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A UNIFORM OPERATOR ERGODIC THEOREM

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1. Introduction. The purpose of this note is to prove a uniform operator ergodic theorem for mean convergence of differences of right continuous stochastic processes. Our result contains a difference version of the Glivenko-Cantelli theorem for infinite invariant measures. We also state a pointwise convergence theorem valid in the presence of a positive fixed point, which generalizes a result of Burke [1].

Let $(\Omega, \mathcal{A}, P)$ be a probability space; let L_1 be the class of integrable functions on $(\Omega, \mathcal{A}, P)$; let L_1^+ be the class of non-negative integrable functions and let T be a Markovian operator mapping L_1 into itself. A set A is *closed* if for each $f \varepsilon L_1$, $f = 0$ on A^c implies that $Tf = 0$ on A^c. The class of closed sets forms the *invariant sigma field* $\mathcal{J}$. T is ergodic if $\mathcal{J}$ is trivial. Let T_∞ denote the (formal) operator $I + T + T^2 + \cdots$. Hopf's decomposition states that $\Omega = C + D$ where for every $f \varepsilon L_1^+$, $T_\infty f = 0$ or ∞ on C and $T_\infty f < \infty$ on D. If $\Omega = C$, T is conservative. We now state an ergodic theorem required in the sequel. We use the notation:

$$E(f) = \int_\Omega f\, dP. \tag{1.1}$$

THEOREM 1.1. *Let T be a conservative ergodic Markovian operator. If $f \varepsilon L_1$ and $E(f) = 0$, then*

$$n^{-1}(f + Tf + \cdots + T^{n-1}f) \tag{1.2}$$

converges to zero in the L_1 topology.

Theorem 1.1 was obtained by Sucheston [6] and independently by Krengel [5].

2. Main results. Let $X(\omega, t)$, $Y(\omega, t)$ be left continuous non-decreasing stochastic processes on $\Omega \times R$ such that for each $t \varepsilon R$ $EX(\omega, t) < \infty$ and $EY(\omega, t) < \infty$. We will omit ω in $X(\omega, t)$ for simplicity. Set $X_n(t) = T^nX(t)$, $Y_n(t) = T^nY(t)$ for $n = 0, 1, \cdots$. We may and do assume that $X_n(t)$ and $Y_n(t)$ are chosen in such a way that outside a null set N independent of t and n, they are nondecreasing and left continuous functions of t. Such a choice is possible by a regularization procedure as used in constructing regular conditional probabilities.

In [1], it was shown that if T is generated by a point transformation which preserves a finite measure, then the cesaro averages of $T^nX(t)$ converge almost everywhere uniformly with respect to t on compact intervals. Here we show that when suitably normalized, $X(t)$ and $Y(t)$ behave similarly in the mean, uniformly on a rectangle.

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THEOREM 2.1. *Let* $0 < c_1 \leqq c_2 < \infty$, $0 < d_1 \leqq d_2 < \infty$; *let*

$$C = \{t: c_2 \geqq EX(t) \geqq c_1\}, \qquad D = \{t: d_2 \geqq EY(t) \geqq d_1\}. \tag{2.1}$$

Let $B = C \times D$ *and*

$$\Delta_n = \sup_{(s,t)\varepsilon B} |n^{-1} \textstyle\sum_{k=0}^{n-1} X_k(s)/EX(s) - Y_k(t)/EY(t)|. \tag{2.2}$$

Then Δ_n *converges to zero in the* L_1 *topology.*

PROOF. We may and do assume that $c_1 = \inf_{t\varepsilon C} EX(t)$, $c_2 = \sup_{t\varepsilon C} EX(t)$, $d_1 = \inf_{t\varepsilon D} EY(t)$, $d_2 = \sup_{t\varepsilon D} EY(t)$. For each fixed integer m and each $j = 1, 2, \cdots m - 1$, we let s_{mj}, t_{mj} be the smallest real numbers such that:

$$\begin{aligned} EX(s_{mj}) &\leqq c_1 + j(c_2 - c_1)/m \leqq EX(s_{mj} + 0), \\ EY(t_{mj}) &\leqq d_1 + j(d_2 - d_1)/m \leqq EY(t_{mj} + 0). \end{aligned} \tag{2.3}$$

Further set $s_{m0} = \inf C$, $s_{mm} = \sup C$, $t_{m0} = \inf D$, $t_{mm} = \sup D$. For each pair $(s, t) \varepsilon B$, we define

$$\delta_n(s, t) = X_n(s)/EX(s) - Y_n(t)/EY(t). \tag{2.4}$$

It follows from Theorem 1.1 applied to $\delta_0(s, t)$ that for fixed s, t, $\delta_n(s, t)$ converges cesaro in the L_1 topology to zero. Since positive linear operators are order preserving, for $s_{m,i-1} < s \leqq s_{mi}$, $t_{m,j-1} < t \leqq t_{mj}$ we have

$$\begin{aligned} X_k(s_{m,i-1} + 0)/EX(s_{mi}) &\leqq X_k(s)/EX(s) \leqq X_k(s_{m,i})/EX(s_{m,i-1} + 0), \\ Y_k(t_{m,j-1} + 0)/EY(t_{m,j}) &\leqq Y_k(t)/EY(t) \leqq Y_k(t_{m,j})/EY(t_{m,j-1} + 0) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} X_k(s_{m,i-1} + 0)/EX(s_{mi}) - Y_k(t_{m,j})/EY(t_{m,j-1} + 0) &\leqq \delta_k(s, t) \\ \leqq X_k(s_{mi})/EX(s_{m,i-1} + 0) - Y_k(t_{m,j-1} + 0)/EY(t_{m,j}). \end{aligned} \tag{2.6}$$

From (2.3) it follows that

$$EX(s_{mi})/EX(s_{m,i-1} + 0) \leqq 1 + c/m,$$

$$EY(t_{m,j-1} + 0)/EY(t_{mj}) \geqq 1 - d/m,$$

where $c = (c_2 - c_1)/c_1$, $d = (d_2 - d_1)/d_1$. Therefore

$$\begin{aligned} \delta_k(s, t) \leqq{} & (1 + c/m)\delta_k(s_{mi}, t_{m,j-1} + 0) \\ & + (c/m + d/m)Y_k(t_{m,j-1} + 0)/EY(t_{m,j-1} + 0). \end{aligned} \tag{2.7}$$

By similar arguments, we obtain a lower bound

$$\begin{aligned} -\delta_k(s, t) \leqq{} & -(1 - c/m)\delta_k(s_{m,i-1} + 0, t_{mj}) \\ & + (c/m + d/m)Y_k(t_{m,j-1} + 0)/EY(t_{m,j-1} + 0). \end{aligned} \tag{2.8}$$

Since T is Markovian, the integral of $Y_k(s_{m,j-1} + 0)/EY(s_{m,j-1} + 0)$ is one and it follows that

$$E|\Delta_n| \leqq E \max (\Delta_n^{(1)}, \Delta_n^{(2)}) + (c + d)/m \tag{2.9}$$

where

$$\Delta_n^{(1)} = (1 + c/m) \max_{0 \leqq i \leqq m, 0 \leqq j \leqq m} n^{-1} \textstyle\sum_{k=0}^{n-1} \delta_k(s_{mi}, t_{m,j-1} + 0), \tag{2.10}$$
$$\Delta_n^{(2)} = -(1 - c/m) \max_{0 \leqq i \leqq m, 0 \leqq j \leqq m} n^{-1} \textstyle\sum_{k=0}^{n-1} \delta_k(s_{m,i-1} + 0, t_{mj}).$$

Each of the terms over which max is taken in $\Delta_n^{(1)}$ and $\Delta_n^{(2)}$ converge to zero in the L_1 topology by Theorem 1.1. Therefore,

$$\lim_n \sup \int |\Delta_n| \leqq (c + d)/m \tag{2.11}$$

and since m is arbitrary, convergence in the L_1 topology follows.

If the operator T admits of a fixed point $f \varepsilon L_1^+$, we may obtain pointwise convergence. In this case, the role of Theorem 1.1. may be played by Hopf's operator ergodic theorem [2].

THEOREM 2.2. *Let* $T1 = 1$; $0 \leqq c_1 < c_2 < \infty$; $C = \{t: c_1 < EX(t) < c_2\}$; *and*

$$\Delta_n = \sup_{t \varepsilon C} n^{-1} | \textstyle\sum_{k=0}^{n-1} X_k(t) - EX(t)|. \tag{2.12}$$

Then for almost every $\omega \varepsilon \Omega$,

$$\lim_{n \to \infty} \Delta_n = 0. \tag{2.13}$$

The proof of this theorem is similar to Theorem 2.1 and is omitted. For the next theorem, we permit P to be sigma finite on $\mathfrak{A}$. Let τ be a measure preserving, conservative, ergodic point transformation. τ generates a Markovian operator T by means of the relation $Tf = f \circ \tau$. This correspondence preserves the notions of ergodicity and conservativity of an operator. Let X_0, Y_0 be fixed real-valued measurable functions on Ω and for $n = 1, 2, \cdots$, let $X_n = X_0 \circ \tau^n$, $Y_n = Y_0 \circ \tau^n$. If s, x, t, y are extended real numbers, let

$$F_n^s(x) = 1_{(s,x)} \circ X_n, \qquad G_n^t(y) = 1_{(t,y)} \circ Y_n, \qquad n = 0, 1 \cdots, \tag{2.14}$$

and

$$F^s(x) = E(F_0^s(x)), \qquad G^t(y) = E(G_0^t(y)). \tag{2.15}$$

Theorem 2.1 contains the following difference version of the Glivenko-Cantelli theorem for infinite invariant measures. A ratio version of this theorem was proved in [3].

THEOREM 2.3. *Let* $s, t \varepsilon \bar{R}$ *(extended real line). Let* C *and* D *be sets in* $\bar{R}$ *such that for some positive constants* c_1, c_2, d_1, d_2

$$C = \{x: c_2 \geqq F^s(x) \geqq c_1\}, \qquad D = \{y: d_2 \geqq G^t(y) \geqq d_1\}. \tag{2.16}$$

Let $B = C \times D$ *and*

$$\Delta_n = \sup_{(x,y) \varepsilon B} n^{-1} | \textstyle\sum_{i=0}^{n-1} F_i^s(x)/F^s(x) - G_i^t(y)/G^t(y)|. \tag{2.17}$$

Then Δ_n *converges to* 0 *in the* L_1 *topology.*

3. The non-ergodic case. With suitable modifications, Theorems 2.1 and 2.2 remain valid even though the invariant σ-field $\mathcal{I}$ is non-trivial. Theorem 1.1 was actually proved under the weaker condition $E(f \mid \mathcal{I}) = 0$. Therefore, we may now state Theorem 2.1 valid in the case when $\mathcal{I}$ is not trivial. This theorem is based on an idea of Tucker [7].

THEOREM 3.1. *Let* $c_1(\omega)$, $c_2(\omega)$, $d_1(\omega)$, $d_2(\omega)$ *be* $\mathcal{I}$*-measurable; let*

$$(3.1)\quad C = \{t: c_2 \geqq E(X_0(t) \mid \mathcal{I}) \geqq c_1) \qquad D = \{t: d_2 \geqq E(Y_0(t) \mid \mathcal{I}) \geqq d_1),$$

the inequalities holding except on a null set N independent of t. Let $B = C \times D$ *and*

$$(3.2)\quad \Delta_n = \sup_{(s,t)\varepsilon B} n^{-1}\left|\sum_{k=0}^{n-1} X_k(s)/E(X_0(s) \mid \mathcal{I}) - Y_k(t)/E(Y_0(t) \mid \mathcal{I})\right|$$

Then $\lim_{n\to\infty} \int |\Delta_n| = 0$.

PROOF. We merely sketch the proof since it is similar to Theorem 2.1. We may and do assume that

$$c_2 = \sup_{t\varepsilon C} E(X(t) \mid \mathcal{I}), \qquad c_1 = \inf_{t\varepsilon C} E(X(t) \mid \mathcal{I}),$$
$$d_2 = \sup_{t\varepsilon D} E(Y(t) \mid \mathcal{I}), \qquad d_1 = \inf_{t\varepsilon D} E(Y(t) \mid \mathcal{I}).$$

For each fixed integer m and each $j = 0, 1, \cdots, m-1$, we let $s_{mj}(\omega)$, $t_{mj}(\omega)$ be the smallest real numbers such that

$$(3.4)\quad E(X(s_{mj}) \mid \mathcal{I}) \leqq c_1 + j(c_2 - c_1)/m \leqq E(X(s_{mj}+0) \mid \mathcal{I}),$$
$$E(Y(t_{mj}) \mid \mathcal{I}) \leqq d_1 + j(d_2 - d_1)/m \leqq E(Y(t_{mj}+0) \mid \mathcal{I}).$$

The functions s_{mj}, t_{mj} are measurable on the sigma field generated by $\mathcal{I}$ and are ordered: $s_{m,j-1} \leqq s_{mj}$, $t_{m,j-1} \leqq t_{mj}$, $j = 0, \cdots, m$. The arguments in the proof of Theorem 2.1 apply with the preceding changes.

Theorem 2.3 also extends to the non-ergodic case except that the conditional expectation is not defined if $\mathcal{I}$ contains an atom of infinite measure. We may however, compute conditional expectation with respect to an equivalent probability measure (see [4]).

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