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UNBIASEDNESS OF SOME TEST CRITERIA FOR THE EQUALITY OF ONE OR TWO COVARIANCE MATRICES[1]

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1. Introduction. The main purpose of this paper is to answer the question stated in Anderson and Das Gupta [2], of whether the modified likelihood ratio test (= modified LR test) for the equality of two covariance matrices is unbiased or not. We shall answer this question affirmatively in Section 3, by generalizing the method in Pitman [6]. The same idea can be applied to prove the unbiasedness of the modified LR test for the equality of a covariance matrix to a given one in Section 2 and also of the LR test for sphericity in Section 4, for the equality of a mean and a covariance matrix to some given ones in Section 5. Some generalizations of these results will be also stated.

The derivation of these test criteria can be found in Anderson [1]. Gleser [4] has proved recently the unbiasedness of the LR test for sphericity by reducing the problem to the unbiasedness of the Bartlett test in case of the equal sample sizes. But our method of proof is more direct and somewhat different from his.

2. Unbiasedness of the modified LR test for $\Sigma = \Sigma_0$. Let $p \times 1$ vectors $X_1, X_2, \cdots, X_N$, $(N > p)$, be a random sample from a multivariate normal distribution with unknown mean vector μ and unknown covariance matrix Σ (nonsingular). From this sample we want to test the hypothesis $H_1 : \Sigma = \Sigma_0$ against the alternatives $K_1 : \Sigma \neq \Sigma_0$, where the mean μ is unspecified and Σ_0 is a given $p \times p$ positive definite (= pd) matrix. The acceptance region of the LR test for this problem is given by, as in Anderson ([1], p. 265),

$$(2.1) \qquad \omega_1' = \{S \mid S \text{ is pd and } |S\Sigma_0^{-1}|^{N/2} \operatorname{etr}(-\tfrac{1}{2}\Sigma_0^{-1}S) \geqq c_\alpha\},$$

where the symbol etr means exp tr, $S = \sum_{\alpha=1}^{N} (X_\alpha - \bar{X})(X_\alpha - \bar{X})'$, $\bar{X} = N^{-1}\sum_{\alpha=1}^{N} X_\alpha$ and the constant c_α is determined such that the level of this test is α. In case $p = 1$, this acceptance region ω_1' does not give an unbiased test and further the UMP unbiased test is given by replacing $|S\Sigma_0^{-1}|^{N/2}$ to $|S\Sigma_0^{-1}|^{(N-1)/2}$ in (2.1), which can be seen, for example, in Lehmann ([5], p. 165) by some calculation. After this modification of changing the sample size N to the degrees of freedom $N - 1 = n$, we can prove the unbiasedness in the multivariate case. This is the simplest case in our discussion.

THEOREM 2.1. *For testing the hypothesis $H_1 : \Sigma = \Sigma_0$ against the alternatives $K_1 : \Sigma \neq \Sigma_0$ for unknown mean μ, the modified* LR *test having the following ac-*

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ceptance region with respect to $S = \sum_{\alpha=1}^{N}(X_\alpha - \bar{X})(X_\alpha - \bar{X})'$,

$$\omega_1 = \{S \mid S \text{ is pd and } |S\Sigma_0^{-1}|^{n/2} \operatorname{etr}(-\tfrac{1}{2}\Sigma_0^{-1}S) \geqq c_\alpha\} \tag{2.2}$$

is unbiased.

Proof. Considering the matrix $H'\Sigma_0^{-\frac{1}{2}}S\Sigma_0^{-\frac{1}{2}}H$ instead of S, where H is an orthogonal matrix such that $H'\Sigma_0^{-\frac{1}{2}}\Sigma\Sigma_0^{-\frac{1}{2}}H = \Gamma$ (diagonal matrix) and a symmetric matrix $\Sigma_0^{\frac{1}{2}}$ is defined such that $\Sigma_0^{\frac{1}{2}}\Sigma_0^{\frac{1}{2}} = \Sigma_0$, we can assume without loss of generality that $\Sigma_0 = I$ and $\Sigma = \Gamma$, the diagonal matrix whose diagonal element is composed of p characteristic roots of $\Sigma\Sigma_0^{-1}$. Then the statistic S has the Wishart distribution $W(S \mid \Gamma, n)$ under K_1, so we can express

$$P_K(\omega_1) = c_{p,n}\int_{S\varepsilon\omega_1}|S|^{(n-p-1)/2}|\Gamma|^{-n/2}\operatorname{etr}(-\tfrac{1}{2}\Gamma^{-1}S)\,dS, \tag{2.3}$$

where the range of integration is over the set of all $p \times p$ pd matrices $S = (s_{ij})$ belonging to the region ω_1 and $dS = \prod_{i\leqq j} ds_{ij}$. The constant $c_{p,n}$ is given by

$$c_{p,n}^{-1} = \pi^{p(p-1)/4}2^{np/2}\prod_{i=1}^{p}\Gamma((n-i+1)/2). \tag{2.4}$$

Put $U = \Gamma^{-\frac{1}{2}}S\Gamma^{-\frac{1}{2}}$, then the matrix U is also pd and the Jacobian is given by $|\partial U/\partial S| = |\Gamma|^{-(p+1)/2}$. Hence we have

$$P_K(\omega_1) = c_{p,n}\int_{U\varepsilon\omega_1^*}|U|^{(n-p-1)/2}\operatorname{etr}(-\tfrac{1}{2}U)\,dU, \tag{2.5}$$

where the region ω_1^* means the set of all pd matrices U such that $\Gamma^{\frac{1}{2}}U\Gamma^{\frac{1}{2}}$ belongs to the region ω_1. Under the hypothesis H_1, the region ω_1^* is equal to ω_1. It follows that

$$\begin{aligned} P_H(\omega_1) - P_K(\omega_1) &= c_{p,n}\{\textstyle\int_{U\varepsilon\omega_1} - \int_{U\varepsilon\omega_1^*}\}|U|^{(n-p-1)/2}\operatorname{etr}(-\tfrac{1}{2}U)\,dU \\ &= c_{p,n}\{\textstyle\int_{U\varepsilon\omega_1-\omega_1\cap\omega_1^*} - \int_{U\varepsilon\omega_1^*-\omega_1\cap\omega_1^*}\} \\ &\quad \cdot|U|^{(n-p-1)/2}\operatorname{etr}(-\tfrac{1}{2}U)\,dU. \end{aligned} \tag{2.6}$$

Note that the inequality $|U|^{(n-p-1)/2}\operatorname{etr}(-\frac{1}{2}U) \geqq c_\alpha|U|^{-(p+1)/2}$ holds for $U \varepsilon \omega_1$. Since the integral $\int_{U\varepsilon\omega_1-\omega_1\cap\omega_1^*}|U|^{(n-p-1)/2}\operatorname{etr}(-\frac{1}{2}U)\,dU$ exists, we have

$$\int_{U\varepsilon\omega_1-\omega_1\cap\omega_1^*}|U|^{(n-p-1)/2}\operatorname{etr}(-\tfrac{1}{2}U)\,dU \geqq c_\alpha\int_{U\varepsilon\omega_1-\omega_1\cap\omega_1^*}|U|^{-(p+1)/2}\,dU. \tag{2.7}$$

Also we have

$$-\int_{U\varepsilon\omega_1^*-\omega_1\cap\omega_1^*}|U|^{(n-p-1)/2}\operatorname{etr}(-\tfrac{1}{2}U)\,dU \geqq -c_\alpha\int_{U\varepsilon\omega_1^*-\omega_1\cap\omega_1^*}|U|^{-(p+1)/2}\,dU. \tag{2.8}$$

Combining these two inequalities with $\int_{\omega_1\cap\omega_1^*}|U|^{-(p+1)/2}\,dU < \infty$, we can see that

$$\begin{aligned} P_H(\omega_1) - P_K(\omega_1) &\geqq c_{p,n}c_\alpha\{\textstyle\int_{U\varepsilon\omega_1-\omega_1\cap\omega_1^*} - \int_{U\varepsilon\omega_1^*-\omega_1\cap\omega_1^*}\}|U|^{-(p+1)/2}\,dU \\ &= c_{p,n}c_\alpha\{\textstyle\int_{U\varepsilon\omega_1} - \int_{U\varepsilon\omega_1^*}\}|U|^{-(p+1)/2}\,dU = 0. \end{aligned} \tag{2.9}$$

The last equality holds, since $|U|^{-(p+1)/2}\,dU$ is the invariant measure for the transformation of the pd matrix U to the pd matrix AUA' for any nonsingular matrix A, or equivalently by calculating the Jacobian, we can easily get

$$\int_{U\varepsilon\omega_1^*} |U|^{-(p+1)/2}\, dU = \int_{U\varepsilon\omega_1} |U|^{-(p+1)/2}\, dU.$$

Thus Theorem 2.1 is proved.

By the same argument the following theorems can be proved.

THEOREM 2.2. *For testing the hypothesis* $H_1' : \Sigma = \Sigma_0,\ \mu = \mu_0$ *against the alternatives* $K_1' : \Sigma \neq \Sigma_0,\ \mu = \mu_0$, *the* LR *test having the acceptance region* ω_1' *defined by* (2.1), *where* $S = \sum_{\alpha=1}^{N} (X_\alpha - \mu_0)(X_\alpha - \mu_0)'$ *is unbiased.*

We can also generalize Theorem 2.1 to the k-sample case. Let $p \times 1$ vectors $X_{j1}, X_{j2}, \cdots, X_{jN_j}$ $(N_j > p)$ be a random sample from p variate normal distribution with mean μ_j and covariance matrix Σ_j $(j = 1, 2, \cdots, k)$. Put $S_j = \sum_{\alpha=1}^{N_j} (X_{j\alpha} - \bar{X}_j)(X_{j\alpha} - \bar{X}_j)'$, $\bar{X}_j = N_j^{-1} \sum_{\alpha=1}^{N_j} X_{j\alpha}$, and $n_j = N_j - 1$. Then we have the following theorem by the same argument as in the proof of Theorem 2.1.

THEOREM 2.3. *For testing the hypothesis* $H_1'' : \Sigma_j = \Sigma_{0j}$ $(j = 1, 2, \cdots, k)$ *against the alternatives* $K_1'' : \Sigma_i \neq \Sigma_{0i}$ *for some* i, *where the mean* μ_j *is unspecified and* Σ_{0j} *is a given* pd *matrix, the modified* LR *test having the acceptance region*

$$(2.10)\quad \omega_1'' = \{(S_1, \cdots, S_k) \mid S_j \text{ is pd } (j = 1, 2, \cdots, k) \text{ and } \prod_{j=1}^{k} (|S_j\Sigma_{0j}^{-1}|^{n_j/2} \operatorname{etr}[-\tfrac{1}{2}\Sigma_{0j}^{-1}S_j]) \geqq c_\alpha\}$$

is unbiased.

3. Unbiasedness of the modified LR test for $\Sigma_1 = \Sigma_2$. Let $p \times 1$ vectors $X_{i1}, X_{i2}, \cdots, X_{iN_i}$, $(N_i > p)$ be a random sample from a p-variate normal distribution with unknown mean vector μ_i and unknown covariance matrix Σ_i $(\det \Sigma_i \neq 0)$ for $i = 1, 2$. From these samples we want to test the hypothesis $H_2 : \Sigma_1 = \Sigma_2$ against the alternatives $K_2 : \Sigma_1 \neq \Sigma_2$, where the means μ_1 and μ_2 are unspecified. The acceptance region of the LR test of this problem is given by

$$(3.1)\quad \omega_2' = \{(S_1, S_2) \mid S_1 \text{ and } S_2 \text{ are pd},\ |S_1|^{N_1/2}|S_2|^{N_2/2}|S_1 + S_2|^{-(N_1+N_2)/2} \geqq c_\alpha\},$$

where $S_i = \sum_{\alpha=1}^{N_i} (X_{i\alpha} - \bar{X})(X_{i\alpha} - \bar{X}_i)'$ and $\bar{X}_i = N_i^{-1} \sum_{\alpha=1}^{N_i} X_{i\alpha}$ for $i = 1, 2$. In case $p = 1$, this acceptance region gives an unbiased test if and only if $N_1 = N_2$ as is shown by Brown [3] and further the UMP unbiased test is given by replacing $|S_1|^{N_1/2}|S_2|^{N_2/2}|S_1 + S_2|^{-(N_1+N_2)/2}$ to $|S_1|^{(N_1-1)/2}|S_2|^{(N_2-1)/2}|S_1 + S_2|^{-(N_1+N_2-2)/2}$ in (3.1), which can be seen in Lehmann ([5], p. 170) by some calculation. After this modification of changing the sample size N_i to the degrees of freedom $n_i = N_i - 1$ for $i = 1, 2$, we can prove the unbiasedness in the multivariate case, which was conjectured in Anderson and Das Gupta [2].

THEOREM 3.1. *For testing the hypothesis* $H_2 : \Sigma_1 = \Sigma_2$ *against the alternatives* $K_2 : \Sigma_1 \neq \Sigma_2$ *for unknown* μ_1 *and* μ_2, *the modified* LR *test having the acceptance region*

$$(3.2)\quad \omega_2 = \{(S_1, S_2) \mid S_1 \text{ and } S_2 \text{ are pd},\ |S_1|^{n_1/2}|S_2|^{n_2/2}|S_1 + S_2|^{-(n_1+n_2)/2} \geqq c_\alpha\},$$

where $n_i = N_i - 1$ $(i = 1, 2)$, *is unbiased.*

PROOF. By the invariance of the region ω_2 for the transformation $(S_1, S_2) \to (AS_1A', AS_2A')$ for any nonsingular matrix A, we can assume, without loss of generality, that $\Sigma_1 = \Gamma$ (a diagonal matrix whose diagonal elements are given by p characteristic roots of $\Sigma_1\Sigma_2^{-1}$) and $\Sigma_2 = I$. Then the probability $P_K(\omega_2)$ of the acceptance region ω_2 under the alternative K is given by

$$(3.3)\quad c_{p,n_1}c_{p,n_2}\int_{(S_1,S_2)\varepsilon\omega_2}|S_1|^{(n_1-p-1)/2}\,|S_2|^{(n_2-p-1)/2}|\Gamma|^{-n_1/2}\cdot\operatorname{etr}-\tfrac{1}{2}(\Gamma^{-1}S_1+S_2)\,dS_1\,dS_2.$$

Putting $S_1 = U_1$ and $S_2 = U_1^{\frac{1}{2}}U_2U_1^{\frac{1}{2}}$, where $U_1^{\frac{1}{2}}$ can be defined in some unique way with probability one, such that $U_1^{\frac{1}{2}}U_1^{\frac{1}{2}} = U_1$ and $U_1^{\frac{1}{2}}$ is symmetric, then the Jacobian is given by $|\partial(S_1, S_2)/\partial(U_1, U_2)| = |U_1|^{(p+1)/2}$. So we obtain (3.3) as

$$(3.4)\quad \begin{aligned} &c_{p,n_1}c_{p,n_2}\int_{(I,U_2)\varepsilon\omega_2}|U_1|^{(n_1+n_2-p-1)/2}\\ &\quad\cdot|U_2|^{(n_2-p-1)/2}|\Gamma|^{-n_1/2}\operatorname{etr}[-\tfrac{1}{2}(\Gamma^{-1}+U_2)U_1]\,dU_1\,dU_2\\ &= c_{p,n_1}c_{p,n_2}c_{p,n_1+n_2}^{-1}\int_{(I,U_2)\varepsilon\omega_2}|U_2|^{(n_2-p-1)/2}|\Gamma|^{-n_1/2}|\Gamma^{-1}+U_2|^{-(n_1+n_2)/2}\,dU_2.\end{aligned}$$

Transforming the variables U_2 to $V = \Gamma^{\frac{1}{2}}U_2\Gamma^{\frac{1}{2}}$, we have

$$(3.5)\quad P_K(\omega_2) = c_{p,n_1}c_{p,n_2}c_{p,n_1+n_2}^{-1}\int_{V\varepsilon\omega_2^*}|V|^{(n_2-p-1)/2}|I+V|^{-(n_1+n_2)/2}\,dV,$$

where the region ω_2^* means the set of all $p \times p$ pd matrices V such that $(I, \Gamma^{-\frac{1}{2}}V\Gamma^{-\frac{1}{2}})\ \varepsilon\ \omega_2$. Let $\tilde{\omega}_2$ be the set of all $p \times p$ pd matrices V such that $(I, V)\varepsilon\omega_2$ and put $b = c_{p,n_1}c_{p,n_2}c_{p,n_1+n_2}^{-1}$. Then we can see that $\tilde{\omega}_2 = \{V \mid V \text{ is pd and } |V|^{n_2/2}|I+V|^{-(n_1+n_2)/2} \geqq c_\alpha\}$ and

$$(3.6)\quad \begin{aligned} P_H(\omega_2) - P_K(\omega_2) &= b\{\textstyle\int_{V\varepsilon\tilde{\omega}_2} - \int_{V\varepsilon\omega_2^*}\}|V|^{(n_2-p-1)/2}|I+V|^{-(n_1+n_2)/2}\,dV\\ &= b\{\textstyle\int_{V\varepsilon\tilde{\omega}_2-\omega_2^*\cap\tilde{\omega}_2} - \int_{V\varepsilon\omega_2^*-\tilde{\omega}_2\cap\omega_2^*}\}\\ &\quad\cdot|V|^{n_2/2}|I+V|^{-(n_1+n_2)/2}|V|^{-(p+1)/2}\,dV.\end{aligned}$$

Noting that the integral $\int_{V\varepsilon\tilde{\omega}_2}|V|^{(n_2-p-1)/2}|I+V|^{-(n_1+n_2)/2}\,dV$ exists and the inequality

$$(3.7)\quad \int_{V\varepsilon\omega}|V|^{(n_2-p-1)/2}|I+V|^{-(n_1+n_2)/2}\,dV \geqq c_\alpha\int_{V\varepsilon\omega}|V|^{-(p+1)/2}\,dV$$

holds for any subset ω in $\tilde{\omega}_2$, we can see that $\int_{V\varepsilon\tilde{\omega}_2}|V|^{-(p+1)/2}\,dV < \infty$. It follows that

$$(3.8)\quad \begin{aligned} P_H(\omega_2) - P_K(\omega_2) &\geqq bc_\alpha\{\textstyle\int_{V\varepsilon\tilde{\omega}_2-\omega_2^*\cap\tilde{\omega}_2} - \int_{V\varepsilon\omega_2^*-\tilde{\omega}_2\cap\omega_2^*}\}|V|^{-(p+1)/2}\,dV\\ &= bc_\alpha\{\textstyle\int_{V\varepsilon\tilde{\omega}_2} - \int_{V\varepsilon\omega_2^*}\}|V|^{-(p+1)/2}\,dV = 0.\end{aligned}$$

Thus we have $P_K(\omega_2^c) \geqq P_H(\omega_2^c)$, which implies Theorem 3.1.

It is believed that the modified LR test for the equality of covariance matrices more than two p-variate normal populations is unbiased. This is true when $p = 1$ (Pitman [6]), but we fail to prove this for $p > 1$.

4. Unbiasedness of the LR test for sphericity. Although the following theorem is already obtained by Gleser [4], we shall show in this section that it can be proved from our point of view. We shall use the notation in Section 2 without reference.

THEOREM 4.1. *For testing the hypothesis* $H_3 : \Sigma = \sigma^2 I$ *against the alternatives* $K_3 : \Sigma \neq \sigma^2 I$, *where* σ^2 *is an unspecified positive number, the* LR *test having the acceptance region*

$$\omega_3 = \{S \mid S \text{ is pd and } |S|^{n/2}(\operatorname{tr} S)^{-pn/2} \geqq c_\alpha\} \tag{4.1}$$

for $S = \sum_{\alpha=1}^{N} (X_\alpha - \bar{X})(X_\alpha - \bar{X})'$ *is unbiased.*

PROOF. By the invariance of the acceptance region ω_3 for the transformation $S \to HSH'$ for any orthogonal matrix H, we can assume $\Sigma = \Gamma$ (a diagonal matrix). Then we can write the probability of the acceptance region under K,

$$\begin{aligned} P_K(\omega_3) &= c_{p,n} \int_{S\varepsilon\omega_3} |S|^{(n-p-1)/2}|\Gamma|^{-n/2} \operatorname{etr}(-\tfrac{1}{2}\Gamma^{-1}S)\, dS \\ &= c_{p,n} \int_{\Gamma^{\frac{1}{2}}U\Gamma^{\frac{1}{2}}\varepsilon\omega_3} |U|^{(n-p-1)/2} \operatorname{etr}(-\tfrac{1}{2}U)\, dU. \end{aligned} \tag{4.2}$$

If we put $U = v_{11}V_0$, where the symmetric matrix V_0 is given by

$$V_0 = \begin{pmatrix} 1, & v_{12}, & \cdots, & v_{1p} \\ v_{21}, & v_{22}, & \cdots, & v_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ v_{p1}, & v_{p2}, & \cdots, & v_{pp} \end{pmatrix}, \tag{4.3}$$

then $|\partial U/\partial(v_{11}, V_0)| = v_{11}^{[p(p+1)/2]-1}$. By the invariance of the region ω_3 for the transformation $U \to cU$ for any positive number c, we have

$$\begin{aligned} P_K(\omega_3) &= c_{p,n} \int_{\Gamma^{\frac{1}{2}}V_0\Gamma^{\frac{1}{2}}\varepsilon\omega_3} v_{11}^{(np/2)-1}|V_0|^{(n-p-1)/2} \operatorname{etr}(-\tfrac{1}{2}v_{11}V_0)\, dv_{11}\, dV_0 \\ &= 2^{np/2}c_{p,n}\Gamma(np/2) \int_{\Gamma^{\frac{1}{2}}V_0\Gamma^{\frac{1}{2}}\varepsilon\omega_3} |V_0|^{(n-p-1)/2}(\operatorname{tr} V_0)^{-np/2}\, dV_0. \end{aligned} \tag{4.4}$$

Let ω_3^* be the set of all pd matrices V_0 of the form (4.3) such that $\Gamma^{\frac{1}{2}}V_0\Gamma^{\frac{1}{2}} \varepsilon\, \omega_3$. Then by the same argument as in Section 2, we can easily see that

$$P_H(\omega_3) - P_K(\omega_3) \geqq 2^{np/2}\Gamma(\tfrac{1}{2}np)c_{p,n}c_\alpha\{\textstyle\int_{V_0\varepsilon\omega_3} - \int_{V_0\varepsilon\omega_3^*}\}|V_0|^{-(p+1)/2}\, dV_0. \tag{4.5}$$

Make the transformation $W_0 = \lambda_1^{-1}\Gamma^{\frac{1}{2}}V_0\Gamma^{\frac{1}{2}}$ in the second integration, where $\Gamma = \operatorname{diag}(\lambda_1, \cdots, \lambda_p)$ and pd matrix W_0 is of the form (4.3). The Jacobian is given by $|\partial W_0/\partial V_0| = |\Gamma|^{(p+1)/2}\lambda_1^{-p(p+1)/2}$, which implies

$$\int_{V_0\varepsilon\omega_3^*} |V_0|^{-(p+1)/2}\, dV_0 = \int_{W_0\varepsilon\omega_3} |W_0|^{-(p+1)/2}\, dW_0. \tag{4.6}$$

Thus Theorem 4.1 is proved.

We can also prove the following two theorems by the same argument as in the proof of Theorem 4.1. We shall remark that the LR test for sphericity is unbiased, but in order to prove the unbiasedness for the sphericity test in k sample case, we must modify the LR test by reducing the sample size N_i to the degrees of freedom $n_i = N_i - 1$.

THEOREM 4.2. *For testing the hypothesis* $H_3'' : \Sigma_j = \sigma^2\Sigma_{0j}$ $(j = 1, 2, \cdots, k)$ *against the alternatives* $K_3'' : \Sigma_i \neq \sigma^2\Sigma_{0i}$ *for some* i, *where* σ^2 *is unknown constant and* Σ_{0j} *is* *a given* pd *matrix, the modified* LR *test having the acceptance region*

(4.7) $\omega_3'' = \{(S_1, \cdots, S_k) \mid S_j$ *is* pd $(j = 1, 2, \cdots, k)$ *and*

$$[\prod_{j=1}^k |S_j\Sigma_{0j}^{-1}|^{n_j/2}](\sum_{j=1}^k \operatorname{tr} \Sigma_{0j}^{-1} S_j)^{-np/2} \geqq c_\alpha\}$$

where $S_j = \sum_{\alpha=1}^{N_j} (X_{j\alpha} - \bar{X}_j)(X_{j\alpha} - \bar{X}_j)'$ *and* $n = \sum_{j=1}^k n_j$ *is unbiased.*

THEOREM 4.3. *For testing the hypothesis* $H_3''': \Sigma_j = \sigma_j^2 \Sigma_{0j}$ $(j = 1, 2, \cdots, k)$ *against the alternatives* $K_3''': \Sigma_i \neq \sigma_i^2 \Sigma_{0i}$ *for some* i, *where* σ_j^2 *is unknown constant and* Σ_{0j} *is a given* pd *matrix, the modified* LR *test having the acceptance region*

(4.8) $\omega_3''' = \{(S_1, \cdots, S_k) \mid S_j$ *is* pd $(j = 1, 2, \cdots, k)$ *and*

$$\prod_{j=1}^k [|S_j\Sigma_{0j}^{-1}|^{n_j/2} (\operatorname{tr} S_j\Sigma_{0j}^{-1})^{-n_j p/2}] \geqq c_\alpha\}$$

is unbiased.

5. Unbiasedness of the LR test for $\Sigma = \Sigma_0$ and $\mu = \mu_0$. Using the same sample as in Section 2, we want to test the hypothesis $H_4: \Sigma = \Sigma_0$ and $\mu = \mu_0$ against the alternatives $K_4: \Sigma \neq \Sigma_0$ or $\mu \neq \mu_0$. This problem is different from the problem (H_1', K_1') in that the mean vector μ is unknown. The LR test given in Anderson ([1], p. 268) for this problem is proved to be unbiased. It may be interesting to note that the modification of the LR test is not necessary in this case.

THEOREM 5.1. *For testing the hypothesis* $H_4: \Sigma = \Sigma_0$, $\mu = \mu_0$ *against the alternatives* $K_4: \Sigma \neq \Sigma_0$ *or* $\mu \neq \mu_0$, *the* LR *test having the acceptance region*

(5.1) $\omega_4 = \{(\bar{X}, S) \mid \bar{X}$ *is* $p \times 1$ *vector and* S *is* pd *such that*

$$|S\Sigma_0^{-1}|^{N/2} \operatorname{etr} [-\tfrac{1}{2}\Sigma_0^{-1}\{S + N(\bar{X} - \mu_0)(\bar{X} - \mu_0)'\}] \geqq c_\alpha\}$$

for $S = \sum_{\alpha=1}^N (X_\alpha - \bar{X})(X_\alpha - \bar{X})'$ *and* $\bar{X} = N^{-1} \sum_{\alpha=1}^N X_\alpha$ *is unbiased.*

PROOF. Without loss of generality we may assume $\Sigma_0 = I$ and $\Sigma = \Gamma$ (a diagonal matrix). Then we can write

$$(5.2) \quad P_K(\omega_4) = N^{p/2} c_{p,n} (2\pi)^{-p/2} \cdot \int_{(\bar{X}, S)\varepsilon\omega_4} |S|^{(N-p-2)/2} |\Gamma|^{-N/2} \operatorname{etr} [-\tfrac{1}{2}\Gamma^{-1}\{S + N(\bar{X} - \mu)(\bar{X} - \mu)'\}]\, d\bar{X}\, dS.$$

Put $U = \Gamma^{-\frac{1}{2}} S \Gamma^{-\frac{1}{2}}$ and $\bar{Y} - \mu_0 = \Gamma^{-\frac{1}{2}}(\bar{X} - \mu)$, then $|\partial(\bar{Y}, U)/\partial(\bar{X}, S)| = |\Gamma|^{-(p+2)/2}$. We have

$$(5.3) \quad P_K(\omega_4) = N^{p/2} c_{p,n} (2\pi)^{-p/2} \cdot \int_{(\bar{Y}, U)\varepsilon\omega_4^*} |U|^{(N-p-2)/2} \operatorname{etr} [-\tfrac{1}{2}\{U + N(\bar{Y} - \mu_0)(\bar{Y} - \mu_0)'\}]\, d\bar{Y}\, dU,$$

where the region ω_4^* means the set of all $p \times 1$ vectors $\bar{Y}$ and pd matrices U such that $(\Gamma^{\frac{1}{2}}(\bar{Y} - \mu_0) + \mu, \Gamma^{\frac{1}{2}} U \Gamma^{\frac{1}{2}}) \, \varepsilon \, \omega_4$. By the same argument as in Section 2, we can see that

$$(5.4) \quad P_H(\omega_4) - P_K(\omega_4) \geqq N^{p/2} c_{p,n} c_\alpha (2\pi)^{-p/2} \{\int_{(\bar{Y}, U)\varepsilon\omega_4} - \int_{(\bar{Y}, U)\varepsilon\omega_4^*}\} |U|^{-(p+2)/2}\, d\bar{Y}\, dU.$$

Make the transformation $(\bar{Y}, U)$ to $(\bar{Z}, V)$ by $\bar{Z} = \Gamma^{\frac{1}{2}}(\bar{Y} - \mu_0) + \mu$ and

$V = \Gamma^{\frac{1}{2}} U \Gamma^{\frac{1}{2}}$ in the second integration, then we easily see that

$$\int_{(\bar{Y}, U) \varepsilon \omega_4^*} |U|^{-(p+2)/2} \, d\bar{Y} \, dU = \int_{(\bar{Z}, V) \varepsilon \omega_4} |V|^{-(p+2)/2} \, d\bar{Z} \, dV.$$

Thus Theorem 5.1 is proved.

We can also generalize Theorem 5.1 to the k sample case by the same argument as above.

THEOREM 5.2. *For testing the hypothesis* $H_4': \Sigma_j = \Sigma_{0j}$ *and* $\mu_j = \mu_{0j}$ $(j = 1, 2, \cdots, k)$ *against the alternatives* $K_4': \Sigma_i \neq \Sigma_{0i}$ *for some* i *or* $\mu_j \neq \mu_{0j}$ *for some* j, *where* Σ_{0j} *is a given* pd *matrix and the mean* μ_{0j} *is a given* $p \times 1$ *vector, the* LR *test having the acceptance region*

$$(5.5) \quad \omega_4' = \{(\bar{X}_1, \cdots, \bar{X}_k, S_1, \cdots, S_k) | S_j \text{ is pd } (j = 1, 2, \cdots, k) \text{ and } \textstyle\prod_{j=1}^k [|S_j \Sigma_{0j}^{-1}|^{N_j/2} \operatorname{etr} -\frac{1}{2}\Sigma_{0j}^{-1}\{S_j + N_j(\bar{X}_j - \mu_{0j})(\bar{X}_j - \mu_{0j})'\}] \geqq c_\alpha\}$$

is unbiased.

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