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ON DISTINGUISHING TRANSLATES OF MEASURES[1]

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1. Introduction. Let X denote a completely general real valued stochastic process on an arbitrary parameter set T. Let m be any real valued function on T. The well-known statistical problem of estimating a regression parameter consistently can be formulated as follows.

For any real number α, let P_α denote the probability measure that is induced by the stochastic process $x(t) + \alpha m(t)$ on the set S of all real valued functions on T. Let $\mathcal{A}$ denote the σ-field of subsets of S generated by coordinate functionals. (Thus a typical set in $\mathcal{A}$ is of the form $\{x \mid x(t_1) \varepsilon [0, \frac{1}{2}], x(t_2) \varepsilon [1, \infty)\}$ for $t_1, t_2 \varepsilon T$.) Let $\mathcal{A}_\alpha$ be the completion of $\mathcal{A}$ under P_α. Let $\mathcal{B}$ be the intersection of all the $\mathcal{A}_\alpha$'s. Then one may rigorously restate the question "Can one estimate α consistently" by asking whether there exists a functional f defined on S, measurable with respect to $\mathcal{B}$ and such that for all α, $P_\alpha[f = \alpha] = 1$.

In Section 2 of this paper we show how a criterion that Dudley [2] used to establish the singularity of the measures P_0 and P_1 can in fact be adapted to show the existence of such an f. To describe this criterion we need some more notation. Let S^0 denote the set of all "finitely defined" linear functionals on S. By this is meant that for any $f \varepsilon S^0$ there is a finite sequence $a_1, \cdots, a_n$ of real numbers and another finite sequence $t_1, \cdots, t_n$ of elements of T such that for all $x \varepsilon S, f(x) = \sum_1^n a_i x(t_i)$. Let $\mathfrak{I}$ be the pseudo-metric of convergence in P_0 measure. Then $(S^0, \mathfrak{I})$ is a pseudo-metric linear space. For any $m \varepsilon S$, $f \varepsilon S^0$ let $e_m(f) = f(m)$. The criterion of Dudley is just that e_m be a discontinuous linear functional on $(S^0, \mathfrak{I})$. In fact if this criterion is fulfilled then the functional f that we exhibit will be linear on the vector space S, hence the measures P_α are even "linearly singular."

In Section 3 we consider a certain subclass of processes with independent increments and show that all non trivial m give rise to discontinuous linear functionals on the pseudo-metric linear space just mentioned. In Section 4 we continue to treat processes with independent increments but no longer require that the functional f that distinguishes the measures P_α be linear. Dudley [2] under the hypotheses of Theorem 3 proves that the measures P_0 and P_1 are singular, and Gikhman and Skorokhod [3] under the hypotheses of Theorem 4 do the same. The theorems of this section extend the results of these authors in that the continuum of measures P_α are simultaneously distinguished.

2. Proof that the discontinuity criterion gives rise to a linear way of distinguishing the measures P_α.

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THEOREM 1. *If e_m is discontinuous on $(S^0, \mathfrak{J})$ then there exists a sequence $f_n \varepsilon S^0$ such that $P_\alpha\{x \mid \lim_{n\to\infty} f_n(x) = \alpha\} = 1 \ \forall\ \alpha$.*

PROOF. The hypotheses imply that there are $g_n \varepsilon S^0$, with $g_n \to 0$ in $\mathfrak{J}$ but $g_n(m) \nrightarrow 0$. Taking a subsequence and normalizing we may assume that $g_n(x) \to 0$ for almost all x and that $g_n(m) \to 1$. Hence $P_\alpha\{x \mid g_n(x) \to \alpha\} = P_0\{x \mid g_n(x + \alpha m) \to \alpha\} = P_0\{x \mid g_n(x) + \alpha g_n(m) \to \alpha\} = 1 \ \forall\ \alpha$. $\square$

COROLLARY. *Let $T = \{x \varepsilon S \mid \lim_{n\to\infty} f_n(x)$ exists$\}$. Then T is a linear subspace of S and is measurable w.r.t. $\mathfrak{A}$. Also $P_\alpha(T) = 1 \ \forall\ \alpha$, hence if a linear functional f is defined on all of S by a linear extension of $\lim_{n\to\infty} f_n(x)$ on T, then f is measurable w.r.t. $\mathfrak{A}_\alpha$ for all α and $P_\alpha\{x \mid f(x) = \alpha\} = 1 \ \forall\ \alpha$.*

Let us give some examples of situations when this theorem and its corollary apply. Suppose for instance that $T = Z$, the set of integers. Suppose $E(x_n^2) < \infty \ \forall\ n \varepsilon Z, E(x_n x_m) = 0$ if $n \neq m$, and $E(x_n) = 0$ for all $n \varepsilon Z$. Let $\mu_n = E(x_n^2)$ and let μ stand for the measure on Z such that $\mu(\{n_1, \cdots, n_k\}) = \sum_1^k \mu_{n_i}$. Then any $m \varepsilon S$ $(m = (m_n)n \varepsilon Z)$ such that $m \not\varepsilon L_2(Z, \mu)$ has to give rise to a discontinuous functional e_m on $(S^0, \mathfrak{J})$. For if $f(x) = \sum_k^n a_i x_i$ then $e_m(f) = \sum_k^n a_i m_i$.

For another example suppose that $T = [0, 1]$ and that $x(t)$ is a stochastic process with orthogonal mean zero increments that is continuous in probability. Define the measure μ on $[0, 1]$ by $\mu([t_1, t_2)) = E((x(t_2) - x(t_1))^2)$. Suppose that for some μ integrable function h we have that $m(t) = \int_0^t h(t)\, d\mu(s)$. Then if $h \not\varepsilon L_2([0, 1], \mu)$ then e_m is discontinuous $(S^0, \mathfrak{J})$. For, let $\chi[a, b]$ stand for the characteristic function of the set $[a, b]$. Consider step functions of the form $h' = \sum_1^n a_i \chi_{[t_{i-1}, t_i]}, -\infty < t_0 < \cdots < t_n < \infty$. Then $\int_0^1 h'(t)\, dx(t)$ defines an element f of S^0 and $e_m(f) = \int_0^1 h(t) h'(t)\, d\mu(t)$. Now just pick a sequence h_N' of step functions such that $\int_0^1 (h_N'(t))^2\, d\mu(t) \to 0$ but $\int_0^1 h(t) h_N'(t)\, d\mu(t) \to 1$.

3. Application of Theorem 1 to processes with independent increments.

THEOREM 2. *Suppose $x(t)$, $t \varepsilon [0, 1]$ is a process with independent increments which is continuous in probability and such that its characteristic function $\varphi_t(z)$ has the form*

$$\exp\{\mu([0, t]) \int_{-\infty}^0 (e^{izs} - 1)\, dM(s) + \mu([0, t]) \int_0^\infty (e^{izs} - 1)\, dN(s)\}.$$

Here M and N are positive measures on R such that $M(-\infty, -\epsilon] + N[\epsilon, \infty) < \infty$ for all $\epsilon > 0$ and such that $\int_{-1}^0 |s|\, dM(s) + \int_0^1 s\, dN(s) < \infty$. Also μ is a continuous positive measure on $[0, 1]$ with $\mu([0, 1]) < \infty$. Then for any real valued function m on $[0, 1]$, either e_m is identically zero on $(S^0, \mathfrak{J})$ or e_m is discontinuous on $(S^0, \mathfrak{J})$. (We assume $m(0) = 0$.)

PROOF. We will not use but still mention the fact that almost all sample functions of such a process are continuous except for a countable number of jumps, and are of finite variation. See Breiman [1].

Let us begin the proof of this theorem by noticing that if m is not absolutely continuous with respect to μ, then e_m is discontinuous on $(S^0, \mathfrak{J})$. For suppose that $\sum_1^N |m(s_i) - m(t_i)| > 1$ but $\sum_1^N \mu([t_i, s_i)) < 1/N$, where $[t_i, s_i)$ are disjoint intervals in $[0, 1]$. Without loss of generality we can suppose that in fact all of the terms $m(s_i) - m(t_i)$ are positive. Suppose we can do this for all positive integers N.

Let $f_N \varepsilon S^0$ be defined by $f_N(x) = \sum_1^N x(s_i) - x(t_i)$. (The intervals $[t_i, s_i)$ depend on N.) Then $f_N \to 0$ in $\mathfrak{J}$ but $f_N(m) \nrightarrow 0$. We conclude that if e_m is to be continuous on $(S^0, \mathfrak{J})$ then m must be of the form $m(t) = \int_0^t h(s)\, d\mu(s)$ for some $h \varepsilon L_1([0, 1], \mu)$. Now if $\int_0^1 |h(t)|\, d\mu(t) > 0$, i.e. if $\int_0^1 h(t)h'(t)\, d\mu(t) \neq 0$ for some step function h' (which is the same thing as saying e_m is not identically zero on S^0), then $\lim_{t \to t_0, t > t_0} [m(t) - m(t_0)]/\mu([t_0, t)) \neq 0$ for some $t_0 \varepsilon [0, 1]$ by a familiar theorem from measure theory. At this point we simply notice that the characteristic function of $[x(t) - x(t_0)]/\mu([t_0, t)) \to 1$ in any z neighborhood of 0 as $t \to t_0$, $t > t_0$. To prove this let us first note that the proof will not depend on the point t_0 and for simplicity let us assume $t_0 = 0$.

Now $|\log \varphi_t(z/\mu([0, t]))| \leqq |I_1| + |I_2|$ where

$$I_1 = \mu([0, t]) \int_0^\infty (e^{isz/\mu([0,t])} - 1)\, dN(s) \quad \text{and}$$

$$I_2 = \mu([0, t]) \int_{-\infty}^0 (e^{isz/\mu([0,t])} - 1)\, dM(s).$$

Also $|I_1| \leqq |\mu([0, t]) \int_0^{\epsilon_t} (e^{isz/\mu([0,t])} - 1)\, dN(s)| + 2\mu([0, t])N([\epsilon_t, \infty))$.

Now choose $\epsilon_t \to 0$ such that $\mu([0, t])N([\epsilon_t, \infty)) \to 0$ as $t \to 0$.

Also $|\mu([0, t]) \int_0^{\epsilon_t} (e^{isz/\mu([0,t])} - 1)\, dN(s)| \leqq 2\pi \int_0^{\epsilon_t} |zs|\, dN(s)$.

But this expression goes to zero if $\epsilon_t \to 0$. So I_1 goes to zero as $t \to 0$. Similarly I_2 goes to zero as $t \to 0$.

We conclude that we can choose a sequence of numbers $t_n > t_0$, $t_n \to t_0$ such that $\lim_{t_n \to t_0} [m(t_n) - m(t_0)]/\mu([t_0, t_n]) \neq 0$ but $\lim_{t_n \to t_0} [x(t_n) - x(t_0)]/\mu([t_0, t_n]) = 0$ in probability. So if $f_n(x) = [x(t_n) - x(t_0)]/\mu([t_0, t_n])$ then $f_n \to 0$ in $(S^0, \mathfrak{J})$ but $f_n(m) \nrightarrow 0$. □

In the special case when $x(t)$ of the last theorem is a stable process with $\log \varphi_t(z) = -t|z|^q$, $0 < q < 1$, the theorem is just another proof of the fact that there are no non-trivial continuous linear functionals on $L_q([0, 1])$. See B. Gramsch [4] and Woyczynski and Urbanik [7] for related results.

4. Distinguishing the measures in a non linear fashion.

THEOREM 3. *Let x_n, $n \geqq 1$ be a sequence of independent, identically distributed random variables. Let $m = (m_n)n \geqq 1$ be a sequence of real numbers such that $\sum_1^\infty m_n^2 = \infty$. Then there exists a sequence f_N of functionals defined on S, the space of all real valued sequences, such that f_N are all measurable w.r.t. $\mathfrak{A}$ and such that $f_N(x + \alpha m) \to \alpha$* a.s. $\forall\ \alpha$.

PROOF. Choose $M > 0$ such that $P[|x_n| < M] \geqq \frac{1}{2}$.

Let

$$\begin{aligned} h(s) &= M + 1 && \text{if } s > M + 1, \\ &= s && \text{if } |s| \leqq M + 1, \\ &= -M - 1 && \text{if } s < -M - 1. \end{aligned}$$

Let $b = E(h(x_n))$, $c_n^\alpha = E(h(x_n + \alpha m_n))$, $n \geqq 1$.

Now $\sum_1^\infty m_n^2 = \infty$, so for some subsequence n_i, m_{n_i} all have the same sign and $\sum_1^\infty (m_{n_i})^2 = \infty$. So without loss of generality we assume $m_n \geqq 0 \ \forall\ n$. Now fix a sequence $\lambda_n \geqq 0$, $n \geqq 1$ such that $\sum_1^\infty \lambda_n^2 < \infty$ but $\sum_1^k \lambda_n m_n \to +\infty$.

Notice now that if $\alpha \geqq 0$ then $0 \leqq \frac{1}{2}m_n\alpha \leqq (c_n^\alpha - b) \leqq m_n\alpha$ and if $\alpha \leqq 0$ then $m_n\alpha \leqq (c_n^\alpha - b) \leqq \frac{1}{2}m_n\alpha \leqq 0$.

So if $\alpha \geqq 0$ then $[\sum_1^k \lambda_n(c_n^\alpha - b)]/\sum_1^k \lambda_m m_n$, $k = 1, 2, \cdots$, is an infinite sequence of numbers in the interval $[\frac{1}{2}\alpha, \alpha]$ and if $\alpha \leqq 0$ then it is an infinite sequence of numbers in the interval $[\alpha, \frac{1}{2}\alpha]$. In either case let $g(\alpha)$ be the lim inf of the sequence.

If $\alpha_1 > \alpha_2 \geqq 0$ then $\frac{1}{2}\lambda_n m_n(\alpha_1 - \alpha_2) \leqq \lambda_n(c_n^{\alpha_1} - c_n^{\alpha_2}) \leqq \lambda_n m_n(\alpha_1 - \alpha_2)$ and if $\alpha_1 < \alpha_2 \leqq 0$ then $\lambda_n m_n(\alpha_1 - \alpha_2) \leqq \lambda_n(c_n^{\alpha_1} - c_n^{\alpha_2}) \leqq \frac{1}{2}\lambda_n m_n(\alpha_1 - \alpha_2)$.

So g is strictly increasing with respect to its argument. So its inverse $g^{(-1)}$ is well defined and continuous.

Now consider

$$[\textstyle\sum_1^k \lambda_n(h(x_n + \alpha m_n) - b)]/\sum_1^k \lambda_n m_n$$
$$= \{\textstyle\sum_1^k \lambda_n[(h(x_n + \alpha m_n) - c_n^\alpha) + (c_n^\alpha - b)]\}/\sum_1^k \lambda_n m_n$$

Now $h(x_n + \alpha m_n) - c_n^\alpha$ is a sequence of independent square integrable random variables with mean zero and variance $\leqq (M + 1)^2$. So $\sum_1^N \lambda_n(h(x_n + \alpha m_n) - c_n^\alpha)$ converges almost surely and hence

$$[\textstyle\sum_1^N \lambda_n(h(x_n + \alpha m_n) - c_n^\alpha)]/\sum_1^N \lambda_n m_n \to 0 \quad \text{almost surely.}$$

Let $f_N(x) = g^{(-1)}(\inf_{k \geqq N}\{[\sum_1^k \lambda_n(h(x_n + \alpha m_n) - b)]/\sum_1^k \lambda_n m_n\})$.

Now $\{[\sum_1^k \lambda_n(h(x_n + \alpha m_n) - b)]/\sum_1^k \lambda_n m_n\}_{k \geqq 1}$ has the same lim inf as $\{[\sum_1^k \lambda_n(c_n^\alpha - b)]/\sum_1^k \lambda_n m_n\}_{k \geqq 1}$ almost surely.

So $f_N(x + \alpha m) \to \alpha$ almost surely. □

The above arguments are based on Dudley's [2] proof of a theorem of Shepp [6].

Now let us again consider a process $x(t)$, $t \in [0, 1]$ with independent increments which is continuous in probability and such that $x(0) = 0$ a.s.

As shown in Loève [5]

$$x(t) = \zeta(t) + a(t) + \int_0^t \int_{-\infty}^{+\infty} y[\nu(ds, dy) - (1 + y^2)^{-1}\pi(ds, dy)] \quad \text{a.s.}$$

where $a(t)$ is a continuous real valued function which we shall assume to be zero, $\zeta(0) = 0$ a.s. and $\zeta(t)$ is a continuous gaussian process with independent increments of mean zero, and π is a positive measure on $[0, 1] \times R$ which satisfies (i) $\pi([0, 1] \times \{0\}) = 0$ and (ii) $\int_{-\infty}^{+\infty} y^2(1 + y^2)^{-1}\pi([0, 1] \times dy) < \infty$. In the above formula $\nu(A \times B)$ is a Poisson r.v. with mean and variance equal to $\pi(A \times B)$.

Let P_ζ denote the probability measure that the stochastic process $\zeta(t)$ induces on S. Let $\mathfrak{J}_\zeta$ be the pseudo-metric on S^0 of convergence in P_ζ measure.

THEOREM 4. *If e_m is discontinuous on $(S^0, \mathfrak{J}_\zeta)$, then there exists a sequence f_N of $\mathfrak{A}$ measurable functionals such that $P[\lim_{N\to\infty} f_N(x + \alpha m) \to \alpha] = 1\ \forall\ \alpha$. (We can* *assume that m is a continuous function, otherwise e_m is discontinuous on* $(S^0, \mathfrak{I})$ *and Theorem* 1 *applies.*)

PROOF. Gikhman and Skorokhod [3] consider the measurable mapping $T: S \rightarrow S$ defined by $T(x)(t) = \lim_{k\to\infty} (\lim_{n\to\infty} y_{k.n}(x)(t)$, where

$$y_{k.n}(x)(t) = \left(-\sum_{l/n<t} \psi_{\epsilon_k}(x(l/n) - x(l - 1/n))\right) + x(t) + \int_0^t \int_{|y|>\epsilon_k} y(1 + y^2)^{-1}\pi(ds \times dy).$$

Here $\psi_\epsilon(y) = y$ for $|y| > \epsilon$, $\psi_\epsilon(y) = 0$ for $|y| \leqq \epsilon$ and ϵ_k is a sequence of positive numbers tending to zero for which $\pi([0, 1] \times \{y: |y| = \epsilon_k\}) = 0$. (The limits are taken into the sense of convergence in measure.)

Now $\lim_{n\to\infty} \sum_{l/n<t} \psi_{\epsilon_k}(x(l/n) - x(l - 1/n)) = \int_0^t \int_{|y|>\epsilon_k} y\nu(ds, dy)$ a.s.

Consequently $T(x + \alpha m)(t) = \zeta(t) + \alpha m(t)$ a.s.

Now let $g_N \varepsilon S^0$ be such that $g_N \rightarrow 0$ in $\mathfrak{I}_\zeta$ but $g_N(m) \rightarrow 1$. We can find step functions h_N such that $g_N(x) = \int_0^1 h_N(t)\, dx(t)\ \forall\ x \varepsilon S$. So we have $\int_0^1 h_N(t)\, d\zeta(t) \rightarrow 0$ in quadratic mean.

Now consider $\int_0^1 h_N(t)\, d(y_{k.n}(x + \alpha m)(t))$.

This expression equals $\alpha g_N(m) + \int_0^1 h_N(t)\, d\zeta(t) + z_{k,n}(h_N)$ a.s., where for any step function h, $z_{k.n}(h)$ is a random variable such that $\lim_{k\to\infty} (\lim_{n\to\infty} z_{k.n}(h)) = 0$ in probability.

Now let us choose for each N, $k(N)$ and $n(N)$ so large that

$$\lim_{N\to\infty} z_{k(N),n(N)}(h_N) \rightarrow 0 \text{ a.s.}$$

Hence $\int_0^1 h_N(t)\, d(y_{k(N),n(N)}(x + \alpha m)(t)) \rightarrow \alpha$ a.s. by choosing a subsequence of the sequence h_N if necessary. Now define $f_N(x) = \int_0^1 h_N(t)\, d(y_{k(N),n(N)}(x)(t))$. Then $\lim_{N\to\infty} f_N(x + \alpha m) = \alpha$ a.s. □

COROLLARY. *If* $\zeta(t) \equiv 0\ \forall\ t$, *then we need no condition at all on m for the theorem to hold.*

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