The Annals of Mathematical Statistics
1969, Vol. 40, No. 5, 1850–1851

A BRANCHING PROCESS WITHOUT REBRANCHING

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Associated with a branching process, let $S(t)$ denote the number of particles replacing original particles that split during $[0, t]$, and $C(t)$ denote the number of particles arising from branching and in existence at time t. The branching process is said to be a barely branching process if $P\{C(t') = S(t'), 0 \leqq t' \leqq t\} = 1$. It is shown that the limiting branching process discussed by Stratton and Tucker [2] is a barely branching process. For their limiting process an elementary computation yields (1) the distribution of $S(t)$ for each t and (2) the fact that $S(t)$ has independent increments.

This result was obtained by Bühler [1] whose approach differs from our approach.

Stratton and Tucker [2] consider the following model:

(1) At time 0 there are N particles. All particles are mutually stochastically independent.

(2) For a particle in existence at time t, the probability of its being replaced by k (an integer $\geqq 0$) particles in the time interval $[t, t + h]$ is $\lambda_{k,N}\phi_N(t)h + o(h)$, where $o(h)$ is uniform in t, $0 < \sum_{k=0}^{\infty} \lambda_{k,N} = \lambda_N < \infty$, $\phi_N(t) > 0$ and $\lambda_{k,N} \geqq 0$. The probability of nonreplacement is $1 - \lambda_N\phi_N(t)h + o(h)$, where $o(h)$ is uniform in t.

(3) $\lim_{N\to\infty} \phi_N(t) = \phi(t)$ uniformly over every bounded interval, and $\lim_{N\to\infty} N\lambda_{k,N} = \nu_k \geqq 0$ where $0 < \lim_{N\to\infty} N\lambda_N = \nu = \sum_{k=0}^{\infty} \nu_k < \infty$. The functions $\{\phi_N(\cdot)\}$ are assumed continuous.

Let $Y_N(t)$ be the number of branchings by time t of the original N particles. Let $S_N(t)$ be the number of particles which replaced those $Y_N(t)$. Given $Y_N(t) = y > 0$, then the distribution of $S_N(t)$ is the y-fold convolution of the distribution of a random variable X_N with $P(X_N = k) = \lambda_{k,N}/\lambda_N$.

The usual elementary computations will show that, for fixed t, the distributions of $\{Y_N(t)\}$ converges to the distribution of a Poisson random variable, $Y(t)$, with

$$P(Y(t) = y) = [\nu\Phi(t)]^y e^{-\nu\Phi(t)}/y!,$$

where $\Phi(t) = \int_0^t \phi(x)\, dx$. [The proof uses the fact that $Y_N(t)$ has a binomial distribution with parameters $p_N(t) = 1 - \exp - \lambda_N \int_0^t \phi_N(x)\, dx$ and N. Then (3) implies that for fixed t, $\{Np_N(t)\}$ converges to $\nu\Phi(t)$.]

Also the stochastic processes $\{Y_N(t)\}$ converge in distribution to a nonstationary Poisson process $\{Y(t)\}$.

Received 13 November 1968.

[1] Research supported by contract NONR 988(13) NR 042-003 from the Office of Naval Research.

[2] Research supported by Contract GZ-315 from the National Science Foundation.

Further, given $Y_N(t) = y$, the distribution of $\{S_N(t)\}$ converges to the y-fold convolution of the distribution of X where $P(X = k) = \nu_k/\nu$. So that $\{S_N(t)\}$ converges in distribution to a stochastic process, $\{S(t)\}$, with probability generating function $m_t(\theta)$, where

$$\begin{aligned} m_t(\theta) &= \textstyle\sum_y P(Y(t) = y)(\sum_k \nu^{-1}\nu_k e^{k\theta})^y \\ &= \exp - \nu\Phi(t)(1 - \textstyle\sum_k \nu^{-1}\nu_k e^{k\theta}), \quad \text{which is equivalent to [2(8)].} \end{aligned}$$

And, $\{S(t)\}$ has independent increments. The $\{S(t)\}$ process is barely branching since it entails no rebranching.

Stratton and Tucker consider $C_N(t)$ the number of particles arising from branching and in existence at time t. The event $\{C_N(t') = S_N(t'), 0 \leqq t' \leqq t\}$ occurs when there is no branching of the new particles arising from branching during $[0, t]$. Now if a particle came into existence at time T $(<t)$ then the probability of its not branching by time t is

$$\exp\{-\textstyle\int_T^t \lambda_N\phi_N(x)\,dx\} \geqq \exp\{-\int_0^t \lambda_N\phi_N(x)\,dx\}.$$

So the probability that none of the $S_N(t)$ particles branch by time t is greater than or equal to

$$\exp\{-S_N(t)\textstyle\int_0^t \lambda_N\phi_N(x)\,dx\}.$$

Thus

$$\begin{aligned} &P(S_N(t') = C_N(t'), 0 \leqq t' \leqq t) \\ &\geqq \textstyle\sum_{j=0}^{\infty} P(S_N(t) = j)P(\text{no replacement branches} \mid S_N(t) = j) \\ &\geqq \textstyle\sum_{j=0}^{\infty} P(S_N(t) = j)\exp\{-N^{-1}j\int_0^t N\lambda_N\phi_N(x)\,dx\} \\ &\geqq P(S_N(t) \leqq N^{\frac{1}{2}})\exp\{-N^{-1}N^{\frac{1}{2}}\textstyle\int_0^t N\lambda_N\phi_N(x)\,dx\}. \end{aligned}$$

For increasing N, $\{S_N(t)\}$ has a limiting distribution so $P(S_N(t) \leqq N^{\frac{1}{2}}) \to 1$ and

$$\exp\{-N^{-\frac{1}{2}}\textstyle\int_0^t N\lambda_N\phi_N(x)\,dx\} \to 1 \quad \text{as} \quad N \to \infty,$$

thus

$$P(S_N(t)' = C_N(t'), 0 \leqq t' \leqq t) \to 1 \quad \text{as} \quad N \to \infty.$$

So $\{C_N(t)\}$ converges in distribution to a process which is barely branching.

REFERENCES

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