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ON THE DISTRIBUTIONS OF THE RATIOS OF THE ROOTS OF A COVARIANCE MATRIX AND WILKS' CRITERION FOR TESTS OF THREE HYPOTHESES[1]

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1. Introduction. Let $\mathbf{X}(p \times n)$ be a matrix variate with columns independently distributed as $N(\mathbf{0}, \mathbf{\Sigma})$. Then the distribution of the latent roots, $0 < w_1 \leqq \cdots \leqq w_p < \infty$, of $\mathbf{XX}'$ is first considered in this paper for deriving the distributions of the ratios of individual roots $w_i/w_j (i < j = 2, \cdots, p)$. In particular, the distributions of such ratios are derived for $p = 2$, 3 and 4. The use of these ratios in testing the hypothesis $\delta\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2$, $\delta > 0$ unknown, has been pointed out, where $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ are the covariance matrices of two p-variate normal populations.

Further, the non-central distributions of Wilks' criterion, $\Lambda = W^{(p)} = \prod_{i=1}^p (1 - c_i)$, are obtained in the following cases: (1) test of $\mathbf{\Sigma}_1 = \delta\mathbf{\Sigma}_2$, $\delta > 0$ known, (2) MANOVA and (3) Canonical correlation, where c_i's stand for latent roots of a matrix arising in each of the situations. The density functions are given in terms of Meijer's G-function [12] and for $p = 2$, the density and distribution functions are explicitly evaluated. For Case (2), Pillai and Al-Ani [15] have derived the density for $p = 2$, 3 and 4 using some results on Mellin transforms [2, 3, 4], and Jouris [9] has shown by induction that the G-function can be expressed in an alternate form than given in the paper; this latter form includes as special cases the results of Pillai and Al-Ani [15].

2. Distribution of ratios of the roots of a covariance matrix. The distribution of the latent roots, $0 < w_1 \leqq w_2 \leqq \cdots \leqq w_p < \infty$ of $\mathbf{XX}'$ depends only upon the latent roots of $\mathbf{\Sigma}$ and can be given in the form (James [6])

$$(2.1)\quad K(p, n)|\mathbf{\Sigma}|^{-\frac{1}{2}n} |\mathbf{W}|^m \{\exp(-\tfrac{1}{2}\operatorname{tr}\mathbf{W})\} \cdot \textstyle\prod_{i>j}(w_i - w_j)\, {}_0F_0(\tfrac{1}{2}(\mathbf{I}_p - \mathbf{\Sigma}^{-1}), \mathbf{W}), \quad 0 < w_1 \leqq w_2 \leqq \cdots \leqq w_p < \infty,$$

where

$$m = \tfrac{1}{2}(n - p - 1), \qquad K(p, n) = \Pi^{\frac{1}{2}p^2}/\{2^{\frac{1}{2}pn}\Gamma_p(\tfrac{1}{2}n)\Gamma_p(\tfrac{1}{2}p)\},$$

$$\mathbf{W} = \operatorname{diag}(w_1, \cdots, w_p),$$

$$(2.2)\quad {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; \mathbf{S}, \mathbf{T}) = \textstyle\sum_{k=0}^\infty \sum_\kappa [(a_1)_\kappa \cdots (a_p)_\kappa]/[(b_1)_\kappa \cdots (b_q)_\kappa] \cdot C_\kappa(\mathbf{S})C_\kappa(\mathbf{T})/[C_\kappa(\mathbf{I}_p)k!]$$

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where $a_1, \cdots, a_p, b_1, \cdots, b_q$ are real or complex constants and the multivariate coefficient $(a)_\kappa$ is given by $(a)_\kappa = \prod_{i=1}^{p} (a - \frac{1}{2}(i - 1))_{k_i}$, where $(a)_k = a(a + 1) \cdots (a + k - 1)$. The partition κ of k is such that $\kappa = (k_1, k_2, \cdots, k_p)$, $k_1 \geqq k_2 \geqq \cdots \geqq k_p \geqq 0$, $k_1 + k_2 + \cdots + k_p = k$ and the zonal polynomials, $C_\kappa(\mathbf{S})$, are expressible in terms of elementary symmetric functions (esf) of the latent roots of $\mathbf{S}$, James [7].

It may be pointed out that the form (2.1) can also be viewed as a limiting form of the non-central distribution of the latent roots Khatri [10] associated with the test of the hypothesis: $\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2$, where $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ are the covariance matrices of two p-variate normal populations, when $n_2 \to \infty$, where n_2 is the size of the sample from the second population. Now, if we wish to test instead the null hypothesis $\delta\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2$, $\delta > 0$ unknown, the ratios of the latent roots would be of interest as test criteria. In this context, in the limiting form (2.1), $\mathbf{\Sigma}$ should be replaced by $\delta\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1}$.

Now, let $l_i = w_i/w_p$, $i = 1, \cdots, p - 1$, then the distribution of $l_1, \cdots, l_p, w_p$ can be written in the form

$$(2.3) \quad K(p, n)|\mathbf{\Sigma}|^{-\frac{1}{2}n} w_p^{\frac{1}{2}pn-1} |\mathbf{L}|^m |\mathbf{I} - \mathbf{L}| \prod_{i>j} (l_i - l_j) \exp -\tfrac{1}{2}(w_p \operatorname{tr} \mathbf{L}_1) \cdot [\sum_{k=0}^{\infty} w_p^k/(2^k k!) \sum_\kappa C_\kappa(\mathbf{I}_p - \mathbf{\Sigma}^{-1}) C_\kappa(\mathbf{L}_1)/C_\kappa(\mathbf{I}_p)],$$

where $\mathbf{L} = \operatorname{diag}(l_1, \cdots, l_{p-1})$ and $\mathbf{L}_1 = \operatorname{diag}(l_1, \cdots, l_{p-1}, 1)$. Integrating (2.3) with respect to w_p, then the distribution of $l_1, \cdots, l_{p-1}$ is of the form

$$(2.4) \quad K_1(p, n)|\mathbf{\Sigma}|^{-\frac{1}{2}n} |\mathbf{L}|^m |\mathbf{I} - \mathbf{L}| \prod_{i>j} (l_i - l_j) \cdot [\sum_{k=0}^{\infty} \Gamma(\tfrac{1}{2}pn + k)/k! \sum_\kappa C_\kappa(\mathbf{I}_p - \mathbf{\Sigma}^{-1}) C_\kappa(\mathbf{L}_1)/\{C_\kappa(\mathbf{I}_p)(\operatorname{tr} \mathbf{L}_1)^{\frac{1}{2}pn+k}\}],$$

where $K_1(p, n) = 2^{\frac{1}{2}pn} K(p, n)$. An expansion similar to the above but in a slightly different form has been given by James (See (5.2) and (5.6) of [8]).

CASE 1. Let $p = 2$ in (2.4); then the distribution of $l = w_1/w_2$ is of the form

$$(2.5) \quad K_1(2, n)|\mathbf{\Sigma}|^{-\frac{1}{2}n} l^{\frac{1}{2}(n-3)} (1 - l) \cdot [\sum_{k=0}^{\infty} \Gamma(n + k)/\{k!\,(1 + l)^{n+k}\} \sum_\kappa C_\kappa(\mathbf{I}_2 - \mathbf{\Sigma}^{-1}) C_\kappa\begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}/C_\kappa(\mathbf{I}_2)].$$

Girshick [5] has given the distribution of $L_e = 2l^{\frac{1}{2}}/(1 + l)$, which takes a simpler form.

CASE 2. Putting $p = 3$ in (2.4) and by the use of the results of Khatri and Pillai [11], the distribution of l_1, l_2 can be written in the form

$$(2.6) \quad K_1(3, n)|\mathbf{\Sigma}|^{-\frac{1}{2}n} (l_1 l_2)^{\frac{1}{2}(n-4)} (l_2 - l_1)(1 - l_1)(1 - l_2) \cdot [\sum_{k=0}^{\infty} \Gamma(a_k)/k! \sum_\kappa C_\kappa(\mathbf{I}_3 - \mathbf{\Sigma}^{-1})/C_\kappa(\mathbf{I}_3) \cdot \sum_{i=0}^{k} \sum_\eta b_{\eta,\kappa} C_\eta\begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix} \sum_{r=0}^{\infty} \binom{-a_k}{r} l_1^r (1 + l_2)^{-r-a_k}],$$

where $a_k = (3n/2) + k$, $b_{\eta,\kappa}$ are the constants defined in [11], and η is the partition of i into not more than p elements.

It may be noted that the distribution of l_1 and of l_2 can be found by writing $C_\eta\begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix} = \sum_{i_1+i_2=i} a_{i_1,i_2} l_1^{i_1} l_2^{i_2}$ and expanding $(1 + l_2)^{-r-a_k}$ and integrating l_2 and l_1 respectively.

In (2.6) let $h_1 = l_1/l_2$ from which the distribution of h_1, l_2 can readily be found. Integration with respect to l_2 yields

$$
\begin{aligned}
(2.7)\qquad & K_1(3, n (|\Sigma|^{-\frac{1}{2}n} h_1^{\frac{1}{2}(n-4)} (1 - h_1) \\
& \cdot[\textstyle\sum_{k=0}^{\infty} \Gamma(a_k)/k! \sum_\kappa C_\kappa(\mathbf{I}_3 - \Sigma^{-1})/C_\kappa(\mathbf{I}_3) \sum_{i=0}^{k} \sum_\eta b_{\eta,\kappa} C_\eta\begin{pmatrix} h_1 & 0 \\ 0 & 1 \end{pmatrix} \\
& \cdot \textstyle\sum_{r=0}^{\infty} \binom{-a_k}{r} h_1^r \sum_{h=0}^{\infty} \binom{-r-a_k}{h} \{\beta(a_1', 2) - h_1\beta(a_1' + 1, 2)\}],
\end{aligned}
$$

where $a_1' = n - 1 + i + r + h$.

CASE 3. Let $p = 4$ in (2.4), then the distribution of l_1, l_2, l_3 can be written in the form

$$
\begin{aligned}
(2.8)\qquad & K_1(4, n)|\Sigma|^{-\frac{1}{2}n} \textstyle\prod_{i=1}^{3} \{l_i^{\frac{1}{2}(n-5)}(1 - l_i)\} \prod_{i>j} (l_i - l_j) \\
& \cdot[\textstyle\sum_{k=0}^{\infty} \Gamma(2n + k)/\{k!\,(1 + l_1 + l_2 + l_3)^{2n+k}\} \\
& \cdot \textstyle\sum_\kappa C_\kappa(\mathbf{I}_4 - \Sigma^{-1})/C_\kappa(\mathbf{I}_4) \sum_{i=0}^{k} \sum_\eta b_{\kappa,\eta} C_\eta(\mathbf{L})],
\end{aligned}
$$

where $\mathbf{L} = \operatorname{diag}(l_1, l_2, l_3)$.

Now, in (2.8) let $h_i = l_i/l_3$, $i = 1, 2$ and integrate l_3 from 0 to 1, then the distribution of h_1, h_2 can be obtained as a series involving zonal polynomials of $\mathbf{H}_1 = \operatorname{diag}(h_1, h_2, 1)$. Further, from this series the distribution of h_1 or h_2 can be found using the method outlined in Pillai and Al-Ani [14] and integrating with respect to h_2 or h_1 such that $0 < h_1 \leqq h_2 < 1$.

Now, in the joint distribution of h_1, h_2 let $h_1' = h_1/h_2$, then the distribution of h_1' can be written in the form

$$
\begin{aligned}
(2.9)\qquad & K_1(4, n)|\Sigma|^{-\frac{1}{2}n} h_1'^{\frac{1}{2}(n-5)} (1 - h_1') \\
& \cdot \textstyle\sum_{k=0}^{\infty} \Gamma(2n + k)/k! \sum_\kappa C_\kappa(\mathbf{I}_4 - \Sigma^{-1})/C_\kappa(\mathbf{I}_4) \\
& \cdot \textstyle\sum_{i=0} \sum_\eta b_{\kappa,\eta} \sum_{t=0}^{i} \sum_\tau b'_{i,\tau} C_\tau\begin{pmatrix} h_1' & 0 \\ 0 & 1 \end{pmatrix} \sum_{r=0}^{\infty} \binom{-2n-k}{r} (1 + h_1')^r \\
& \cdot \textstyle\sum_{h=0}^{\infty} \binom{-2n-k-r}{h} \{\beta(b, 2)\beta(C, 2) + h_1'[\beta(C + 2, 2)\beta(b + 2, 2) \\
& - \beta(C + 1, 2)\beta(b, 2)] + (1 + h_1')\beta(b + 1, 2)(h_1'\beta(C + 2, 2) \\
& - \beta(C + 1, 2)) - h_1'^2\beta(b + 2, 2)\beta(C + 3, 2)\},
\end{aligned}
$$

where $b = 3(n - 1)/2 + i + h + r$, $C = n - 2 + t + r$ and constants $b'_{i,\tau}$ and τ are defined in [11].

3. Preliminaries in connection with Wilks' criterion. The non-central distributions of Wilks' criterion for the three cases mentioned in the Introduction will be obtained in the following sections in terms of Meijer's G-function.

Meijer [12] defined the G-function by

$$G_{p,q}^{m,n}\left(x \,\middle|\, {a_1,a_2,\cdots,a_p \atop b_1,b_2,\cdots,b_q}\right) = (2\pi i)^{-1} \int_C [\textstyle\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)] / [\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)] x^s \, ds, \tag{3.1}$$

where an empty product is interpreted as unity and where C is a curve separating the singularities of $\prod_{j=1}^m \Gamma(b_j - s)$ from those of $\prod_{j=1}^n \Gamma(1 - a_j + s)$, $q \geqq 1$, $0 \leqq n \leqq p \leqq q$, $0 \leqq m \leqq q$; $x \neq 0$ and $|x| < 1$ if $q = p$; $x \neq 0$ if $q > p$. It has been shown that [2]

$$G_{2,2}^{2,0}\left(x \,\middle|\, {a_1,a_2 \atop b_1,b_2}\right) = x^{b_1}(1 - x)^{a_1+a_2-b_1-b_2-1}/\Gamma(a_1 + a_2 - b_1 - b_2) \cdot {}_2F_1(a_2 - b_2, a_1 - b_2; a_1 + a_2 - b_1 - b_2; 1 - x), \qquad 0 < x < 1, \tag{3.2}$$

where ${}_2F_1$ here is the Gauss hypergeometric function. The G-function of (3.1) can be expressed as a finite number of generalized hypergeometric functions as follows: [13]

$$\begin{aligned} G_{p,q}^{m,n}\left(x \,\middle|\, {a_1,\cdots,a_p \atop b_1\cdots,b_q}\right) &= \textstyle\sum_{h=1}^m [\prod_{j=1,j\neq h}^m \Gamma(b_j - b_h) \prod_{j=1}^n \Gamma(1 + b_h - a_j)] / \\ &\quad [\textstyle\prod_{j=m+1}^q \Gamma(1 + b_h - b_j) \prod_{j=n+1}^p \Gamma(a_j - b_h)] x^{b_h} \\ &\quad \cdot {}_pF_{q-1}(1 + b_h - a_1, \cdots, 1 + b_h - a_p; \\ &\quad 1 + b_h - b_1, \cdots {}^* \cdots, 1 + b_h - b_q; \ (-1)^{p-m-n} x), \end{aligned} \tag{3.3}$$

where the asterisk denotes that the number $1 + b_h - b_h$ is omitted in the sequence $1 + b_h - b_1, \cdots, 1 + b_h - b_q$.

The above results on G-function will be used in the sequel.

4. The non-central distribution of $W^{(p)}$ in Case 1. Let $\mathbf{X}(p \times n_1)$ and $\mathbf{Y}(p \times n_2)$, $p \leqq n_i$, $i = 1, 2$, be independent matrix variates with the columns of $\mathbf{X}$ independently distributed as $N(0, \mathbf{\Sigma}_1)$ and those of Y independently distributed as $N(0, \mathbf{\Sigma}_2)$. Hence $\mathbf{S}_1 = \mathbf{XX}'$ and $\mathbf{S}_2 = \mathbf{YY}'$ are independently distributed as Wishart $(n_i, p, \mathbf{\Sigma}_i)$, $i = 1, 2$. Let $0 < f_1 < f_2 < \cdots < f_p < \infty$ be the latent roots of the determinantal equation

$$|\mathbf{S}_1 - f\mathbf{S}_2| = 0 \tag{4.1}$$

and $0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_p < \infty$ be the characteristic roots of

$$|\mathbf{\Sigma}_1 - \lambda \mathbf{\Sigma}_2| = 0. \tag{4.2}$$

For testing the hypothesis $H_0: \delta\mathbf{\Lambda} = \mathbf{I}_p$, $\delta > 0$ being given, we will use

$$W^{(p)} = \textstyle\prod_{i=1}^p (1 - e_i) = |\mathbf{I}_p - \mathbf{E}| \tag{4.3}$$

where $\mathbf{\Lambda} = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_p)$, $e_i = \delta f_i/(1 + \delta f_i)$, $i = 1, 2, \cdots, p$, and $\mathbf{E} = \operatorname{diag}(e_1, \cdots, e_p)$.

To find $E[W^{(p)}]^h$ we multiply the density of $\mathbf{E}$ given by Khatri [10] by $|\mathbf{I}_p - \mathbf{E}|^h$, transform $\mathbf{E} \to \mathbf{HVH}'$, where $\mathbf{H}$ is an orthogonal and $\mathbf{V}$ a symmetric matrix, and integrate out $\mathbf{H}$ and $\mathbf{V}$ using (44) and (22) of Constantine [1]. We get

$$(4.4)\quad E[W^{(p)}]^h = [\Gamma_p(\tfrac{1}{2}n)\Gamma_p(\tfrac{1}{2}n_2 + h)]/[\Gamma_p(\tfrac{1}{2}n_2)\Gamma_p(\tfrac{1}{2}n + h)]\,|\delta\mathbf{\Lambda}|^{-\frac{1}{2}n_1} \cdot {}_2F_1(\tfrac{1}{2}n, \tfrac{1}{2}n_1; \tfrac{1}{2}n + h; \mathbf{I}_p - (\delta\mathbf{\Lambda})^{-1}),$$

where $n = n_1 + n_2$, and ${}_2F_1$ is a hypergeometric function of the matrix variate defined in (2.2). Using (2.2), the coefficient of $C_\kappa(\mathbf{I}_p - (\delta\mathbf{\Lambda})^{-1})$ in (4.4) is given by

$$(4.5)\qquad \{C_p(\tfrac{1}{2}n)_\kappa\, (\tfrac{1}{2}n_1)_\kappa \textstyle\prod_{i=1}^p \Gamma(r + b_i)\}/\{k! \prod_{i=1}^p \Gamma(r + a_i)\},$$

where $r = \frac{1}{2}n_2 + h - \frac{1}{2}(p - 1)$, $b_i = \frac{1}{2}(i - 1)$, $a_i = \frac{1}{2}n_1 + k_{p-i+1} + b_i$, and $C_p = \{\Gamma_p(\frac{1}{2}n)/\Gamma_p(\frac{1}{2}n_2)\}|\delta\mathbf{\Lambda}|^{-\frac{1}{2}n_1}$.

Now using results on inverse Mellin transform [2, 3, 4]

$$(4.6)\quad f(W^{(p)}) = C_p \textstyle\sum_{k=0}^\infty \sum_\kappa \{(\frac{1}{2}n)_\kappa(\frac{1}{2}n_1)_\kappa/k!\}\, C_\kappa(\mathbf{I}_p - (\delta\mathbf{\Lambda})^{-1})\{W^{(p)}\}^{\frac{1}{2}(n_2-p-1)} \cdot (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \{W^{(p)}\}^{-r} \prod_{i=1}^p \Gamma(r + b_i)/\prod_{i=1}^p \Gamma(r + a_i)]\, dr.$$

Noting that the integral in (4.6) is in the form of Meijer's G-function we can write the density of $W^{(p)}$ as

$$(4.7)\quad f(W^{(p)}) = C_p\{W^{(p)}\}^{\frac{1}{2}(n_2-p-1)} \cdot \textstyle\sum_{k=0}^\infty \sum_\kappa \{(\frac{1}{2}n)_\kappa(\frac{1}{2}n_1)_\kappa/k!\}\, C_\kappa(\mathbf{I}_p - (\delta\mathbf{\Lambda})^{-1}) G_{p,p}^{p,0}\left(W^{(p)} \,\middle|\, {a_1, a_2, \cdots, a_p \atop b_1, b_2, \cdots b_p}\right).$$

Special Case. Letting $p = 2$ in (4.7) and using (3.2) we obtain

$$(4.8)\quad \begin{aligned} f(W^{(2)}) &= C_2\{W^{(2)}\}^{\frac{1}{2}(n_2-3)} \textstyle\sum_{k=0}^\infty \sum_\kappa (\frac{1}{2}n)_\kappa(\frac{1}{2}n_1)_\kappa/k! \\ &\quad\cdot C_\kappa(I_2 - (\delta\mathbf{\Lambda})^{-1})\{1 - W^{(2)}\}^{n_1+k-1}/\Gamma(n_1 + k) \\ &\quad\cdot {}_2F_1(\tfrac{1}{2}n_1 + k_1, \tfrac{1}{2}(n_1 - 1) + k_2, n_1 + k; 1 - W^{(2)}). \end{aligned}$$

The probability that $W^{(2)} \leqq w(\leqq 1)$ can be obtained by integrating (4.8) by parts $a_1 = \frac{1}{2}n_1 + k_2$ times when n_1 is even. Using the relation [4]

$$(4.9)\quad (d^n/dz^n)[z^{c-1}\,{}_2F_1(a, b; c; z)] = (c - n)_n z^{c-n-1}\,{}_2F_1(a, b; c - n; z),$$

and recalling that $\kappa = (k_1, k_2)$, we obtain the cdf of $W^{(2)}$ as

$$(4.10)\quad \begin{aligned} &\Pr\{W^{(2)} \leqq w\} \\ &= |\delta\mathbf{\Lambda}|^{-\frac{1}{2}n_1} \textstyle\sum_{k=0}^\infty \sum_\kappa (\frac{1}{2}n_1)_\kappa C_\kappa(\mathbf{I}_2 - (\delta\mathbf{\Lambda})^{-1}) w^{\frac{1}{2}(n_2-1)}/k! \\ &\quad\cdot\{\Gamma_2(\tfrac{1}{2}n)(\tfrac{1}{2}n)_\kappa/[\Gamma_2(\tfrac{1}{2}n_2)\Gamma(n_1 + k)] \\ &\quad\cdot \textstyle\sum_{r=0}^a \{(n_1 + k - r)_r/\{\frac{1}{2}(n_2 - 1)\}_{r+1}\} w^r (1 - w)^{n_1+k-r-1} \\ &\quad\cdot {}_2F_1(\tfrac{1}{2}n_1 + k_1, \tfrac{1}{2}(n_1 - 1) + k_2, n_1 + k - r; 1 - w) + I_w(\tfrac{1}{2}n_2, b)\}, \end{aligned}$$

where $a = a_1 - 1$ and $b = a_2 - b_2$, $a_2 = \frac{1}{2}n_1 + k_1 + \frac{1}{2}$ and $b_2 = \frac{1}{2}$. When n_1 is odd, after integrating (4.8) by parts a_2 times, the cdf of $W^{(2)}$ is (4.10) with $a = a_2 - 1$ and $b = a_1 - b_2$.

5. The non-central distribution of $W^{(p)}$ in Case 2. Let $\Lambda = W^{(p)} = \prod_{i=1}^{p} (1 - g_i)$ where $g_1, g_2, \cdots, g_p$ are the latent roots of the determinantal equation

$$|\mathbf{S}_1 - g(\mathbf{S}_1 + \mathbf{S}_2)| = 0 \tag{5.1}$$

where $\mathbf{S}_1$ is a (p × p) matrix distributed as non-central Wishart with s degrees of freedom, $\mathbf{\Omega}$ is a matrix of non-centrality parameters and $\mathbf{S}_2$ has the Wishart distribution with t degrees of freedom, the covariance matrix in each case being $\mathbf{\Sigma}$. Constantine [1] has given the $E[W^{(p)}]^h$ in this case in the following form: (Writing $n = s + t$) $E[W^{(p)}]^h = \Gamma_p(h + \frac{1}{2}t)\Gamma_p(\frac{1}{2}n)/[\Gamma_p(\frac{1}{2}t)\Gamma_p(h + \frac{1}{2}n)] \cdot {}_1F_1(h; h + \frac{1}{2}n; -\Omega)$, and hence using (3.1)

$$\begin{aligned} (5.2)\quad & f(W^{(p)}) \\ &= C_p\{W^{(p)}\}^{\frac{1}{2}(t-p-1)} \sum_{k=0}^{\infty} \sum_{\kappa} \{(\tfrac{1}{2}n)_\kappa C_\kappa(\mathbf{\Omega})/k!\}\, G_{p,p}^{p,0}\left(W^{(p)} \,\middle|\, {}^{a_1, a_2, \cdots, a_p}_{b_1, b_2, \cdots, b_p}\right), \end{aligned}$$

where $C_p = \Gamma_p(\frac{1}{2}n)/\Gamma_p(\frac{1}{2}t) \exp(-\operatorname{tr} \mathbf{\Omega})$, $b_i = \frac{1}{2}(i - 1)$, $a_i = \frac{1}{2}s + k_{p-i+1} + b_i$. The probability that $W^{(2)} \leqq w(\leqq 1)$ can be obtained by using (3.2) in (5.2), integrating by parts a_1 times when s is even, then using (4.9) we get the cdf of $W^{(2)}$ as

$$\begin{aligned} & \Pr\{W^{(2)} \leqq w\} \\ &= \exp(-\operatorname{tr} \mathbf{\Omega} \textstyle\sum_{k=0}^{\infty} \sum_{\kappa} \{C_\kappa(\mathbf{\Omega})/k!\}\, w^{\frac{1}{2}(t-1)} \\ (5.3)\quad & \cdot\{\Gamma_2(\tfrac{1}{2}n)(\tfrac{1}{2}n)_\kappa/[\Gamma_2(\tfrac{1}{2}t)\Gamma(s + k)] \\ & \cdot \textstyle\sum_{r=0}^{a} \{(s + k - r)_r/\{\tfrac{1}{2}(t - 1)\}_{r+1}\} w^r (1 - w)^{s+k-r-1} \\ & \cdot {}_2F_1(\tfrac{1}{2}s + k_1, \tfrac{1}{2}(s - 1) + k_2; s + k - r; 1 - w) + I_w(\tfrac{1}{2}t, b)\} \end{aligned}$$

where $a = \frac{1}{2}s + k_2 - 1$, $b = \frac{1}{2}s + k_1$. When s is odd, we integrate (5.2) by parts a_2 times and find the cdf is (5.3) with $a = \frac{1}{2}s + k_1 - \frac{1}{2}$, $b = \frac{1}{2}(s - 1) + k_2$.

6. The non-central distribution of $W^{(p)}$ in Case 3. Let the columns of $\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}$ be independent normal $(p + q)$–variates ($p \leqq q$, $p + q \leqq n$, n is the sample size) with zero means and covariance matrix

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{12}' & \mathbf{\Sigma}_{22} \end{pmatrix}. \tag{6.1}$$

Let $\mathbf{R}^2 = \operatorname{diag}(r_1^2, r_2^2, \cdots, r_p^2)$ where r_i^2 are the latent roots of

$$|\mathbf{X}_1\mathbf{X}_2'(\mathbf{X}_2\mathbf{X}_2')^{-1}\mathbf{X}_2\mathbf{X}_1' - r^2\mathbf{X}_1\mathbf{X}_1'| = 0 \tag{6.2}$$

and $\mathbf{P}^2 = \operatorname{diag}(\rho_1^2, \rho_2^2, \cdots, \rho_p^2)$ where ρ_i^2 are the latent roots of

$$|\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{12}' - \rho^2\mathbf{\Sigma}_{11}| = 0. \tag{6.3}$$

The density of $r_1^2, r_2^2, \cdots, r_p^2$ has been obtained by Constantine [1] and to find $E[W^{(p)}]^h$ where $W^{(p)} = \prod_{i=1}^p (1 - r_i^2)$, we multiply that density by $|\mathbf{I}_p - \mathbf{R}^2|^h$, proceed as in Section 4 for Case 1 and we find

$$(6.4)\quad E[W^{(p)}]^h = \Gamma_p(\tfrac{1}{2}n)\Gamma_p(\tfrac{1}{2}(n-q)+h)/[\Gamma_p(\tfrac{1}{2}(n-q))\Gamma_p(\tfrac{1}{2}n+h)] \cdot |\mathbf{I}_p - \mathbf{P}^2|^{\frac{1}{2}n}\, {}_2F_1(\tfrac{1}{2}n, \tfrac{1}{2}n; \tfrac{1}{2}n+h; \mathbf{P}^2).$$

Noting that (6.4) can be obtained from (4.4) by substituting

$$(6.5)\qquad (n_2, n_1, (\delta\mathbf{\Lambda})^{-1}) \to (n-q, n, \mathbf{I}_p - \mathbf{P}^2)$$

it can be verified that the density of $W^{(p)}$ in this case is

$$(6.6)\quad f(W^{(p)}) = C_p\{W^{(p)}\}^{\frac{1}{2}(n-q-p-1)} \cdot \sum_{k=0}^{\infty}\sum_{\kappa}\{(\tfrac{1}{2}n)_\kappa(\tfrac{1}{2}n)_\kappa C_\kappa(\mathbf{P}^2)/k!\}\, G_{p,p}^{p,0}\left(W^{(p)} \,\middle|\, {}^{a_1,a_2,\cdots,a_p}_{b_1,b_2,\cdots,b_p}\right)$$

where

$$Cp = \{\Gamma_p(\tfrac{1}{2}n)/\Gamma_p(\tfrac{1}{2}(n-q))\}|\mathbf{I}_p - \mathbf{P}^2|^{\frac{1}{2}n},\quad a_i = \tfrac{1}{2}q + k_{p-i+1} + b_i,\quad b_i = \tfrac{1}{2}(i-1).$$

7. Remark. The densities of $W^{(p)}$ obtained above in the three cases can be put in a single general form given by

$$(7.1)\quad f(W^{(p)}) = \{\Gamma_p(\tfrac{1}{2}n)/\Gamma_p(\tfrac{1}{2}\gamma)\}\alpha\{W^{(p)}\}^{\frac{1}{2}(\gamma-p-1)} \cdot \sum_{k=0}^{\infty}\sum_{\kappa}\{(\tfrac{1}{2}n)_\kappa\beta/k!\}\, C_\kappa(\mathbf{M})G_{p,p}^{p,0}\left(W^{(p)} \,\middle|\, {}^{a_1,a_2,\cdots,a_p}_{b_1,b_2,\cdots,b_p}\right),$$

where $a_i = \frac{1}{2}(n-\gamma) + k_{p-i+1} + b_i$ and $b_i = \frac{1}{2}(i-1)$
and

	Case 1	Case 2	Case 3
$\gamma =$	n_2	t	$n-q$
$\beta =$	$(\frac{1}{2}n_1)_\kappa$	1	$(\frac{1}{2}n)_\kappa$
$\alpha =$	$\lvert\delta\mathbf{\Lambda}\rvert^{-\frac{1}{2}n_1}$	$\exp(-\mathrm{tr}\,\mathbf{\Omega})$	$\lvert\mathbf{I}_p - \mathbf{P}^2\rvert^{\frac{1}{2}n}$
$\mathbf{M} =$	$\mathbf{I}_p - (\delta\mathbf{\Lambda})^{-1}$	$\mathbf{\Omega}$	$\mathbf{P}^2$.

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