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SOME STRIKING PROPERTIES OF BINOMIAL AND BETA MOMENTS[1]

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1. Introduction. For each positive integer n, let M_n denote the convex body of n-tuples $(C_1, C_2, \cdots, C_n)$ with

$$C_i = \int_{[0,1]} x^i \, d\sigma(x), \qquad i = 1, 2, \cdots, n,$$

where σ is allowed to vary over the class of all probability measures on the Borel subsets of the unit interval [0, 1]. Detailed treatment of these spaces may be found in [2] and [3]. For the moment sequence $(C_1, C_2, \cdots)$ corresponding to an arbitrary σ, write

$$\nu_n(C_1, C_2, \cdots) = C_n$$
$$\nu_n^{\pm}(C_1, C_2, \cdots) = {}^{\max}_{\min} \{d: (C_1, C_2, \cdots, C_{n-1}, d) \, \varepsilon \, M_n\},$$

and take $R_n = \nu_n^+ - \nu_n^-$.

Let $M_n^0 (n > 1)$ denote that subset of M_n whose points $(C_1, \cdots, C_{n-1}, C_n)$ have $(C_1, \cdots, C_{n-1})$ interior to M_{n-1}; $M_1^0 = M_1$. In [5], we defined "normalized" moments $p_n = 1 - q_n$ on M_n^0 by taking

$$p_n = (\nu_n - \nu_n^-)/R_n. \tag{1.1}$$

Note that $p_1, p_2, \cdots, p_{n-1}$ may be viewed as functions on M_n^0 and in appropriate context below they will be so regarded (in this connection see Corollary 1.1, page 107 of [3] and its proof). In [5] it was proved that everywhere on M_n^0

$$R_{n+1} = \prod_{i=1}^{n} p_i q_i. \tag{1.2}$$

If we define the right hand side to be zero on $M_n - M_n^0$, (1.2) holds there as well. (1.2) and certain of its corollaries which appear in [6] (in particular Corollary 2) exhibit in an emphatic manner the fundamental nature of the normalization (1.1). The somewhat startling form of normalized binomial moments (Theorem 2), the direct connection between moments of even index and the distribution's support, and the simple form of normalized Beta moments (Theorem 3) tend strongly to reinforce this judgment.

In Section 4, two general theorems are given. The first exhibits the connection between two normalized moment sequences whose corresponding distributions on [0, 1] are symmetrically related. The second theorem exhibits the invariance of moment normalization under the standard 1–1 mapping that takes the class of all distributions on a finite interval $[a, b]$ onto the class of all distributions on [0, 1].

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The theorems proved in Sections 2 and 3 and some additional propositions there remarked upon rest squarely upon Theorems 1, 2 and Corollary 5 of [6]. We restate these here for convenience and completeness as a single theorem.

THEOREM 1. *Let*

$$1 - p_0 = q_0 = S_{0j} = 1, \qquad j = 1, 2, \cdots. \tag{1.3}$$

Take

$$\zeta_j = q_{j-1}p_j, \qquad j = 1, 2, \cdots, \tag{1.4}$$

and recursively define

$$S_{ij} = \sum_{k=i}^{j} \zeta_{k-i+1}S_{i-1,k}, \qquad j = i, i+1, \cdots; i = 1, 2, \cdots, \tag{1.5}$$

then the following functional equalities hold.

$$\nu_n = S_{nn} = \sum_{i=0}^{[n/2]} S_{i,n-i}^2 \prod_{j=1}^{n-2i} \zeta_j = \sum_{i=0}^{n-1} S_{i,n-1} \prod_{j=1}^{n-i} \zeta_j. \tag{1.6}$$

ν_n^- *is obtained by putting* $p_n = 0$ *in* S_{nn} *and deleting the term corresponding to index* $i = 0$ *in the two sums.* $[n/2]$ *denotes the integral part of* $n/2$.

Theorem 1 shows that the vector-valued onto map,

$$\mathbf{p}_n = (p_1, \cdots, p_n), \qquad \mathbf{p}_n: M_n^0 \to I_n^0,$$

where

$$I_n^0 = \{(a_1, \cdots, a_{n-1}, a_n): 0 < a_i < 1, i = 1, \cdots, n-1, \quad 0 \leqq a_n \leqq 1\},$$

is one-one, and in essence it exhibits the inverse map in three distinct forms. These yield three distinct identities on I_n^0.

Some additional remarks concerning Theorem 1 may be of interest. By Corollaries 4 and 1 of [6], we have for $i = 1, 2, \cdots$,

$$\zeta_i = (\nu_i - \nu_i^-)/(\nu_{i-1} - \nu_{i-1}^-), \qquad \prod_{j=1}^{i} \zeta_j = \nu_i - \nu_i^-,$$

(taking the denominator on the right hand side of the first equation to be 1 when $i = 1$). The equalities (1.6) derive in part from the fact that ν_n may be represented in the form

$$\nu_n = (\nu_{[\frac{1}{2}(n+1)]}, \cdots, \nu_n)V_n^{-1}(\nu_{[\frac{1}{2}(n+1)]}, \cdots, \nu_n)',$$

where V_n is the lower Hankel matrix of order n. (See the definition following Theorem 2 of [6]). Reduction of this quadratic form by a method due to Lagrange yields a sum whose form is essentially that of the sum on the right hand side of the second equation in (1.6). (See Theorem 3 of [6]). The squared terms in that sum are squares of ratios of certain minor determinants of the Hankel matrix. In Lemma 2 of [6], these ratios are shown to have the form $S_{i,n-i}$. The recursive nature of the S_{ij} is a reflection of the recursive nature of these ratios.

The author apologizes in advance for the inductive nature of the proofs of Lemmas 2 and 3 with the hope that the results (namely, Theorems 2 and 3) will compensate for the means by which they are derived. Of course, the results in question were discovered before they were proved.

2. The binomial moments.

LEMMA 1. *Let N be a positive integer, $0 \leqq \theta \leqq 1$. Define*

$$(2.1)\qquad C_i^*(N, \theta) = \sum_{j=0}^{N} (j/N)^i \binom{N}{j} \theta^j (1-\theta)^{N-j}, \qquad i = 1, 2, \cdots,$$

then

$$(2.2)\qquad C_i^*(N, \theta) = N^{-i} \sum_{k=1}^{i} g_i^{(k)} N^{(k)} \theta^k, \qquad i = 1, 2, \cdots$$

where

$$N^{(k)} = N(N-1) \cdots (N-k+1), \qquad k = 1, 2, \cdots$$

and the $g_i^{(k)}$ are Stirling numbers of the second kind.

PROOF. This may be seen directly if we expand $(1 - \theta)^{N-j}$ on the right hand side of (2.1), interchange the order of summation, and observe the well-known closed form for Stirling numbers of the second kind; see 24.1.4, A and C, page 824 of [4].

THEOREM 2. *Let N, θ, $C_i^*(N, \theta)$, $i = 1, 2, \cdots$ be given as in Lemma 1 and for each positive integer n, take*

$$\mathbf{C}_n^*(N, \theta) = (C_1^*(N, \theta), C_2^*(N, \theta), \cdots, C_n^*(N, \theta)).$$

We then have that

$$(2.3)\qquad p_{2i-1}(\mathbf{C}_{2i-1}^*(N, \theta)) = \theta, \qquad p_{2i}(\mathbf{C}_{2i}^*(N, \theta)) = i/N, \quad i = 1, 2, \cdots, N.$$

Thus $C_n^*(N, \theta)$ is interior to M_n for $n = 1, 2, \cdots, 2N - 1$, on the boundary of M_n for $n = 2N, 2N + 1, \cdots$; and the binomial (N, θ) law with support at 0, $1/N$, $2/N$, $\cdots$, 1 is the unique distribution on the unit interval with moments $\mathbf{C}_{2N}^*(N, \theta)$.

PROOF. Let

$$(2.4)\qquad \mathbf{p}_{2N}^*(N, \theta) = (\theta, 1/N, \theta, 2/N, \cdots, \theta, 1).$$

In view of Theorem 1, (2.3) will be proved if we show that

$$(2.5)\qquad \mathbf{C}_{2N}^*(N, \theta) = \mathbf{p}_{2N}^{-1}(\mathbf{p}_{2N}^*(N, \theta)).$$

But this result (using Theorem 1 and Lemma 1) is in essence Corollary 1 to Lemma 2 that we prove below. (2.5) and (2.4) imply moreover that $\mathbf{C}_n^*(N, \theta)$ is interior to or on the boundary of M_n according as n is less than or n is equal to $2N$. It follows (see Theorem 20.1 on page 64 of [2]) that it is also on the boundary for $n > 2N$, and that the uniqueness property in the statement of the theorem holds.

Without further comment we note at this point the striking fact that by the above theorem the set of even indexed normalized binomial moments (taking the normalized moment with index zero to be zero) is precisely the support of the distribution on [0, 1]. We observe moreover that the even indexed normalized moments are independent of θ; the odd, independent of N.

Throughout the following we hold N and θ fixed (as given in Lemma 1). Let

$$p_{2i-1}^* = \theta, \quad p_{2i}^* = i/N, \qquad i = 1, 2, \cdots, N \tag{2.6}$$

Take

$$p_0^* = 0, \quad \zeta_k^* = (1 - p_{k-1}^*)p_k^*, \qquad k = 1, 2, \cdots$$

following (1.3), (1.4), so that by (2.6)

$$\zeta_{2k-1}^* = (N - k + 1)\theta/N, \quad \zeta_{2k}^* = k(1 - \theta)/N, \qquad k = 1, 2, \cdots, N, \tag{2.7}$$

and similarly, following (1.3), (1.4), (1.5), let S_{ij}^* (for $j = i, i + 1, \cdots, 2N$; $i = 1, 2, \cdots, 2N$) denote the values of S_{ij} that correspond. Take $S_{ij}^* = 0$, when $j < i$. For integers m and n we adopt the conventions

$$g_n^{(m)} = 0 = \binom{n}{m} \quad \text{when} \quad 0 < n < m \quad \text{or} \quad m < 0 < n \tag{2.8}$$

and

$$n^{(-1)} = n^{(0)} = 1. \tag{2.9}$$

A summation is taken to be zero when the upper limit of its index is smaller than the lower. We shall have occasion below to use the well-known relationship

$$g_{n+1}^{(m)} = mg_n^{(m)} + g_n^{(m-1)}, \qquad 1 \leqq m \leqq n, \tag{2.10}$$

and the simple facts that

$$g_n^{(n)} = 1, \quad g_n^{(n-1)} = \binom{n}{2} \tag{2.11}$$

(See page 825 of [4].)

LEMMA 2. *For* $i = 1, 2, \cdots, 2N$,

$$N^i S_{i,i+2j}^* = \sum_{k=0}^{i} \binom{j+k}{k} g_{j+1}^{(j+k)} (N - j)^{(k)} \theta^k, \qquad j = 0, 1, \cdots, [N - \tfrac{1}{2}i]. \tag{2.12}$$

$$N^i S_{i,i+2j-1}^* = \sum_{k=0}^{i} \binom{j+k-1}{k} g_{j+i}^{(j+k)} (N - j)^{(k)} \theta^k, \qquad j = 1, 2, \cdots, [N - \tfrac{1}{2}i - \tfrac{1}{2}]. \tag{2.13}$$

PROOF. By the definition (1.5)

$$S_{1,2j}^* = \sum_{k=1}^{j} (\zeta_{2k-1}^* + \zeta_{2k}^*), \qquad j = 1, 2, \cdots, N.$$

Hence by (2.7)

$$NS_{1,2j}^* = \sum_{k=1}^{j} [(N - k + 1)\theta + k(1 - \theta)] = \binom{j+1}{2} + j(N - j)\theta.$$

Moreover

$$NS_{1,1+2j}^* = N(S_{1,2j}^* + \zeta_{2j+1}^*) = \binom{j+1}{2} + (j + 1)(N - j)\theta.$$

Thus, by (2.11), the lemma holds when $i = 1$. Now suppose the lemma to hold for an arbitrary positive integer i less than $2N$. We show first that (2.13) and then that (2.12) must hold for $i + 1$.

Using (1.5), we have for

(2.14) $$j = 0, 1, 2, \cdots, [N - \tfrac{1}{2}i]; \qquad i = 1, 2, \cdots, 2N - 1,$$

that

$$S^*_{i+1,i+2j} = \sum_{l=1}^{j} (\zeta^*_{2l-1} S^*_{i,i+2l-1} + \zeta^*_{2l} S^*_{i,i+2l})$$

Using (2.7) and multiplying both sides by N, we get

$$NS^*_{i+1,i+2j} = \sum_{l=1}^{j} \{lS^*_{i,i+2l} + \theta[(N - l + 1)S^*_{i,i+2l-1} - lS^*_{i,i+2l}]\}.$$

If we multiply both sides by N^i, use the induction hypothesis, and change the order of summation, this equation takes the form

(2.15) $$N^{i+1}S^*_{i+1,i+2j} = \sum_{k=0}^{i+1} (\sum_{l=1}^{j} \psi^{(i)}_{k,l})\theta^k,$$

where for i, j satisfying (2.14) and for

$$l = 1, 2, \cdots, j; \qquad k = 0, 1, \cdots, i + 1,$$

and mindful of the conventions (2.8) and (2.9),

(2.16) $$\psi^{(i)}_{k,l} = l\binom{l+k}{k} g^{(l+k)}_{l+i} (N - l)^{(k)} + [(N - l + 1)\binom{l+k-2}{k-1} - l\binom{l+k-1}{k-1}] g^{(l+k-1)}_{l+i} (N - l)^{(k-1)}.$$

In view of (2.15), it will suffice (in order to prove (2.13) with i replaced by $i + 1$) to show that for i, j which satisfy (2.14)

(2.17) $$\sum_{l=1}^{j} \psi^{(i)}_{kl} = \binom{j+k-1}{k} g^{(j+k)}_{j+i+1} (N - j)^{(k)}, \qquad k = 0, 1, \cdots, i + 1.$$

But to prove (2.17) we need only show that for all the indices in question

$$\psi^{(i)}_{k,l} = \binom{l+k-1}{k} g^{(l+k)}_{l+i+1} (N - l)^{(k)} - \binom{l+k-2}{k} g^{(l+k-1)}_{l+i} (N - l + 1)^{(k)}.$$

By (2.16) and after simple manipulation, this equation takes the form

$$[(N - l + 1)\binom{l+k-1}{k} - l\binom{l+k-1}{k-1}] g^{(l+k-1)}_{l+i} = (N - l - k + 1)\binom{l+k-1}{k}[g^{(l+k)}_{l+i+1} - (l + k) g^{(l+k)}_{l+i}].$$

It may easily be seen that

$$(N - l + 1)\binom{l+k-1}{k} - l\binom{l+k-1}{k-1} = (N - l - k + 1)\binom{l+k-1}{k}.$$

But then the desired equality holds by (2.10). Thus (2.17) and hence (2.13) (with i replaced by $i + 1$) is true.

Finally, using (1.5), we have

$$S^*_{i+1,i+1+2j} = S^*_{i+1,i+2j} + \zeta^*_{2j+1} S^*_{i,i+1+2j},$$

so that using the above result together with the induction hypothesis we have

$$N^{i+1}S^*_{i+1,i+1+2j} = \sum_{k=0}^{i+1} \binom{j+k-1}{k} g^{(j+k)}_{j+i+1} (N - j)^{(k)}\theta^k + (N - j)\theta \sum_{k=0}^{i} \binom{j+k}{k} g^{(j+k+1)}_{j+i+1} (N - j - 1)^{(k)}\theta^k.$$

This easily yields the desired result that (2.12) is true with i replaced by $i + 1$. □

COROLLARY 1.

$$N^i S_{i,i}^* = \sum_{k=1}^{i} g_i^{(k)} N^{(k)} \theta^k, \qquad i = 1, 2, \cdots, 2N.$$

PROOF. Put $j = 0$ in (2.12) and note that $g_i^{(0)} = 0$, $i > 0$.

COROLLARY 2. *Let N, θ be given as in Lemma 1. Then for $m = 1, 2, \cdots, N$, we have that*

$$R_{2m}(\mathbf{C}_{2m-1}^*(N, \theta)) = N^{-2m+1} N^{(m)} (m-1)!\,[\theta(1-\theta)]^m$$

$$R_{2m+1}(\mathbf{C}_{2m}^*(N, \theta)) = N^{-2m-1} N^{(m+1)} m!\,[\theta(1-\theta)]^m.$$

PROOF. This is an immediate consequence of (1.2) and Theorem 2.

Note that by (1.1), Theorem 2, and the above corollary

$$(2.18) \qquad C_{2m}^*(N, \theta) - \bar{\nu}_{2m}(\mathbf{C}_{2m-1}^*(N, \theta)) = N^{-2m} N^{(m)} m!\,[\theta(1-\theta)]^m$$

$$(2.19) \qquad C_{2m+1}^*(N, \theta) - \bar{\nu}_{2m+1}(\mathbf{C}_{2m}^*(N, \theta)) = N^{-2m-1} N^{(m+1)} m!\,\theta^{m+1}(1-\theta)^m.$$

As an unlooked for bonus which is derived from Lemma 2 used in conjunction with Theorem 1, one may obtain several families of identities which involve Stirling numbers of the second kind. One such family (see (2.17) and (2.16)) has already been demonstrated in the process of proving Lemma 2. Other identities may be obtained by substituting the results of Lemmas 1 and 2 into the functional equations of Theorem 1. For example, let $n = 2m$. By the second equality of (1.6),

$$C_{2m}^*(N, \theta) = \sum_{i=0}^{m} S_{m-i,m+i}^{*2} \prod_{s=1}^{2i} \zeta_s^*.$$

By (2.18) and corollary 1 of [6],

$$\prod_{s=1}^{2i} \zeta_s^* = N^{-2i} N^{(i)} i!\,[\theta(1-\theta)]^i.$$

Thus by (2.2) and (2.12) we have for $m = 1, 2, \cdots, N; N = 1, 2, \cdots; 0 \leqq \theta \leqq 1$, that

$$(2.20) \qquad \sum_{i=1}^{2m} g_{2m}^{(i)} N^{(i)} \theta^i$$

$$= \sum_{i=0}^{m} \left(\sum_{k=0}^{m-i} \binom{i+k}{k} g_m^{(i+k)} (N-i)^{(k)} \theta^k\right)^2 i!\, N^{(i)} \theta^i (1-\theta)^i.$$

Note that if we delete the terms with index $i = 0$ on the right hand side of (2.20), the resulting expression is by Theorem 1 precisely $N^{2m} \bar{\nu}_{2m}(\mathbf{C}_{2m-1}^*(N, \theta))$.

If we equate coefficients of equal powers of θ on both sides of (2.20) this yields for $i = 1, \cdots, 2m$

$$(2.21) \qquad g_{2m}^{(i)} N^{(i)} = \sum_{j=\max(0,i-m)}^{\min(i,m)} \sum_{k=0}^{j} \sum_{l=\max(0,i-j-k)}^{\min(i-j,m-k)} T_{j,k,l}^{(2m,i)}$$

where

$$T_{j,k,l}^{(2m,i)} = (-1)^{i+j+l} \frac{[j^{(k)}(k+l)^{(k)}]}{[(i-j-l)!(j+k+l-i)!]} N^{(j)} (N-k)^{(l)} g_m^{(j)} g_m^{(k+l)}.$$

For example, if we take $i = 2m$, (2.21) yields as a special case that

$$N^{(2m)} = N^{(m)} \sum_{k=0}^{m} (-1)^k \binom{m}{k} m^{(k)} (N-k)^{(m-k)}.$$

This may in turn be put in the form

$$\binom{N-m}{m} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{N-k}{m-k},$$

which is itself a special case of (12.15), page 62 of [1].

Other families of identities analogous to (2.20) and (2.21) may be obtained by starting with $n = 2m - 1$ or by using the other relationships of Theorem 1.

3. The beta moments. Let a and b denote positive real numbers. Define

$$\hat{C}_i(a, b) = \Gamma(a+b)\{\Gamma(a)\Gamma(b)\}^{-1} \int_0^1 x^{i+a-1}(1-x)^{b-1}\, dx, \qquad i = 1, 2, \cdots$$

where Γ denotes the Gamma function. It is well-known and easily seen that these moments take the form

$$\hat{C}_i(a, b) = \prod_{k=0}^{i-1} \{(a+k)(a+b+k)^{-1}\}, \qquad i = 1, 2, \cdots. \tag{3.1}$$

THEOREM 3. *For each positive integer n, take*

$$\hat{\mathbf{C}}_n(a, b) = (\hat{C}_1(a, b), \hat{C}_2(a, b), \cdots, \hat{C}_n(a, b))$$

then

$$p_n(\hat{\mathbf{C}}_n(a, b)) = \tfrac{1}{2}(1 - \epsilon_n(a, b)), \qquad n = 1, 2, \cdots,$$

where

$$\begin{aligned}(a + b - 1 + n)\epsilon_n(a, b) &= b - a && \textit{when } n \textit{ is odd,}\\ &= a + b - 1 && \textit{when } n \textit{ is even.}\end{aligned}$$

PROOF. Let

$$\hat{p}_i = \tfrac{1}{2}(1 - \epsilon_i(a, b)), \qquad i = 1, 2, \cdots \tag{3.2}$$

and take

$$\hat{\mathbf{p}}_n = (\hat{p}_1, \hat{p}_2, \cdots, \hat{p}_n), \qquad n = 1, 2, \cdots.$$

Following the pattern of Theorem 2, it will suffice to prove for any positive integer n that $\hat{\mathbf{C}}_n(a, b) = \mathbf{p}_n^{-1}(\hat{\mathbf{p}}_n)$. But this by Theorem 1 is in essence the substance of Corollary 3 to Lemma 3 which follows below.

We note as of some interest at this point that $\epsilon_1(a, b), \epsilon_2(a, b), \cdots$ is a simple, explicitly defined null sequence for each positive a and b, so that identically for $a, b > 0$, we have that $\lim_{n\to\infty} p_n(\hat{\mathbf{C}}_n(a, b)) = \tfrac{1}{2}$.

Let $\hat{\zeta}_k = (1 - \hat{p}_{k-1})\hat{p}_k$, $k = 1, 2, \cdots$, and take $d = a + b$. By (3.2)

$$\hat{\zeta}_{2j} = \frac{(b+j-1)j}{(d+2j-2)(d+2j-1)}, \qquad \hat{\zeta}_{2j+1} = \frac{(d+j-1)(a+j)}{(d+2j-1)(d+2j)}, \qquad j = 0, 1, \cdots. \tag{3.3}$$

Using (1.3), (1.4), (1.5), let $\hat{S}_{ij}$ (for $j = i, i + 1, \cdots; i = 1, 2, \cdots$) denote the values of S_{ij} that correspond.

LEMMA 3. *For each positive integer i and each non-negative integer j we have that*

$$\hat{S}_{i,i+2j} = \binom{i+j}{j} \prod_{k=0}^{i-1} V(j, k), \qquad \hat{S}_{i,i+2j+1} = \binom{i+j}{j} \prod_{k=1}^{i} V(j, k),$$

where

$$V(j, k) = (a + j + k)/(d + 2j + k).$$

PROOF. We show first that the lemma holds when $i = 1$, i.e., that

$$(3.4) \quad \hat{S}_{1,2j+1} = (j + 1)V(j, 0), \qquad \hat{S}_{1,2j+2} = (j + 1)V(j, 1), \quad j = 0, 1, 2, \cdots.$$

By (1.3), (1.5), (3.3),

$$(3.5) \qquad \hat{S}_{11} = \hat{\zeta}_1 = \hat{p}_1 = a/d = V(0, 0),$$
$$\hat{S}_{12} = \hat{S}_{11} + \hat{\zeta}_2 = a/d + b/(d(d + 1)) = V(0, 1).$$

Now let j be a positive integer and suppose (3.4) to hold with j replaced by $j - 1$. We then have that

$$\hat{S}_{1,2j+1} = jV(j - 1, 1) + \hat{\zeta}_{2j+1} = \frac{j(a + j)}{d + 2j - 1} + \frac{(d + j - 1)(a + j)}{(d + 2j - 1)(d + 2j)}$$
$$= (j + 1)V(j, 0)$$
$$\hat{S}_{1,2j+2} = (j + 1)V(j, 0) + \hat{\zeta}_{2j+2} = \frac{(j + 1)(a + j)}{d + 2j} + \frac{(j + 1)(b + j)}{(d + 2j)(d + 2j + 1)}$$
$$= (j + 1)V(j, 1).$$

Now suppose the lemma holds for an arbitrary positive integer i. We show that it must then continue to hold for $i + 1$, i.e., that for $j = 0, 1, 2, \cdots$,

$$(3.6) \qquad \hat{S}_{i+1,i+2j+1} = \binom{i+j+1}{j} \prod_{k=0}^{i} V(j, k),$$
$$\hat{S}_{i+1,i+2j+2} = \binom{i+j+1}{j} \prod_{k=1}^{i+1} V(j, k).$$

Again by (1.3), (1.5), (3.3), and the induction hypothesis,

$$\hat{S}_{i+1,i+1} = \hat{\zeta}_1 \hat{S}_{i,i+1} = V(0, 0) \prod_{k=1}^{i} V(0, k).$$

Moreover

$$\hat{S}_{i+1,\, i+2} = \hat{S}_{i+1,\, i+1} + \hat{\zeta}_2 \hat{S}_{i,\, i+2}$$
$$= \prod_{k=0}^{i} V(0, k) + \frac{b(i + 1)}{d(d + 1)} \prod_{k=0}^{i-1} V(1, k)$$
$$= \frac{a(d + i + 1) + b(i + 1)}{d(a + i + 1)} \prod_{k=1}^{i+1} V(0, k).$$

But the ratio in front on the right hand side is 1 and the product is precisely the right hand side of the second equation in (3.6) when $j = 0$. Thus (3.6) holds for $j = 0$. Now let j be a positive integer and suppose (3.6) to hold for $j - 1$. To complete the proof we need only show that it must then continue to hold for j. We have by (1.3), (1.5), (3.3), and both induction hypotheses that

$$\begin{aligned}\hat{S}_{i+1,\,i+2j+1} &= \hat{S}_{i+1,\,i+2j} + \hat{\zeta}_{2j+1}\hat{S}_{i,\,i+2j+1} \\ &= \tbinom{i+j}{j-1} \textstyle\prod_{k=1}^{i+1} V(j-1, k) + \dfrac{(d+j-1)(a+j)}{(d+2j-1)(d+2j)} \\ &\qquad \cdot \tbinom{i+j}{j} \textstyle\prod_{k=1}^{i} V(j, k) \\ &= \dfrac{j(d+2j+i) + (i+1)(d+j-1)}{(i+j+1)(d+2j-1)} \tbinom{i+j+1}{j} \textstyle\prod_{k=0}^{i} V(j, k).\end{aligned}$$

The ratio in front on the right hand side above is 1 which verifies the first equation of (3.6). But then in a similar way we have that

$$\begin{aligned}\hat{S}_{i+1,\,i+2j+2} &= \hat{S}_{i+1,\,i+2j+1} + \hat{\zeta}_{2j+2}\hat{S}_{i,\,i+2j+2} \\ &= \tbinom{i+j+1}{j} \textstyle\prod_{k=0}^{i} V(j, k) + \dfrac{(j+1)(b+j)}{(d+2j)(d+2j+1)} \\ &\qquad \cdot \tbinom{i+j+1}{j+1} \textstyle\prod_{k=0}^{i-1} V(j+1, k) \\ &= \tbinom{i+j+1}{j} \textstyle\prod_{k=1}^{i+1} V(j, k).\end{aligned}$$

which verifies the second equation of (3.6) and completes the proof.

COROLLARY 3. *For each positive integer* i

$$\hat{S}_{ii} = \textstyle\prod_{k=0}^{i-1} V(0, k) = \prod_{k=0}^{i-1} \{(a+k)(d+k)^{-1}\}.$$

PROOF. Put $j = 0$ in the first equation of Lemma 3.

By Theorem 1 and (1.2) we have

COROLLARY 4. *For* $m = 1, 2, \cdots$

$$R_{2m+1}(\hat{\mathbf{C}}_{2m}(a, b)) = \textstyle\prod_{i=0}^{m-1} (i+1)(d+i)(a+i)(b+i)/\{(d+2i)^2(d+2i+1)^2\}.$$

For $m = 0, 1, 2, \cdots$ *multiplication of the right hand side above by* $(a+m)(b+m)/(d+2m)^2$ *yields* $R_{2m+2}(\hat{\mathbf{C}}_{2m+1}(a, b))$. (*When* $m = 0$, *the above product is taken to be* 1.)

Expressions analogous to (2.18) and (2.19) are easily obtained using (1.1) and Theorem 3.

In the following corollary we examine several special cases of the Beta distribution which are perhaps of some interest.

COROLLARY 5. *For the arcsine distribution, we have*

$$p_n(\hat{\mathbf{C}}_n(\tfrac{1}{2}, \tfrac{1}{2})) = \tfrac{1}{2}, \qquad n = 1, 2, \cdots$$

For the uniform distribution

$$p_{2i-1}(\hat{\mathbf{C}}_{2i-1}(1, 1)) = \tfrac{1}{2}, \qquad p_{2i}(\hat{\mathbf{C}}_{2i}(1, 1)) = i/(2i+1), \qquad i = 1, 2, \cdots.$$

The first part of this corollary is a known result. See Theorem 25.3 page 78 of [2].

4. Two General Theorems. Let $(p_1, p_2, \cdots)$ denote a sequence of normalized moments corresponding to some probability measure P defined on the Borel subsets of [0, 1]. Let

$$f(x) = 1 - x, \qquad 0 \leqq x \leqq 1;$$

define $P^{(s)}$ by taking for each Borel subset A of [0, 1],

$$P^{(s)}(A) = P(f^{-1}(A));$$

and let $(p_1^{(s)}, p_2^{(s)}, \cdots)$ denote the sequence of normalized moments corresponding to $P^{(s)}$. We then have

THEOREM 4. $$p_{2i}^{(s)} = p_{2i}, \qquad p_{2i+1}^{(s)} = q_{2i+1}.$$

PROOF. This is an immediate consequence of Theorem 26.5, page 85 plus formulas (18.1), (18.2) page 59 of [2] together with the definition of normalized moment. We sketch below a proof which is somewhat more elementary and direct. Let $C_1, C_2, \cdots; C_1^{(s)}, C_2^{(s)}, \cdots$ denote the moment sequences which correspond respectively to P and $P^{(s)}$. Let $\mathbf{C}_i$, $\mathbf{C}_i^{(s)}$ denote the obvious i-tuples for each positive integer i. Clearly

$$C_i^{(s)} = \int_{[0,1]} (1 - x)^i \, dP(x), \qquad C_i = \int_{[0,1]} (1 - x)^i \, dP^{(s)}(x) \quad i = 1, 2, \cdots,$$

Thus after suitable manipulation we may write

$$C_i^{(s)} = K_i(\mathbf{C}_{i-1}^{(s)}) + (-1)^i C_i,$$

where

$$K_i(\mathbf{C}_{i-1}^{(s)}) = \textstyle\sum_{j=0}^{i-1} (-1)^{i-j+1} \binom{i}{j} C_j^{(s)}.$$

Hence

$$\nu_{2i}^{\pm}(\mathbf{C}_{2i-1}^{(s)}) = K_{2i}(\mathbf{C}_{2i-1}^{(s)}) + \nu_{2i}^{\pm}(\mathbf{C}_{2i-1}),$$

while on the other hand

$$\nu_{2i+1}^{\pm}(\mathbf{C}_{2i}^{(s)}) = K_{2i+1}(\mathbf{C}_{2i}^{(s)}) - \nu_{2i+1}^{\mp}(\mathbf{C}_{2i}).$$

The conclusions now follow directly from the definition of normalized moments.

COROLLARY 6. *When* $P^{(s)} = P$, *we have that*

$$p_{2i-1} = p_{2i-1}^{(s)} = \tfrac{1}{2}, \qquad i = 1, 2, \cdots.$$

Thus, each distribution on [0, 1], symmetric about $\frac{1}{2}$ has each of its odd indexed moments at the midpoint of the range permitted by the moments which precede it. The special cases of Corollary 5 exhibit this property.

Let $a < b$ denote two finite real numbers, and let $M_n^{[a,b]}$ denote the space of the first n moments for the class of all probability measures defined on the Borel subsets of the interval $[a, b]$. Thus the space defined in the first paragraph of Section 1 is $M_n^{[0,1]}$. Clearly, maximum, minimum, and normalized moments $\nu_n^{\pm,[a,b]}$ and $p_n^{[a,b]}$, say, may be defined relative to $M_n^{[a,b]}$ in precise analogy to their definition for $M_n^{[0,1]}$.

Now let Y denote a random variable distributed on the interval $[a, b]$. $(Y - a)/(b - a)$ is then distributed on $[0, 1]$ and this transformation yields in effect a one-one correspondence between distributions on $[a, b]$ and those on $[0, 1]$. We may now state the following invariance principal as

THEOREM 5. *Let* $p_n^{[a,b]}(Y)$ *denote the* nth *normalized moment of the distribution of* Y *on* $[a, b]$. *Then*

$$p_n^{[a,b]}(Y) = p_n^{[0,1]}((Y - a)/(b - a)).$$

PROOF. Let $X = (Y - a)/(b - a)$; then

$$EY^n = \sum_{i=0}^{n-1} (-1)^{n-i+1} \binom{n}{i} a^{n-1} EY^i + (b - a)^n EX^n \tag{4.1}$$

for each positive integer n. Let $(\eta_1, \eta_2, \cdots)$ denote the moment sequence corresponding to some arbitrary but fixed distribution on $[a, b]$. It is easily seen that

$$\nu_n^{\pm,[a,b]}(\eta_1, \eta_2, \cdots, \eta_{n-1})$$

are respectively the maximum and minimum values of EY^n as Y varies over the class of all random variables which are distributed on $[a, b]$ with the property that

$$EY^i = \eta_i, \qquad i = 1, 2, \cdots, n - 1.$$

Thus by (4.1)

$$\nu_n^{\pm,[a,b]}(\eta_1, \eta_2, \cdots, \eta_{n-1}) = \sum_{i=0}^{n-1} (-1)^{n-i+1} \binom{n}{i} a^{n-i} \eta_i + (b - a)^n \nu_n^{\pm,[0,1]}(\xi_1, \xi_2, \cdots, \xi_{n-1}).$$

where

$$\xi_i = (b - a)^i \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} a^{i-j} \eta_j, \qquad i = 1, 2, \cdots.$$

As an immediate consequence (letting $\eta_i = EY^i$, $i = 1, \cdots, n - 1$, for the particular Y under consideration) we now have that

$$EY^n - \nu_n^{\pm,[a,b]}(\eta_1, \eta_2, \cdots, \eta_{n-1}) = (b - a)^n (EX^n - \nu_n^{\pm,[0,1]}(\xi_1, \xi_2, \cdots, \xi_{n-1})).$$

When we divide by the range of the nth moment, $(b - a)^n$ cancels and we have the desired result. □

Relative to the example considered in Section 2, we may now remark, using the above theorem, that the normalized binomial (N, θ) moments relative to the class of all distributions on $[0, N]$ are precisely the same as those (relative to the distributions on $[0, 1]$) of the binomial (N, θ) distribution with support at $\{0, 1/N, \cdots, 1\}$ that we considered.

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