag{2.5}$$

where $\mathbf{B}^{(k)}(t)$ is the k-fold convolution of $\mathbf{B}(t)$ with itself, i.e. is defined iteratively from $\mathbf{B}^{(k+1)}(t) = \int_0^t \mathbf{B}(t - t')\, d\mathbf{B}^{(k)}(t')$. Let $\boldsymbol{\beta}(s) = \int_0^\infty e^{-st}\, d\mathbf{B}(t)$. We note that the non-arithmetic support of $\mathbf{B}(x)$ implies [5] that $\sum |\beta_{ij}(s)| < 1$, in $D = \{s \mid \mathrm{Re}(s) \geqq 0,\ s \neq 0\}$ for one or more values of i. We have, therefore, [1], that the spectral radius of $\boldsymbol{\beta}(s)$ is less than unity in the domain D. Hence, in D, $[\mathbf{I} - \boldsymbol{\beta}(s)]$ is non-singular and

$$\mathfrak{L}\{\mathbf{H}(t)\} = s^{-1} \textstyle\sum_1^\infty \boldsymbol{\beta}^k(s) = s^{-1}\boldsymbol{\beta}(s)[\mathbf{I} - \boldsymbol{\beta}(s)]^{-1}. \tag{2.6}$$

From (2.4) we may then write, for limits taken along paths in the interior of D,

$$\mathbf{a}_2 = \lim_{s\to 0+} s^2\{s^{-1}\boldsymbol{\beta}(s)[\mathbf{I} - \boldsymbol{\beta}(s)]^{-1}\}, \tag{2.6$'$}$$

since $s^2\mathfrak{L}\{\mathbf{a}_1 + \boldsymbol{\varepsilon}(t)\} \to \mathbf{0}$ as $s \to 0+$. Moreover, from $s\mathfrak{L}\{\boldsymbol{\varepsilon}(t)\} \to \mathbf{0}$ as $s \to 0+$ and (2.4) we also have

$$\mathbf{a}_1 = \lim_{s\to 0+} [s\{s^{-1}\boldsymbol{\beta}(s)[\mathbf{I} - \boldsymbol{\beta}(s)]^{-1} - a_2/s^2\}]. \tag{2.7}$$

Evaluation of $\mathbf{a}_1$ and $\mathbf{a}_2$ from (2.6′) and (2.7) requires study of the behavior of $[I - \boldsymbol{\beta}(s)]^{-1}$ in the neighborhood of $s = 0$.

3. On the structure of $[\mathbf{I} - \boldsymbol{\beta}(s)]^{-1}$ near $s = 0$. The matrix characteristic function $\boldsymbol{\beta}(s)$ is analytic in the domain Re $(s) > 0$, and continuous in the closed right half-plane Re $(s) \geqq 0 = D \cup \{0\}$. It has been shown by Keilson and Wishart [2, page 549] that when $\mathbf{B}_2 < \infty$, $\boldsymbol{\beta}(s)$ has a simple eigenvalue $\lambda_0(s)$ larger in magnitude than all other eigenvalues for values s on the pure imaginary s-axis in some neighborhood of $s = 0$. Moreover, $|\lambda_0(s)| \leqq 1$ and $\lambda_0(s)$ is continuous in that neighborhood and differentiable twice there. By virtue of the analyticity of $\boldsymbol{\beta}(s)$ in the interior of D, a similar argument may be employed to show that for some closed neighborhood N of the origin in the right half s-plane, i.e. for some set $N = \{s \mid \mathrm{Re}\ (s) \geqq 0,\ |s| \leqq \delta\}$, $\boldsymbol{\beta}(s)$ has a simple maximal eigenvalue $\lambda_0(s)$, differentiable twice at *every* point s_0 of N, in the sense that the defining limits are independent of all possible paths taken within N, and $\lambda_0'(s)$ and $\lambda_0''(s)$ are continuous at all points of N.

Associated with $\lambda_0(s)$ in N, will be left and right eigenvectors $\mathrm{X}_0(s)$ and $\mathrm{Y}_0(s)$, and a corresponding dyadic matrix $\mathbf{J}(s)$ with components

$$J_{ij}(s) = Y_{0i}(s)X_{0j}(s)/\textstyle\sum_j X_{0j}(s)Y_{0j}(s),$$

such that

$$\mathbf{J}(s)\boldsymbol{\beta}(s) = \lambda_0(s)\mathbf{J}(s), \tag{3.1}$$

$$\boldsymbol{\beta}(s)\mathbf{J}(s) = \lambda_0(s)\mathbf{J}(s), \tag{3.2}$$

$$\mathbf{J}^2(s) = \mathbf{J}(s). \tag{3.3}$$

Correspondingly one will always have available in N, the decomposition

$$\beta(s) = \lambda_0(s)\mathbf{J}(s) + \mathbf{L}(s) \tag{3.4}$$

where $\mathbf{L}(s) = \beta(s) - \lambda_0(s)\mathbf{J}(s)$. From (3.1), (3.2) and (3.3) it follows that $\mathbf{J}(s)\mathbf{L}(s) = \mathbf{L}(s)\mathbf{J}(s) = \mathbf{0}$. Hence from (3.4) and the idempotence of $\mathbf{J}$ we have $\beta^k(s) = \lambda_0^k(s)\mathbf{J}(s) + \mathbf{L}^k(s)$. The spectral radius of $\mathbf{L}(s)$ is smaller than one for all points of N, and hence $[\mathbf{I} - \mathbf{L}(s)]$ will be non-singular in N. It follows from (3.4) that

$$[\mathbf{I} - \beta(s)]^{-1} = \lambda_0(s)[1 - \lambda_0(s)]^{-1}\mathbf{J}(s) + [\mathbf{I} - \mathbf{L}(s)]^{-1}, \quad s \varepsilon N, \ s \neq 0. \tag{3.5}$$

The principle dyadic $\mathbf{J}(s)$ and its derivative $\mathbf{J}'(s)$ will also be analytic and continuous on N. This follows from an argument almost identical to that of Theorem 2.5 of Keilson and Wishart [2]. Hence $\mathbf{J}(s) \to \mathbf{J}(0) = \mathbf{J}_0$ with elements $J_{0ij} = e_j$ where $\mathbf{e}$ is the left eigenvector of $\mathbf{B}_0$ corresponding to the ergodic state probabilities for the associated chain. Also $\mathbf{L}(s) \to \mathbf{B}_0 - \mathbf{J}_0$ and $[\mathbf{I} - \mathbf{L}(s)]^{-1} \to [\mathbf{I} - \mathbf{B}_0 + \mathbf{J}_0]^{-1} = \mathbf{Z}$, the fundamental matrix for the associated chain.

4. Differentiation of the principal eigenvalue and principal dyadic at $s = 0$. If we differentiate (3.1) at $s = 0$, and use $\lambda_0(0) = 1$, $\beta(0) = \mathbf{B}_0$ and $\beta'(0) = -\mathbf{B}_1$, we obtain

$$-\mathbf{J}(0)\mathbf{B}_1 + \mathbf{J}'(0)\mathbf{B}_0 = \mathbf{J}'(0) + \lambda_0'(0)\mathbf{J}(0). \tag{4.1}$$

If we next premultiply (4.1) by $\mathbf{e}$, postmultiply by $\mathbf{1}$, and use $\mathbf{eJ}(0) = \mathbf{e}$, $\mathbf{B}_0\mathbf{1} = \mathbf{1}$ and $\mathbf{e}\cdot\mathbf{1} = 1$, we obtain as in Keilson and Wishart [2]

$$-\lambda_0'(0) = \mathbf{eB}_1\mathbf{1} = m. \tag{4.2}$$

We next obtain an expression for $\mathbf{J}'(0)$, from differentiation of (3.1), (3.2) and (3.3). If we add $\mathbf{J}'(0)\mathbf{J}(0)$ to both sides of (4.1) and use $\mathbf{Z} = \{\mathbf{I} - \mathbf{B}_0 + \mathbf{J}(0)\}^{-1}$, we find

$$\{\mathbf{J}'(0)\mathbf{J}(0) + m\mathbf{J}(0) - \mathbf{J}(0)\mathbf{B}_1\}\mathbf{Z} = \mathbf{J}'(0). \tag{4.3}$$

Similarly, differentiation of (3.2) and the same procedure gives

$$\mathbf{Z}\{\mathbf{J}(0)\mathbf{J}'(0) + m\mathbf{J}(0) - \mathbf{B}_1\mathbf{J}(0)\} = \mathbf{J}'(0). \tag{4.4}$$

We next add (4.3) and (4.4). We observe that $\mathbf{ZJ}(0) = \mathbf{J}(0)$ and $\mathbf{J}(0)\mathbf{Z} = \mathbf{J}(0)$. Moreover, from (3.3), $\mathbf{J}(0)\mathbf{J}'(0) + \mathbf{J}'(0)\mathbf{J}(0) = \mathbf{J}'(0)$. Hence we obtain

$$\mathbf{J}'(0) = 2m\mathbf{J}(0) - \mathbf{J}(0)\mathbf{B}_1\mathbf{Z} - \mathbf{ZB}_1\mathbf{J}(0) \tag{4.5}$$

where everything on the right of (4.5) is known.

An evaluation of $\lambda_0''(0)$ is also of interest. If we differentiate (3.2) twice and set $s = 0$, we find

$$\mathbf{B}_0\mathbf{J}''(0) - 2\mathbf{B}_1\mathbf{J}'(0) + \mathbf{B}_2\mathbf{J}(0) = \mathbf{J}''(0) - 2m\mathbf{J}'(0) + \lambda_0''(0)\mathbf{J}(0).$$

Premultiplication by $\mathbf{e}$, postmultiplication by 1, and use of $\mathbf{eB}_1 1 = m$ (4.2) then gives

$$\lambda_0''(0) = 2m\mathbf{e}\mathbf{J}'(0)1 - 2\mathbf{e}\mathbf{B}_1\mathbf{J}'(0)\mathbf{1} + \mathbf{e}\mathbf{B}_2\mathbf{1}. \tag{4.6}$$

From (4.5), however $\mathbf{e}\mathbf{J}'(0)\mathbf{1} = 0$, since $\mathbf{e}\mathbf{Z} = e$, and $\mathbf{Z1} = \mathbf{1}$. Also from (4.5) $\mathbf{J}'(0)\mathbf{1} = m\mathbf{1} - \mathbf{Z}\mathbf{B}_1\mathbf{1}$. Substituting in (4.6), we find

$$\lambda_0''(0) = \mathbf{e}\mathbf{B}_2\mathbf{1} - 2m^2 + 2\mathbf{e}\mathbf{B}_1\mathbf{Z}\mathbf{B}_1\mathbf{1}. \tag{4.7}$$

Hence

$$\lambda_0''(0) - \{\lambda'(0)\}^2 = \mathbf{e}\mathbf{B}_2\mathbf{1} - 3m^2 + 2\mathbf{e}\mathbf{B}_1\mathbf{Z}\mathbf{B}_1\mathbf{1}. \tag{4.8}$$

The expression on the left of (4.8) was identified previously in Keilson and Wishart [2] as the asymptotic variance per increment for the Central Limit Theorem for additive processes defined on a finite Markov chain, but the authors were unable to evaluate the expression by direct differentiation. The term on the right of (4.8) was obtained subsequently in Keilson and Wishart [3] as an expression for the asymptotic variance by purely probabilistic considerations. The evaluation of $\lambda''(0) - \{\lambda'(0)\}^2$ by our present technique provides a check on these older results.

5. Evaluation of $\mathbf{a}_2$ and $\mathbf{a}_1$. We next evaluate $\mathbf{a}_2$ from (2.6) with the help of (3.5). Since $\boldsymbol{\beta}(s) \to \mathbf{B}_0$, $\mathbf{J}(s) \to \mathbf{J}_0$, and $[\mathbf{I} - \mathbf{L}(s)]^{-1} \to \mathbf{Z}$, we have from (2.6), and $\mathbf{B}_0\mathbf{J}_0 = \mathbf{J}_0$,

$$\mathbf{a}_2 = \lim_{s\to 0+} \frac{s}{1 - \lambda_0(s)} \mathbf{J}_0 = \frac{-1}{\lambda_0'(0)} \mathbf{J}_0, \tag{5.1}$$

or

$$\mathbf{a}_2 = m^{-1}\mathbf{J}_0. \tag{5.2}$$

To evaluate $\mathbf{a}_1$, we employ (2.7) and (5.2) in the following manner. The term appearing on the right of (2.7) may be rewritten as

$$\begin{aligned}\mathbf{a}_1 = \lim_{s\to 0+} [\{s[\mathbf{I} - \boldsymbol{\beta}(s)]^{-1}\}\{s^{-1}[\boldsymbol{\beta}(s) - \boldsymbol{\beta}(0)] \\ + (ms^2)^{-1}[\boldsymbol{\beta}(s) - \boldsymbol{\beta}(0) - \boldsymbol{\beta}'(0)s]\mathbf{J}_0\}] \\ + \lim_{s\to 0+} [\{\mathbf{I} - \boldsymbol{\beta}(s)\}^{-1}\{\boldsymbol{\beta}(0) + m^{-1}\boldsymbol{\beta}'(0)\mathbf{J}_0\}].\end{aligned} \tag{5.3}$$

The finiteness of $\mathbf{B}_2$ assures the availability of the Taylor Expansion $\boldsymbol{\beta}(s) = \boldsymbol{\beta}(0) + \boldsymbol{\beta}'(0)s + \boldsymbol{\beta}''(0)s^2/2 + s^2\boldsymbol{\theta}(s)$ where $\boldsymbol{\theta}(s) \to 0$ as $s \to 0+$. Since $\boldsymbol{\beta}'(0) = -\mathbf{B}_1$ and $\boldsymbol{\beta}''(0) = \mathbf{B}_2$, (5.3) becomes

$$\mathbf{a}_1 = m^{-1}\mathbf{J}_0\{\tfrac{1}{2}m^{-1}\mathbf{B}_2\mathbf{J}_0 - \mathbf{B}_1\} + \lim_{s\to 0+} [\{\mathbf{I} - \boldsymbol{\beta}(s)\}^{-1}\{\mathbf{B}_0 - m^{-1}\mathbf{B}_1\mathbf{J}_0\}]. \tag{5.4}$$

We note that

$$\mathbf{J}(0)\{\mathbf{B}_0 - m^{-1}\mathbf{B}_1\mathbf{J}_0\} = 0 \tag{5.5}$$

and employ (3.5) to write the second term in (5.4) as

$$\lim_{s\to 0+} \left[\frac{\lambda_0(s)}{1 - \lambda_0(s)} \{\mathbf{J}(s) - \mathbf{J}(0)\}\{\mathbf{B}_0 - m^{-1}\mathbf{B}_1\mathbf{J}_0\}\right] + \mathbf{Z}\{\mathbf{B}_0 - m^{-1}\mathbf{B}_1\mathbf{J}_0\}.$$

Hence

(5.6) $\mathbf{a}_1 = m^{-1}\mathbf{J}(0)\{\frac{1}{2}m^{-1}\mathbf{B}_2\mathbf{J}(0) - \mathbf{B}_1\} + \{m^{-1}\mathbf{J}'(0) + \mathbf{Z}\}\{\mathbf{B}_0 - m^{-1}\mathbf{B}_1\mathbf{J}(0)\}.$

If we now substitute the expression (4.5) for $\mathbf{J}'(0)$ into (5.6) and again use (5.5) we find that

(5.7) $\mathbf{a}_1 = m^{-1}\mathbf{J}_0\{\frac{1}{2}m^{-1}\mathbf{B}_2\mathbf{J}_0 - \mathbf{B}_1\} + \{\mathbf{Z} - m^{-1}\mathbf{J}_0\mathbf{B}_1\mathbf{Z}\}\{\mathbf{B}_0 - m^{-1}\mathbf{B}_1\mathbf{J}_0\}.$

6. The mean and variance of the regeneration time distributions. Let us suppose that at $t = 0$, the semi-renewal process has just experienced a renewal event of class $\mathcal{E}_j$. Let $R_j(x)$ be the regeneration time distribution for events of class j, i.e., the distribution of times elapsing between successive events of class $\mathcal{E}_j$. Then, from (1.2) and (1.3),

(6.1) $$H_{jj}(t) = \sum_{k=1}^{\infty} R_j^{(k)}(t) = m^{-1}e_jt + a_{1jj} + \epsilon_{jj}(t).$$

By comparison with (1.1) we have for the mean m_j and variance σ_j^2 of $R_j(x)$

(6.2) $$m_j = m/e_j$$

and $(\sigma_j^2 - m_j^2)/2m_j^2 = a_{1jj}$. Hence $\sigma_j^2 = m^2e_j^{-2}(1 + 2a_{1jj})$.
From (1.4) we then have the desired expression for σ_j^2.

7. The mean time from an epoch of class $\mathcal{E}_i$ to the next epoch of class $\mathcal{E}_j$. A comparison of (2.1) and (2.2) leads to the simple equation

(7.1) $$E[\tau_{ij}]/m_j = a_{1jj} - a_{1ij} + 1,$$

where $m_j = m/e_j$ as in Section 6, and a_{1ij} is again given by (1.4). We note that $(\mathbf{J}_0\mathbf{Q})_{jj} - (\mathbf{J}_0\mathbf{Q})_{ij} = 0$ for any matrix Q. Hence from (1.4), (7.1) simplifies to

(7.2) $$E[\tau_{ij}] = m_j(q_{jj} - q_{ij} + 1)$$

where q_{ij} is the ijth element of the matrix

(7.3) $$\mathbf{Q} = \mathbf{Z}\{\mathbf{B}_0 - m^{-1}\mathbf{B}_1\mathbf{J}_0\}.$$

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